Analysis of Stokes equations by penalty method

Y. Saito (Waseda Univ.)

joint work with T.Kubo (Univ. of Tsukuba) and Y.Shibata (Waseda Univ.)

2010/03/08

Stokes equations

Stokes equations

$$\begin{cases} \partial_t \mathbf{u} - \Delta \mathbf{u} + \nabla \pi = 0 & \text{in } (0,\infty) \times \Omega, \\ \nabla \cdot \mathbf{u} = 0 & \text{in } (0,\infty) \times \Omega, \\ \mathbf{u}(t,x) = 0 & \text{on } (0,\infty) \times \partial\Omega, \\ \mathbf{u}(0,x) = \mathbf{a}(x) & \text{in } \Omega. \end{cases} \tag{GS}$$

$\mathbf{u} = (u_1, \cdots, u_n)$: Velocity field
π : Scalar pressure
$\mathbf{a}$: Initial data
Ω : Some domains ($\mathbb{R}^n$, $\mathbb{R}^n_+$, bounded domains)

Penalty method

$\alpha > 0$: Constant

Stokes equations approximated by Penalty method

$$\begin{cases} \partial_t \mathbf{u} - \Delta \mathbf{u} + \nabla \pi = 0 & \text{in } (0,\infty) \times \Omega, \\ \nabla \cdot \mathbf{u} = -\pi/\alpha & \text{in } (0,\infty) \times \Omega, \\ \mathbf{u}(t,x) = 0 & \text{on } (0,\infty) \times \partial\Omega, \\ \mathbf{u}(0,x) = \mathbf{a}(x) & \text{in } \Omega. \end{cases} \tag{PS}$$

Penalty method

$\alpha > 0$: Constant

Stokes equations approximated by Penalty method

$$\begin{cases} \partial_t \mathbf{u} - \Delta \mathbf{u} + \nabla \pi = 0 & \text{in } (0, \infty) \times \Omega, \\ \nabla \cdot \mathbf{u} = -\pi/\alpha & \text{in } (0, \infty) \times \Omega, \\ \mathbf{u}(t, x) = 0 & \text{on } (0, \infty) \times \partial\Omega, \\ \mathbf{u}(0, x) = \mathbf{a}(x) & \text{in } \Omega. \end{cases} \tag{PS}$$

Merit of Penalty method

We have $\pi = -\alpha \nabla \cdot \mathbf{u}$, and obtain the following equation:

$$\partial_t \mathbf{u} - \Delta \mathbf{u} - \alpha \nabla (\nabla \cdot \mathbf{u}) = 0.$$

Therefore we can eliminate π without Helmholtz decomposition.

Resolvent problem

In order to get resolvent estimate,
I consider the corresponding resolvent problem.

Resolvent problem

$$\begin{cases} \lambda \mathbf{u} - \Delta \mathbf{u} + \nabla \pi = \mathbf{f} & \text{in } \Omega, \\ \nabla \cdot \mathbf{u} = -\pi/\alpha & \text{in } \Omega, \\ \mathbf{u}(x) = 0 & \text{on } \partial\Omega. \end{cases} \tag{PRS}$$

Resolvent problem

In order to get resolvent estimate,
I consider the corresponding resolvent problem.

Resolvent problem

$$\begin{cases} \lambda \mathbf{u} - \Delta \mathbf{u} + \nabla \pi = \mathbf{f} & \text{in } \Omega, \\ \nabla \cdot \mathbf{u} = -\pi/\alpha & \text{in } \Omega, \\ \mathbf{u}(x) = 0 & \text{on } \partial\Omega. \end{cases} \tag{PRS}$$

Resolvent problem corresponding to Stokes equations

$$\begin{cases} \lambda \mathbf{u} - \Delta \mathbf{u} + \nabla \pi = \mathbf{f} & \text{in } \Omega, \\ \nabla \cdot \mathbf{u} = 0 & \text{in } \Omega, \\ \mathbf{u}(x) = 0 & \text{on } \partial\Omega. \end{cases} \tag{RS}$$

Resolvent problem

In order to get resolvent estimate,
I consider the corresponding resolvent problem.

