Free Movement of a Rigid Body in a Generalized Newtonian Fluid

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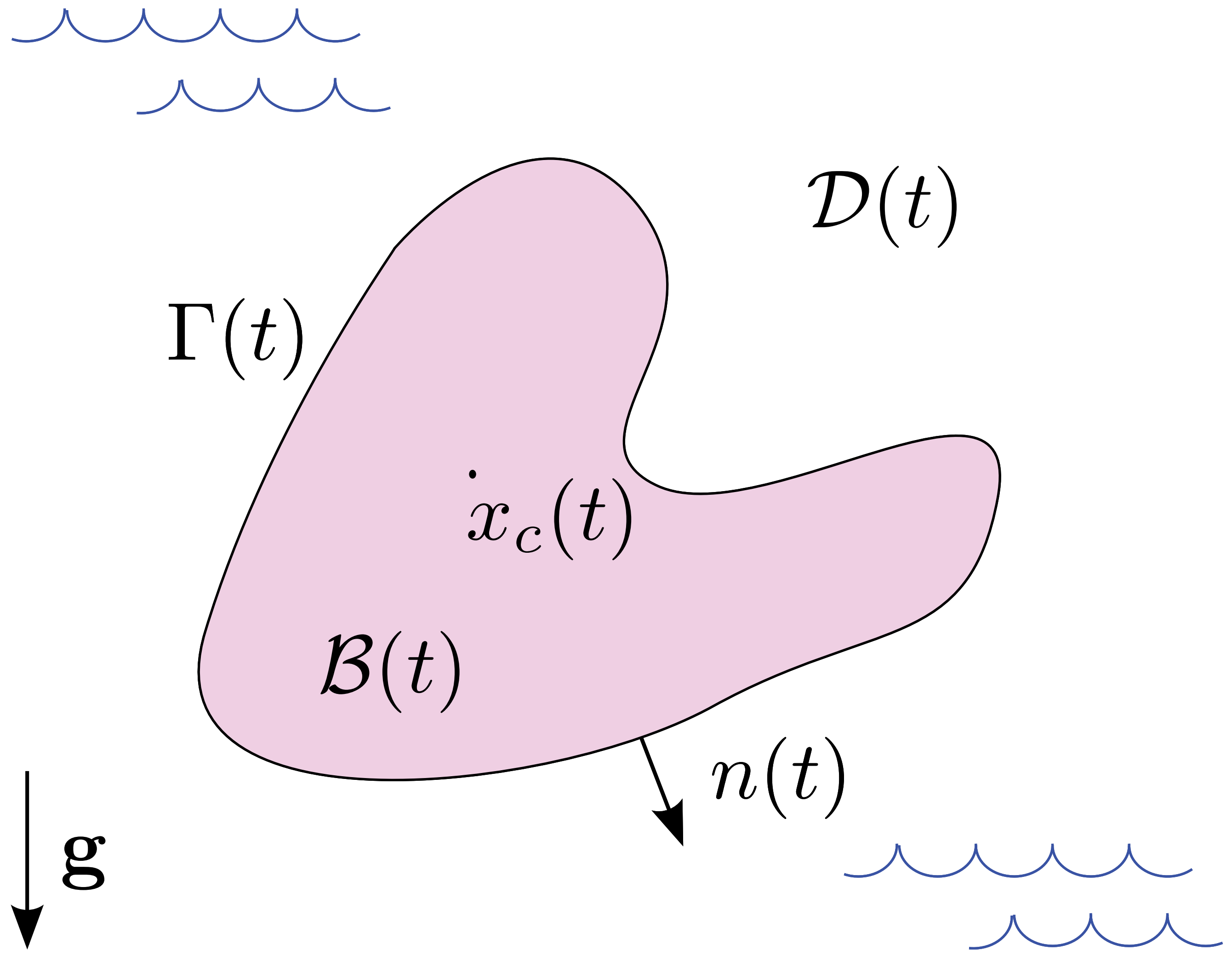
$\mathcal{D}(t)$
$\Gamma(t)$
$x_c(t)$
$\mathcal{B}(t)$
$n(t)$
$\mathbf{g}$

Model

Fluid: Generalized Navier-Stokes Equations

$$\begin{cases} v_t - \operatorname{div} \mathbf{T}(v,q) + v \cdot \nabla v & = & \mathbf{g}, & \text{in } \mathbb{R}_+ \times \mathcal{D}(\cdot), \\ \operatorname{div} v & = & 0, & \text{in } \mathbb{R}_+ \times \mathcal{D}(\cdot), \\ v & = & \eta + \omega \times (x - x_c), & \text{on } \mathbb{R}_+ \times \Gamma(\cdot), \\ v(0) & = & v_0, & \text{in } \mathcal{D}(0). \end{cases}$$

