Limitation of the Brinkman model for porous media flows

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March 11, 2010
Simple and Complicated Flow Domains

- The Navier-Stokes equations captures the physics of incompressible, viscous fluid dynamics

- Von Karman vortex street developed off Guadalupe Island
Simple and Complicated Flow Domains

- Solving Navier-Stokes equations is infeasible

Darcy’s Law:

\[ u = -\nu^{-1}K(\nabla p + f) \]

- sub-surface fluid flow (bedrock, sand, clay, …)
Very porous media

Pebble bed nuclear reactors

- > 350,000 uranium fuel pellets per core
- Fuel pellets, 70 mm-diameter
- Porosity \( \approx 0.50 \)
Very porous media

Wind farms

- $114 million DOE grant for Texas wind farm development
- Complicated flow obstacles (rotating turbine blades)
Very porous media

Open-angle glaucoma

- ~85% of Glaucoma cases
- Related to flow obstruction from eye through the *trabecular meshwork*
The Brinkman Flow Model

Problem

(\textit{Brinkman 1947a, 1947b}) Find \((u,p)\) in \(\Omega\) satisfying

\[
-\nabla \cdot \tilde{\nu} \mathcal{D}(u) + \nabla p + \nu K^{-1} u = f
\]

\[
\mathcal{D}(u) = 0.5(\nabla u + \nabla u^t)
\]

\(u\) = filtration velocity
\(p\) = pressure
\(\nu\) = kinematical viscosity
\(K\) = permeability
\(\tilde{\nu}\) = Brinkman (effective) viscosity
\(f\) = body forces
The Brinkman Flow Model

Problem

Find \((u,p)\) in \(\Omega\) satisfying

\[
\begin{align*}
    u \cdot \nabla u - \nabla \cdot \tilde{\mathbb{D}}(u) + \nabla p + \nu K^{-1}u &= f \\
    \nabla \cdot u &= \phi
\end{align*}
\]

Boundary, Initial conditions

Model validation:

- Marcenko, Hruslov (1974) - homogenization technique
- Rubenstein (1986) - probabilistic approach
The Brinkman Flow Model

Problem

Find \( u \in H^1(\Omega) \) so that \( u|_{\partial \Omega} = \tilde{u} \) and \( \nabla \cdot u = \phi \),

\[
(u \cdot \nabla u, v) + (\tilde{v} \nabla u, \nabla v) + (\nu K^{-1} u, v) = (f, v), \quad \forall v \in V
\]

\( V = \{ w \in H^1_0(\Omega) : \nabla \cdot w = 0 \} \).

Compatibility condition:

\[
\int_{\Omega} \phi = \int_{\partial \Omega} \tilde{u} \cdot \hat{n}
\]
Existence of steady Brinkman solutions

Theorem

There exists \((u, p) \in H^1(\Omega) \times L^2_0(\Omega)\) satisfying

\[ u|_{\partial \Omega} = \tilde{u} \in H^{1/2}(\partial \Omega), \quad \nabla \cdot u = \phi \in L^2(\Omega) \]

under small data constraint on \(\tilde{u}\) and \(\phi\).

Remark

- \(\phi \in L^2_0(\Omega)\) \(\Rightarrow\) no small data required on \(\tilde{u}\)
- a priori estimate + Leray-Schauder \(\Rightarrow\) existence
Existence of steady Brinkman solutions

\[ \nabla \cdot u = 0 \]

\( \partial \Omega \)-connected:

- **Existence** in \( H^1 \) for any data
- **Uniqueness** for small data

\( \partial \Omega \)-not connected:

- **Existence** guaranteed only if \( |\tilde{u}|_{\Gamma_i} \) is small
- Represents **sources** and/or **sinks**
Problem reformulation

Lemma
Fix $\varepsilon > 0$ and $\tilde{u} \in H^{1/2}(\partial \Omega)$, $\phi \in L^2(\Omega)$

- There exists $\tilde{u}_0 \in H^1(\Omega)$ such that $\tilde{u}_0|_{\partial \Omega} = \tilde{u}$ and

$$||\tilde{u}_0||_{L^4} < \varepsilon$$

Note: $\phi \equiv 0 \Rightarrow$ there exists $\nabla \cdot \tilde{u}_0 = 0$
(ref. Hopf (1941, 1957), Galdi (1994))

- There exists $\tilde{u}_* \in H^1_0(\Omega)$ such that

$$\nabla \cdot (\tilde{u}_{ext}) = \phi, \quad \tilde{u}_{ext} := \tilde{u}_* + \tilde{u}_0$$

and

$$||\tilde{u}_*||_{H^1(\Omega)} \leq C(||\tilde{u}||_{H^{1/2}(\partial \Omega)} + ||\phi||)$$
Problem reformulation

Problem

(Reformulated Brinkman) Find $U = u - \tilde{u}_{\text{ext}} \in V$

$$(\tilde{v} \nabla U, \nabla v) + (U \cdot \nabla U, v) + (\nu K^{-1} U, v)$$

$$= (f, v) - (\tilde{v} \nabla \tilde{u}_{\text{ext}}, \nabla v) - (\nu K^{-1} \tilde{u}_{\text{ext}}, v)$$

$$- (\tilde{u}_{\text{ext}} \cdot \nabla \tilde{u}_{\text{ext}}, v) - (U \cdot \nabla \tilde{u}_{\text{ext}}, v) - (\tilde{u}_{\text{ext}} \cdot \nabla U, v)$$
Existence of steady Brinkman solutions

Energy equation: \( \nu = U \Rightarrow \)

\[
\| \tilde{\nu}^{1/2} \nabla U \|^2 + \nu \| K^{-1/2} U \|
\]

\[
= (f, U) - (\tilde{\nu} \nabla \tilde{u}_{ext}, \nabla U) - (\nu K^{-1} \tilde{u}_{ext}, U)
\]

\[
- (U \cdot \nabla \tilde{u}_{ext}, U) - (\tilde{u}_{ext} \cdot \nabla \tilde{u}_{ext}, U) - (\tilde{u}_{ext} \cdot \nabla U, U)
\]
Existence of steady Brinkman solutions

Crucial estimate

\[\int_{\Omega} U \cdot \nabla \tilde{u}_{ext} \cdot U = - \int_{\Omega} U \cdot \nabla U \cdot \tilde{u}_{ext} \leq C \|\tilde{u}_{ext}\|_{L^4} \|\nabla U\|^2\]

\[\int_{\Omega} \tilde{u}_{ext} \cdot \nabla U \cdot U = - \int_{\Omega} \phi |U|^2 \leq C \|\phi\| \|\nabla U\|^2\]
Existence of steady Brinkman solutions

**Conclusion** Terms must be absorbed into LHS

- $\phi \equiv 0 \Rightarrow$

  $$\|\tilde{u}_{\text{ext}}\|_{L^4} < \varepsilon,$$  
  (Hopf extension)

- Otherwise, require

  $$C(\|\tilde{u}_{\text{ext}}\|_{L^4} + \|\phi\|) \leq \frac{\nu}{2},$$  
  (smallness condition)
THANK YOU FOR YOUR ATTENTION. QUESTIONS?