Resolvent problem

$$\begin{cases} \lambda \mathbf{u}_\alpha - \Delta \mathbf{u}_\alpha + \nabla \pi_\alpha = \mathbf{f} & \text{in } \Omega, \\ \nabla \cdot \mathbf{u}_\alpha = -\pi_\alpha / \alpha & \text{in } \Omega, \\ \mathbf{u}_\alpha(x) = 0 & \text{on } \partial\Omega. \end{cases} \tag{PRS}$$

Resolvent problem corresponding to Stokes equations

$$\begin{cases} \lambda \mathbf{u} - \Delta \mathbf{u} + \nabla \pi = \mathbf{f} & \text{in } \Omega, \\ \nabla \cdot \mathbf{u} = 0 & \text{in } \Omega, \\ \mathbf{u}(x) = 0 & \text{on } \partial\Omega. \end{cases} \tag{RS}$$

Main results

Let $n \geq 2$, $1 < q < \infty$, $0 < \varepsilon < \pi/2$.
Set $\Sigma_\varepsilon = \{\lambda \in \mathbb{C}\backslash\{0\} \mid |\arg\lambda| < \pi - \varepsilon\}$.

Theorem (Resolvent estimates for $\Omega = \mathbb{R}^n, \mathbb{R}^n_+$)

For every $\lambda \in \Sigma_\varepsilon$,
there exists a unique solution $(\mathbf{u}_\alpha, \pi_\alpha) \in W^2_q(\Omega)^n \times W^1_q(\Omega)$ of (PRS).
The solution $(\mathbf{u}_\alpha, \pi_\alpha)$ satisfies the following estimates:

$$\left\| \left(|\lambda|\mathbf{u}_\alpha, |\lambda|^{\frac{1}{2}}\nabla\mathbf{u}_\alpha, \nabla^2\mathbf{u}_\alpha, \nabla\pi_\alpha \right) \right\|_{q,\Omega} \leq C_{n,q,\varepsilon,\Omega}\|\mathbf{f}\|_{q,\Omega},$$

$$|\lambda|^{\frac{1}{2}} \|\pi_\alpha\|_{q,\Omega} \leq C_{n,q,\varepsilon,\Omega}(1+\alpha)^{\frac{1}{2}}\|\mathbf{f}\|_{q,\Omega}.$$

Main results

Let $n \geq 2$, $1 < q < \infty$, $0 < \varepsilon < \pi/2$.
Set $\Sigma_\varepsilon = \{\lambda \in \mathbb{C}\backslash\{0\} \mid |\arg\lambda| < \pi - \varepsilon\}$.

Theorem (Resolvent estimates for $\Omega = \mathbb{R}^n, \mathbb{R}^n_+$)

For every $\lambda \in \Sigma_\varepsilon$,
there exists a unique solution $(\mathbf{u}_\alpha, \pi_\alpha) \in W^2_q(\Omega)^n \times W^1_q(\Omega)$ of (PRS).
The solution $(\mathbf{u}_\alpha, \pi_\alpha)$ satisfies the following estimates:

$$\left\| \left(|\lambda|\mathbf{u}_\alpha, |\lambda|^{\frac{1}{2}}\nabla\mathbf{u}_\alpha, \nabla^2\mathbf{u}_\alpha, \nabla\pi_\alpha \right) \right\|_{q,\Omega} \leq C_{n,q,\varepsilon,\Omega}\|\mathbf{f}\|_{q,\Omega},$$

$$|\lambda|^{\frac{1}{2}} \|\pi_\alpha\|_{q,\Omega} \leq C_{n,q,\varepsilon,\Omega}(1+\alpha)^{\frac{1}{2}}\|\mathbf{f}\|_{q,\Omega}.$$

cf. Resolvent estimate of Stokes equations (Farwig-Sohr.94)