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Rigid Body: Balance of momentum and angular momentum

$$\begin{cases} \mathrm{m}\eta' &= \mathrm{m}\mathbf{g} - \int_{\Gamma(t)} \mathbf{T}(v,q)n \,\mathrm{d}\sigma, & \text{in } \mathbb{R}_+, \\ (J\omega)' &= -\int_{\Gamma(t)} (x - x_c) \times \mathbf{T}(v,q)n \,\mathrm{d}\sigma, & \text{in } \mathbb{R}_+, \\ \eta(0) = \eta_0 \ \text{ und } \ \omega(0) = \omega_0. \end{cases}$$

Solve for: v, q, η, ω

Generalized Newtonian Fluids

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Stress tensor: $\mathbf{T}(v,q) := 2\mu(|E^v|_2^2)E^v - q\mathrm{Id}$

Deformation tensor: $E^v := \frac{1}{2}(\nabla v + (\nabla v)^T)$

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Viscosity:

$$\mu \in C^{1,1}(\mathbb{R}_+)$$

$$\mu(s) > 0$$

$$\mu(s) + 2s\mu'(s) > 0$$

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Special case: fluids of "power-law" type ($d \geq 1$)

$$\mu(s) = \mu_0(1+s)^{(d-2)/2}$$

shear-thinning ($d < 2$), shear-thickening ($d > 2$)

Generalized Stokes Problem

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Quasi-linear "Fluid Operator":

$$
\begin{aligned}
(\,A(v)v)_i \quad &:= \quad \operatorname{div}(\,2\mu(|E^v|_2^2)E^v)_i \\
&= \quad \mu(|\,E^v\,|_2^2)\Delta v_i + 4\mu'(|\,E^v\,|_2^2)\sum_{j,k,l=1}^{3} e_{ij}^v\, e_{kl}^v\;\; \partial_j\partial_l v_k
\end{aligned}
$$

$$\mu = \text{const.} \;\Rightarrow\; A(v) = \mu\Delta$$

Generalized Stokes Problem

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Quasi-linear "Fluid Operator":

$$(A(u_*)v)_i := \operatorname{div}(2\mu(|E^v|_2^2)E^v)_i$$

$$= \mu(|E^{u_*}|_2^2)\Delta v_i + 4\mu'(|E^{u_*}|_2^2) \sum_{j,k,l=1}^{3} e_{ij}^{u_*} e_{kl}^{u_*} \partial_j \partial_l v_k$$

$$\mu = \text{const.} \Rightarrow A(v) = \mu\Delta$$

Generalized Stokes Problem

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$$\mu = \text{const.} \;\; \Rightarrow \;\; A(v) = \mu\Delta$$

- Bothe/Prüss '07:

$$\begin{cases} v_t - A(u_*)v + \nabla q &= \quad f, \quad \text{in } (0,T)\times\mathcal{D}, \\ \mathrm{div}\, v &= \quad 0, \quad \text{in } (0,T)\times\mathcal{D}, \\ v &= \quad h, \quad \text{on } (0,T)\times\Gamma, \\ v(0) &= \quad v_0, \quad \text{in } \mathcal{D}. \end{cases}$$

has mazimal L^p -regularity for $p > 5$.