I found the following estimate in the paper of Farwig and Sohr:

$$\left\| \left(|\lambda|\mathbf{u}, |\lambda|^{\frac{1}{2}}\nabla\mathbf{u}, \nabla^2\mathbf{u}, \nabla\pi \right) \right\|_{q,\Omega} \leq C_{n,q,\varepsilon,\Omega}\|\mathbf{f}\|_{q,\Omega}.$$

Main results

Ω : a bounded domain
$\partial\Omega$: compact $C^{2,1}$ hypersurface

Theorem (Resolvent estimates for Ω is a bounded domain)

There exists a constant $\lambda_0 \geq 1$ such that for every $\lambda \in \Sigma_\varepsilon$ $(|\lambda| \geq \lambda_0)$, there is a unique solution $(\mathbf{u}_\alpha, \pi_\alpha) \in W_q^2(\Omega)^n \times W_q^1(\Omega)$ of (PRS). The solution $(\mathbf{u}_\alpha, \pi_\alpha)$ satisfies the following estimates:

$$\left\| \left(|\lambda| \mathbf{u}_\alpha, |\lambda|^{\frac{1}{2}} \nabla \mathbf{u}_\alpha, \nabla^2 \mathbf{u}_\alpha, \nabla \pi_\alpha \right) \right\|_{q,\Omega} \leq C_{n,q,\varepsilon,\Omega} \|\mathbf{f}\|_{q,\Omega},$$
$$\|\pi_\alpha\|_{q,\Omega} \leq C_{n,q,\varepsilon,\Omega} \|\mathbf{f}\|_{q,\Omega}.$$

The solution of (RS) has similar estimates:

$$\left\| \left(|\lambda| \mathbf{u}, |\lambda|^{\frac{1}{2}} \nabla \mathbf{u}, \nabla^2 \mathbf{u}, \nabla \pi \right) \right\|_{q,\Omega} \leq C_{n,q,\varepsilon,\Omega} \|\mathbf{f}\|_{q,\Omega} \quad \text{(Solonnikov and Giga)},$$
$$|\lambda|^{\frac{1}{2}} \|\pi\|_{q,\Omega} \leq C_{n,q,\varepsilon,\Omega} \|\mathbf{f}\|_{q,\Omega} \quad \text{(Hieber and Saal)}.$$

Outline of the proof of main results

Case 1 $\Omega = \mathbb{R}^n$

By $\pi_\alpha = -\alpha\nabla\cdot\mathbf{u}_\alpha$, I rewrite (PRS) as follows:

$$\lambda\mathbf{u}_\alpha - \Delta\mathbf{u}_\alpha - \alpha\nabla(\nabla\cdot\mathbf{u}_\alpha) = \mathbf{f}.$$

Applying the Fourier transform, I have

$$\lambda\widehat{u}_j + |\xi|^2\widehat{u}_j + \alpha\sum_{i=1}^{n}\xi_j\xi_i\widehat{u}_i = \widehat{f}_j \quad (j = 1, \ldots, n).$$

Outline of the proof of main results

Case 1 $\Omega = \mathbb{R}^n$

By $\pi_\alpha = -\alpha\nabla\cdot\mathbf{u}_\alpha$, I rewrite (PRS) as follows:

$$\lambda\mathbf{u}_\alpha - \Delta\mathbf{u}_\alpha - \alpha\nabla(\nabla\cdot\mathbf{u}_\alpha) = \mathbf{f}.$$

Applying the Fourier transform, I have

$$\lambda\widehat{u}_j + |\xi|^2\widehat{u}_j + \alpha\sum_{i=1}^{n}\xi_j\xi_i\widehat{u}_i = \widehat{f}_j \quad (j = 1,\ldots,n).$$

Then I obtain the solution formula:

$$u_j = \mathcal{F}^{-1}\left[\frac{1}{(\lambda+|\xi|^2)}\widehat{f}_j\right] - \sum_{i=1}^{n}\mathcal{F}^{-1}\left[\left(\frac{\xi_i\xi_j|\xi|^2}{\lambda+|\xi|^2} + \frac{\xi_i\xi_j}{\lambda+|\xi|^2+\alpha|\xi|^2}\right)\widehat{f}_i\right],$$

$$\pi = -i\sum_{i=1}^{n}\mathcal{F}^{-1}\left[\frac{\alpha\xi_i}{\lambda+|\xi|^2+\alpha|\xi|^2}\widehat{f}_i\right].$$

Outline of the proof of main results

In order to estimate $\mathbf{u}_\alpha$ and π_α, I use the Fourier multiplier theorem.

Resolvent estimates

$$\left\|\left(|\lambda|\mathbf{u}_\alpha, |\lambda|^{\frac{1}{2}}\nabla\mathbf{u}_\alpha, \nabla^2\mathbf{u}_\alpha, \nabla\pi_\alpha\right)\right\|_{q,\mathbb{R}^n} \leq C_{n,q,\varepsilon,\Omega}\|\mathbf{f}\|_{q,\mathbb{R}^n},$$

$$|\lambda|^{\frac{1}{2}}\left\|\pi_\alpha\right\|_{q,\mathbb{R}^n} \leq C_{n,q,\varepsilon,\Omega}(1+\alpha)^{\frac{1}{2}}\|\mathbf{f}\|_{q,\mathbb{R}^n}.$$

Outline of the proof of main results

In order to estimate $\mathbf{u}_\alpha$ and π_α, I use the Fourier multiplier theorem.

Resolvent estimates

$$\left\| \left(|\lambda| \mathbf{u}_\alpha, |\lambda|^{\frac{1}{2}} \nabla \mathbf{u}_\alpha, \nabla^2 \mathbf{u}_\alpha, \nabla \pi_\alpha \right) \right\|_{q,\mathbb{R}^n} \leq C_{n,q,\varepsilon,\Omega} \|\mathbf{f}\|_{q,\mathbb{R}^n},$$

$$|\lambda|^{\frac{1}{2}} \|\pi_\alpha\|_{q,\mathbb{R}^n} \leq C_{n,q,\varepsilon,\Omega} (1+\alpha)^{\frac{1}{2}} \|\mathbf{f}\|_{q,\mathbb{R}^n}.$$

The estimate of π_α implies that

$$\|\nabla \cdot \mathbf{u}_\alpha\|_{q,\mathbb{R}^n} = \left\| -\frac{\pi_\alpha}{\alpha} \right\|_{q,\mathbb{R}^n} \leq C \frac{(1+\alpha)^{\frac{1}{2}}}{\alpha} \|\mathbf{f}\|_{q,\mathbb{R}^n} \to 0 \quad (\alpha \to \infty).$$

So I see that penalty method is justified in $\mathbb{R}^n$.

Outline of the proof of main results

Case 2 $\Omega = \mathbb{R}^n_+$

Let $\mathbf{v}$ be a solution to the whole space resolvent problem with $\mathbf{F} = (f_1^e, \ldots, f_{n-1}^e, f_n^o)$:

$$\lambda \mathbf{v} - \Delta \mathbf{v} - \alpha \nabla(\nabla \cdot \mathbf{v}) = \mathbf{F} \qquad \text{in } \mathbb{R}^n,$$

where f_k^e is even extension of the function f_k, and f_n^o is odd extension of f_n.