Overview

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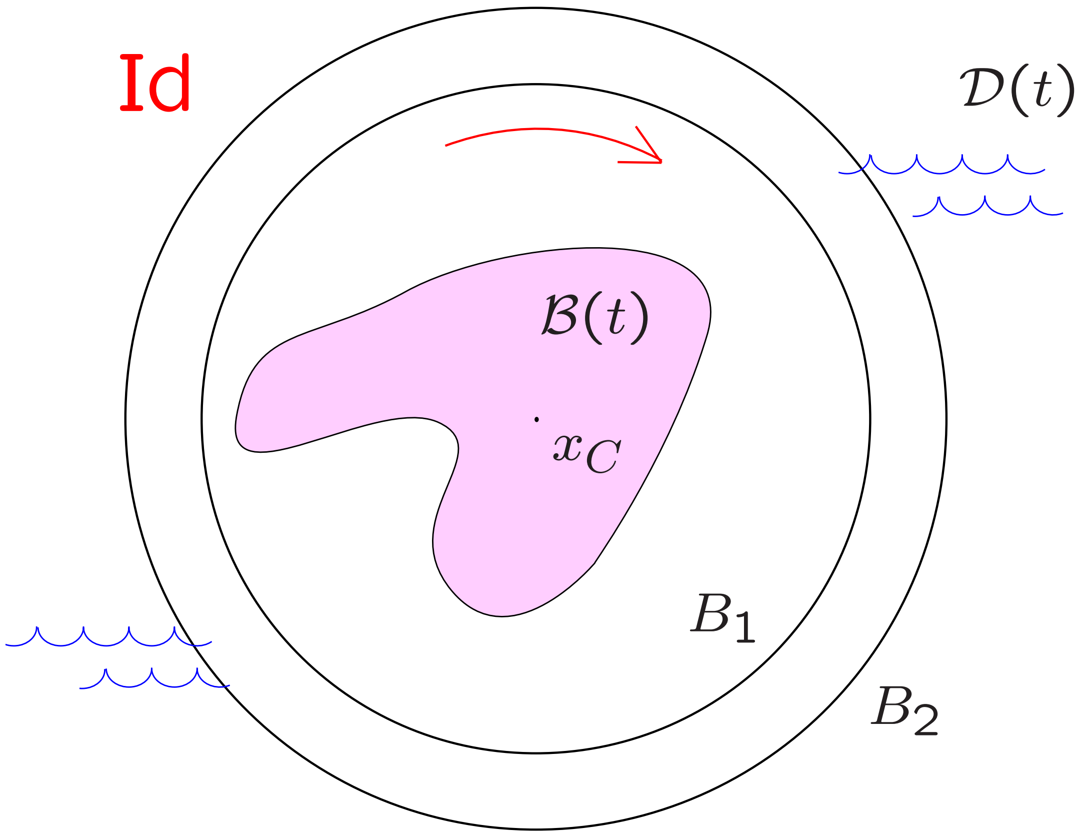
Outline

- Change of Coordinates
- Linear transformed coupled problem
  - Bothe/Prüss result
- Contraction mapping argument

Known Results

- Feireisl/Hillairet/Nečasová ’08
- Takahashi ’03
- Galdi/Silvestre ’02

Coordinate Transform (non-linear, local)



- Inoue/Wakimoto ’77
- Takahashi ’03

Transformed System of Equations

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$$\begin{cases} u_t - A(u_*)u + \nabla p = \mathcal{A}(u+u_*)(u+u_*) - A(u_*)u & \text{in } (0,T)\times\mathcal{D}, \\ \qquad +(\nabla - \mathcal{G})p - \mathcal{N}(u) + l.o.t., & \\ \operatorname{div} u = 0, & \text{in } (0,T)\times\mathcal{D}, \\ u = \xi + \Omega \times y, & \text{on } (0,T)\times\Gamma, \\ u(0) = v_0, & \text{in } \mathcal{D}, \\ \mathrm{m}\xi' + \int_\Gamma \mathbf{T}_*(u,p)n\,\mathrm{d}\sigma = \int_\Gamma (\mathbf{T}_* - \mathcal{T})(u,p)n\,\mathrm{d}\sigma + l.o.t., & \text{in } (0,T), \\ I\Omega' + \int_\Gamma y \times \mathbf{T}_*(u,p)n\,\mathrm{d}\sigma = \int_\Gamma y \times (\mathbf{T}_* - \mathcal{T})(u,p)n\,\mathrm{d}\sigma + l.o.t, & \text{in } (0,T), \\ (\xi(0),\Omega(0)) = (\eta_0,\omega_0), & \end{cases}$$

- $\mathcal{A},\mathcal{N},\mathcal{G},\mathcal{T}$ transformed differential operators, e.g.

$$(\mathcal{A}(u)u)_i = \mu(|\mathcal{E}|_2^2)(\Sigma_{j,k=1}^3 g^{jk}\partial_{jk}^2 u_i + 2\Sigma_{j,k,l=1}^3 g^{kl}\Gamma_{kl}^i \partial_l u_j + \ldots) + \ldots$$

Transformed System of Equations

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- $t = 0 \;\Rightarrow\; \mathcal{A}(u+u_*) = A(u+u_*),\; \mathcal{G} = \nabla,\; \dots$
- $[A(u_*) - A(u+u_*)](u+u_*),\; \mathcal{N}(u), \dots$ “quadratic”