Outline of the proof of main results

Case 2 $\Omega = \mathbb{R}^n_+$

Let $\mathbf{v}$ be a solution to the whole space resolvent problem with $\mathbf{F} = (f_1^e, \ldots, f_{n-1}^e, f_n^o)$:

$$\lambda \mathbf{v} - \Delta \mathbf{v} - \alpha \nabla (\nabla \cdot \mathbf{v}) = \mathbf{F} \qquad \text{in } \mathbb{R}^n,$$

where f_k^e is even extension of the function f_k, and f_n^o is odd extension of f_n. Setting $\mathbf{u} = \mathbf{v} + \mathbf{w}$, I have the following equations for $\mathbf{w}$:

Resolvent problem

$$\begin{cases} \lambda \mathbf{w} - \Delta \mathbf{w} - \alpha \nabla (\nabla \cdot \mathbf{w}) = 0 & \text{in } \Omega, \\ w_j(x', 0) = -v_j(x', 0) & (j = 1, \ldots, n-1), \\ w_n(x', 0) = 0. \end{cases}$$

Outline of the proof of main results

Case 2 $\Omega = \mathbb{R}^n_+$

Let $\mathbf{v}$ be a solution to the whole space resolvent problem with $\mathbf{F} = (f_1^e, \ldots, f_{n-1}^e, f_n^o)$:

$$\lambda \mathbf{v} - \Delta \mathbf{v} - \alpha \nabla(\nabla \cdot \mathbf{v}) = \mathbf{F} \qquad \text{in } \mathbb{R}^n,$$

where f_k^e is even extension of the function f_k, and f_n^o is odd extension of f_n. Setting $\mathbf{u} = \mathbf{v} + \mathbf{w}$, I have the following equations for $\mathbf{w}$:

Resolvent problem

$$\begin{cases} \lambda \mathbf{w} - \Delta \mathbf{w} - \alpha \nabla(\nabla \cdot \mathbf{w}) = 0 \quad \text{in } \Omega, \\ w_j(x', 0) = -v_j(x', 0) \quad (j = 1, \ldots, n-1), \\ w_n(x', 0) = 0. \end{cases}$$

To solve these equations, I apply the partial Fourier transform with respect to x' variables.

Outline of the proof of main results

Case 3

Ω : a bounded domain

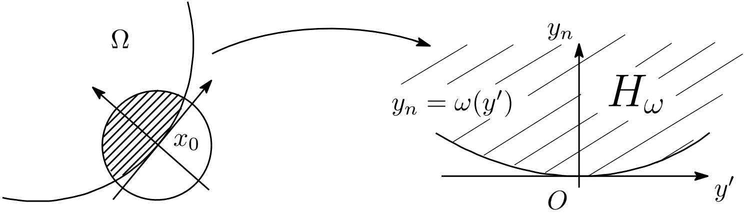
$\partial\Omega$: compact $C^{2,1}$ hypersurface

Outline of the proof of main results

Case 3

Ω : a bounded domain
$\partial\Omega$: compact $C^{2,1}$ hypersurface

By using cutoff technic, (PRS) is reduced to the bent half space problem.



Definition of bent half space

$$H_\omega = \{(x', x_n) \in \mathbb{R}^n \mid x_n > \omega(x')\}$$

Result for bent half space

Let $\omega \in C^{2,1}(\mathbb{R}^{n-1})$. Set $K_1 = \sum_{|\beta'|=1} \left\| \partial_{x'}^{\beta'} \omega \right\|_{L^\infty(\mathbb{R}^{n-1})}$.

Lemma (Resolvent estimates for $\Omega = H_\omega$)

There exist constants $\kappa \leq 1$ and $\lambda_0 \geq 1$ such that if $K_1 \leq \kappa$, then for every $\lambda \in \Sigma_\varepsilon$ ($|\lambda| \geq \lambda_0$) there is a unique solution $(\mathbf{u}_\alpha, \pi_\alpha) \in W_q^2(H_\omega)^n \times W_q^1(H_\omega)$ of (PRS).

The solution $(\mathbf{u}_\alpha, \pi_\alpha)$ satisfies the following estimates:

$$\left\| \left(|\lambda| \mathbf{u}_\alpha, |\lambda|^{\frac{1}{2}} \nabla \mathbf{u}_\alpha, \nabla^2 \mathbf{u}_\alpha, \nabla \pi_\alpha \right) \right\|_{q,H_\omega} \leq C_{n,q,\varepsilon,\Omega} \|\mathbf{f}\|_{q,H_\omega},$$

$$|\lambda|^{\frac{1}{2}} \|\pi_\alpha\|_{q,H_\omega} \leq C_{n,q,\varepsilon,\Omega} (1+\alpha)^{\frac{1}{2}} \|\mathbf{f}\|_{q,H_\omega}.$$

cf. It is well known that the usual Stokes problem has similar resolvent estimate. (Farwig-Sohr.94)

Outline of the proof of lemma

Standard change of variable

$$y' = x', \quad y_n = x_n - \omega(x')$$

By changing of the variable, (PRS) is reduced to the following half space problem.

$$\begin{cases} \lambda \mathbf{u} - \Delta \mathbf{u} + \nabla \pi = \mathbf{f} + \mathbf{G}(\mathbf{u}, \pi) & \text{in } \mathbb{R}^n_+, \\ \nabla \cdot \mathbf{u} = -\pi/\alpha + \sum_{i=1}^{n-1} (\partial_i \omega) \partial_n u_i & \text{in } \mathbb{R}^n_+, \\ \mathbf{u}(x) = 0 & \text{on } \partial \mathbb{R}^n_+. \end{cases}$$

Outline of the proof of lemma

Standard change of variable

$$y' = x', \quad y_n = x_n - \omega(x')$$

By changing of the variable, (PRS) is reduced to the following half space problem.

$$\begin{cases} \lambda \mathbf{u} - \Delta \mathbf{u} + \nabla \pi = \mathbf{f} + \mathbf{G}(\mathbf{u}, \pi) & \text{in } \mathbb{R}^n_+, \\ \nabla \cdot \mathbf{u} = -\pi/\alpha + \sum_{i=1}^{n-1} (\partial_i \omega) \partial_n u_i & \text{in } \mathbb{R}^n_+, \\ \mathbf{u}(x) = 0 & \text{on } \partial \mathbb{R}^n_+. \end{cases}$$

I would like to apply the result of $\mathbb{R}^n_+$ to this equations directly.

But I have serious difficulty because of $\sum_{i=1}^{n-1} (\partial_i \omega) \partial_n u_i$.

Outline of the proof of lemma

Standard change of variable

$$y' = x', \quad y_n = x_n - \omega(x')$$

By changing of the variable, (PRS) is reduced to the following half space problem.

$$\begin{cases} \lambda \mathbf{u} - \Delta \mathbf{u} + \nabla \pi = \mathbf{f} + \mathbf{G}(\mathbf{u}, \pi) & \text{in } \mathbb{R}^n_+, \\ \nabla \cdot \mathbf{u} = -\pi/\alpha + \sum_{i=1}^{n-1} (\partial_i \omega) \partial_n u_i & \text{in } \mathbb{R}^n_+, \\ \mathbf{u}(x) = 0 & \text{on } \partial \mathbb{R}^n_+. \end{cases}$$

I would like to apply the result of $\mathbb{R}^n_+$ to this equations directly.

But I have serious difficulty because of $\sum_{i=1}^{n-1} (\partial_i \omega) \partial_n u_i$.

So I use Solonnikov transform.

Outline of the proof of lemma

Solonnikov transform

I choose unknown function $\mathbf{v} = (v_1, \ldots, v_n)$ and θ as follows:

$$v_j = u_j \quad (j = 1, \ldots, n-1), \quad v_n = u_n - \sum_{i=1}^{n-1} (\partial_i \omega) u_i,$$
$$\theta = \pi.$$

Then,

$$\begin{cases} \lambda \mathbf{v} - \Delta \mathbf{v} + \nabla \theta = \mathbf{f} + \mathbf{H}(\mathbf{v}, \theta) & \text{in } \mathbb{R}^n_+, \\ \nabla \cdot \mathbf{v} = -\theta/\alpha & \text{in } \mathbb{R}^n_+, \\ \mathbf{v}(x) = 0 & \text{on } \partial\mathbb{R}^n_+. \end{cases}$$

I can apply the result of $\mathbb{R}^n_+$.