Linearized System

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$$
\begin{cases}
u_t - A(u_*)u + \nabla p = F_0, & \text{in } (0,T)\times\mathcal{D},\\
\operatorname{div} u = 0, & \text{in } (0,T)\times\mathcal{D},\\
u = \xi + \Omega\times y, & \text{on } (0,T)\times\Gamma,\\
u(0) = v_0, & \text{in } \mathcal{D},\\
\mathrm{m}\xi' + \int_\Gamma \mathbf{T}_*(u,p)n\,\mathrm{d}\sigma = F_1, & \text{in } (0,T),\\
I\Omega' + \int_\Gamma y\times\mathbf{T}_*(u,p)n\,\mathrm{d}\sigma = F_2, & \text{in } (0,T),\\
(\xi(0),\Omega(0)) = (\eta_0,\omega_0), &
\end{cases}
$$

Linearized System

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$\xi, \Omega \in W^{1,p}(0,T)$

$u(\xi,\Omega), p(\xi,\Omega)$

Linearized System

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$\xi, \Omega \in W^{1,p}(0,T)$

$$\mathbb{I}\begin{pmatrix} \xi' \\ \Omega' \end{pmatrix} = \int_\Gamma \begin{pmatrix} \mathbf{T}_*(u(\xi,\Omega), p(\xi,\Omega))n \\ y \times \mathbf{T}_*(u(\xi,\Omega), p(\xi,\Omega))n \end{pmatrix} \mathrm{d}\sigma + \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}$$

Linearized System

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$$\left\{\begin{aligned} u_t - A(u_*)u + \nabla p &= F_0, && \text{in } (0,T)\times\mathcal{D},\\ \operatorname{div} u &= 0, && \text{in } (0,T)\times\mathcal{D},\\ u &= \xi + \Omega\times y, && \text{on } (0,T)\times\Gamma,\\ u(0) &= v_0, && \text{in } \mathcal{D},\\ \mathrm{m}\xi' + \int_\Gamma \mathbf{T}_*(u,p)n\,\mathrm{d}\sigma &= F_1, && \text{in } (0,T),\\ I\Omega' + \int_\Gamma y\times\mathbf{T}_*(u,p)n\,\mathrm{d}\sigma &= F_2, && \text{in } (0,T),\\ (\xi(0),\Omega(0)) &= (\eta_0,\omega_0), \end{aligned}\right.$$

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$$p_T(\xi,\Omega) \quad + \quad p_N(\xi,\Omega)$$

Linearized System

$$
\left\{
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\end{aligned}
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$$

$\xi, \Omega \in W^{1,p}(0,T)$

$$
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$$

$$p_T(\xi,\Omega) \;+\; p_N(\xi,\Omega)$$

✓ $\|p_T\| \le C\|u\|$

Linearized System

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$$p_T(\xi,\Omega) \quad + \quad p_N(\xi,\Omega)$$

✓ $\|p_T\| \le C\|u\|$

Solve NP:

$$\begin{cases} \Delta v = 0 & \text{in } \mathcal{D}, \\ -\frac{\partial v}{\partial n} = (\xi + \Omega\times y)\cdot n & \text{on } \Gamma. \end{cases}$$

✓ $p_N = v' = \mathbb{M}\begin{pmatrix}\xi' \\ \Omega'\end{pmatrix}$

✓ $(\mathbb{I}+\mathbb{M})$ invertible

Main Result

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Assume

- $p > 5$,
- $\mathcal{B}$ bounded $C^{2,1}$ -domain,
- $\eta_0, \omega_0 \in \mathbb{R}^3$ and $v_0 \in W^{2-2/p,p}(\mathcal{D})$,
- $\operatorname{div} v_0 = 0$, on Γ : $v_0(x) = \eta_0 + \omega_0 \times x$.

Then there exists a unique solution

$$
\begin{aligned}
v &\in L^p(0,T_0;W^{2,p}(\mathcal{D}(\cdot))) \cap W^{1,p}(0,T_0;L^p(\mathcal{D}(\cdot))) \\
q &= q_0 + \mathbf{g} \cdot Y,\ q_0 \in L^p(0,T_0;\widehat{W}^{1,p}(\mathcal{D}(\cdot))),\ Y \in C^1(0,T_0;C^\infty(\mathcal{D}(\cdot))), \\
\eta, \omega &\in W^{1,p}(0,T_0;\mathbb{R}^3),
\end{aligned}
$$

on a maximal interval $(0,T_0)$, $T_0 > 0$.