Outline of the proof of main results

By compactness of $\partial\Omega$, there exists a covering $\{B_r(x_j)\}_{j=1}^N$ such that

$$\partial\Omega \subset \bigcup_{j=1}^N B_r(x_j).$$

And I choose a partition of unity $\{\varphi_j\}_{j=0}^N$ such that

$$\varphi_0 \in C_0^\infty(\Omega), \quad \varphi_j \in C_0^\infty(B_r(x_j)), \quad \sum_{j=0}^N \varphi_j = 1.$$

Outline of the proof of main results

By compactness of $\partial\Omega$, there exists a covering $\{B_r(x_j)\}_{j=1}^N$ such that

$$\partial\Omega \subset \bigcup_{j=1}^{N} B_r(x_j).$$

And I choose a partition of unity $\{\varphi_j\}_{j=0}^N$ such that

$$\varphi_0 \in C_0^\infty(\Omega), \quad \varphi_j \in C_0^\infty(B_r(x_j)), \quad \sum_{j=0}^{N} \varphi_j = 1.$$

Localization

If $(\mathbf{u}, \pi)$ is a solution of (PRS), $(\varphi_j \mathbf{u}, \varphi_j \pi)$ satisfies the following equations:

$$\begin{cases} \lambda(\varphi_j \mathbf{u}) - \Delta(\varphi_j \mathbf{u}) + \nabla(\varphi_j \pi) = \varphi_j \mathbf{f} + \mathbf{F}(\mathbf{u}, \pi) & \text{in } \Omega \cap B_r(x_j), \\ \nabla \cdot (\varphi_j \mathbf{u}) = -\dfrac{\varphi_j \pi}{\alpha} + (\nabla \varphi_j) \cdot \mathbf{u} & \text{in } \Omega \cap B_r(x_j), \\ \gamma_0(\varphi_j \mathbf{u}) = 0. \end{cases}$$

Outline of the proof of main results

Generalized Bogovskiĭ operator

Let $1 < q < \infty$, integer $m \geq 0$,
Let Ω be a bounded domain with $C^{m,1}$ -boundary $(m \geq 1)$.
There exists a bounded linear operator $\mathbb{B} : W^m_{q,0}(\Omega) \longrightarrow W^{m+1}_{q,0}(\Omega)$
which have the folowing properties:

(1) There exists $\rho \in C_0^\infty(\Omega)$ such that $\rho \geq 0$, $\int_\Omega \rho dx = 1$ and

$$\nabla \cdot \mathbb{B}[f] = f - \rho \int_\Omega f dx.$$

Furthermore f satisfies $\nabla \cdot \mathbb{B}[f] = f$ if $\int_\Omega f dx = 0$.

(2) If f is given by $f = D_k g \ \ (g \in W^{m+1}_{q,0}(\Omega))$, there exists a constant $C_{m,q,\Omega} > 0$ such that

$$\|\mathbb{B}[D_k g]\|_{m,q,\Omega} \leq C_{m,q,\Omega} \|g\|_{m,q,\Omega}.$$

Outline of the proof of main results

$$
\begin{aligned}
\nabla \cdot \mathbb{B}[(\nabla \varphi_j) \cdot \mathbf{u}] &= (\nabla \varphi_j) \cdot \mathbf{u} - \rho \int_{\Omega \cap B_r(x_j)} (\nabla \varphi_j) \cdot \mathbf{u} dx \\
&= (\nabla \varphi_j) \cdot \mathbf{u} - \frac{\rho}{\alpha} \int_{\Omega \cap B_r(x_j)} \varphi_j \pi dx
\end{aligned}
$$

Outline of the proof of main results

$$\begin{aligned}\nabla\cdot\mathbb{B}[(\nabla\varphi_j)\cdot\mathbf{u}] &= (\nabla\varphi_j)\cdot\mathbf{u}-\rho\int_{\Omega\cap B_r(x_j)}(\nabla\varphi_j)\cdot\mathbf{u}dx\\ &= (\nabla\varphi_j)\cdot\mathbf{u}-\frac{\rho}{\alpha}\int_{\Omega\cap B_r(x_j)}\varphi_j\pi dx\end{aligned}$$

$$(\nabla\varphi_j)\cdot\mathbf{u}=\nabla\cdot\mathbf{w}_j+\frac{\rho}{\alpha}\int_{\Omega\cap B_r(x_j)}\varphi_j\pi dx \qquad (\mathbf{w}_j=\mathbb{B}[(\nabla\varphi_j)\cdot\mathbf{u}])$$

Outline of the proof of main results

$$\nabla \cdot \mathbb{B}[(\nabla \varphi_j) \cdot \mathbf{u}] = (\nabla \varphi_j) \cdot \mathbf{u} - \rho \int_{\Omega \cap B_r(x_j)} (\nabla \varphi_j) \cdot \mathbf{u} dx$$
$$= (\nabla \varphi_j) \cdot \mathbf{u} - \frac{\rho}{\alpha} \int_{\Omega \cap B_r(x_j)} \varphi_j \pi dx$$

$$(\nabla \varphi_j) \cdot \mathbf{u} = \nabla \cdot \mathbf{w}_j + \frac{\rho}{\alpha} \int_{\Omega \cap B_r(x_j)} \varphi_j \pi dx \qquad (\mathbf{w}_j = \mathbb{B}[(\nabla \varphi_j) \cdot \mathbf{u}])$$

$$\begin{cases} \lambda(\varphi_j \mathbf{u} - \mathbf{w}_j) - \Delta(\varphi_j \mathbf{u} - \mathbf{w}_j) + \nabla \left(\varphi_j \pi - \rho \int_{\Omega \cap B_r(x_j)} \varphi_j \pi dx \right) \\ \qquad = \varphi_j \mathbf{f} + \mathbf{F}(\mathbf{u}, \pi) - \lambda \mathbf{w}_j + \Delta \mathbf{w}_j - (\nabla \rho) \int_{\Omega \cap B_r(x_j)} \varphi_j \pi dx, \\ \nabla \cdot (\varphi_j \mathbf{u} - \mathbf{w}_j) = -\frac{1}{\alpha} \left(\varphi_j \pi - \rho \int_{\Omega \cap B_r(x_j)} \varphi_j \pi dx \right) \\ \gamma_0(\varphi_j \mathbf{u} - \mathbf{w}_j) = 0. \end{cases}$$

Outline of the proof of main results

Applying the result of bent half space problem, I get the following estimate:

$$\left\| \left(|\lambda| \mathbf{u}, |\lambda|^{\frac{1}{2}} \nabla \mathbf{u}, \nabla^2 \mathbf{u}, \nabla \pi \right) \right\|_{q,\Omega} \leq C \left(\|\mathbf{f}\|_{q,\Omega} + \|\pi\|_{q,\Omega} \right).$$

Outline of the proof of main results

Applying the result of bent half space problem, I get the following estimate:

$$\left\| \left(|\lambda|\mathbf{u}, |\lambda|^{\frac{1}{2}}\nabla\mathbf{u}, \nabla^2\mathbf{u}, \nabla\pi \right) \right\|_{q,\Omega} \leq C \left(\|\mathbf{f}\|_{q,\Omega} + \|\pi\|_{q,\Omega} \right).$$

By contradiction argument, I obtain

$$\|\pi\|_{q,\Omega} \leq C\|\mathbf{f}\|_{q,\Omega}.$$

And I get

$$\left\| \left(|\lambda|\mathbf{u}, |\lambda|^{\frac{1}{2}}\nabla\mathbf{u}, \nabla^2\mathbf{u}, \nabla\pi \right) \right\|_{q,\Omega} \leq C\|\mathbf{f}\|_{q,\Omega}.$$