Limitation of the Brinkman model for porous media flows

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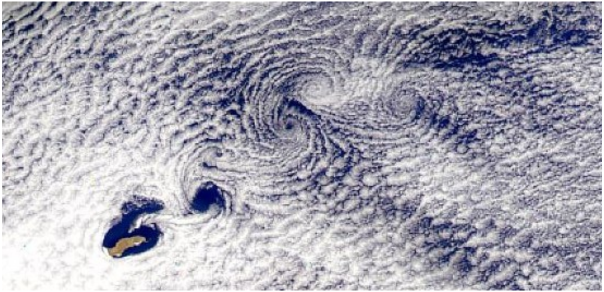
Simple and Complicated Flow Domains

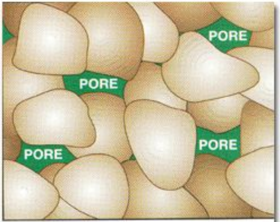
- The Navier-Stokes equations captures the physics of incompressible, viscous fluid dynamics



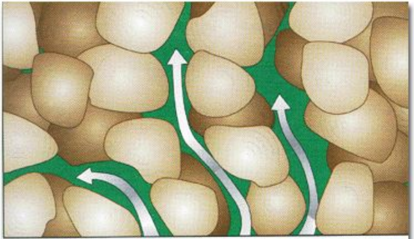
- Von Karman vortex street developed off Guadalupe Island

Simple and Complicated Flow Domains

- Solving Navier-Stokes equations is infeasible



A pore is a small open space

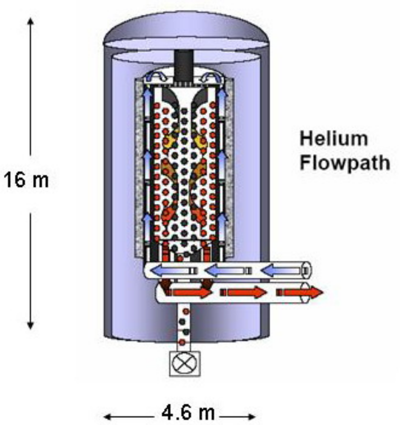

Connected pores give a rock permeability.

- Darcy's Law:

$$u = -\nu^{-1} K(\nabla p + f)$$

- sub-surface fluid flow (bedrock, sand, clay, ...)

Very porous media



Pebble bed nuclear reactors

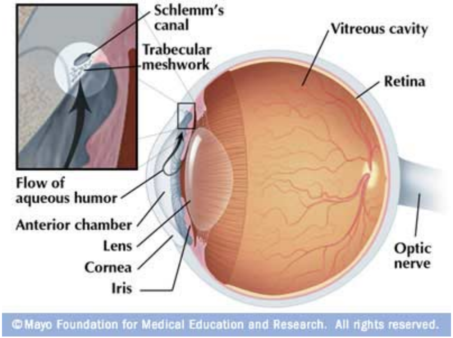
- $> 350,000$ uranium fuel pellets per core
- Fuel pellets, $70mm$-diameter
- Porosity ≈ 0.50

Very porous media

Wind farms

- $114 million DOE grant for Texas wind farm development
- Complicated flow obstacles (rotating turbine blades)

Very porous media



Open-angle glaucoma

- $\sim$ 85% of Glaucoma cases
- Related to flow obstruction from eye through the *trabecular meshwork*

The Brinkman Flow Model

Problem

(Brinkman 1947a, 1947b) Find (u,p) in Ω satisfying

$$-\nabla \cdot \tilde{\nu}\mathbb{D}(u) + \underbrace{\nabla p + \overbrace{\nu K^{-1} u}^{\text{Darcy Drag}}}_{\text{Darcy's Law}} = f$$

$$\mathbb{D}(u) = 0.5(\nabla u + \nabla u^t)$$

u	$=$	filtration velocity
p	$=$	pressure
ν	$=$	kinematical viscosity
K	$=$	permeability
$\tilde{\nu}$	$=$	Brinkman (effective) viscosity
f	$=$	body forces

The Brinkman Flow Model

Problem

Find (u,p) in Ω satisfying

$$u \cdot \nabla u - \nabla \cdot \tilde{\nu}\mathbb{D}(u) + \nabla p + \nu K^{-1} u = f$$

$$\nabla \cdot u = \phi$$

Boundary, Initial conditions

Model validation:

- Marcenko, Hruslov (1974) - homogenization technique
- Rubenstein (1986) - probabilistic approach

The Brinkman Flow Model

Problem

Find $u \in H^1(\Omega)$ *so that* $u|_{\partial\Omega} = \tilde{u}$ *and* $\nabla \cdot u = \phi$,

$$(u \cdot \nabla u, v) + (\tilde{\nu} \nabla u, \nabla v) + (\nu K^{-1} u, v) = (f, v), \qquad \forall v \in V$$

$$V = \left\{ w \in H_0^1(\Omega) : \nabla \cdot w = 0 \right\}.$$

Compatibility condition:

$$\int_\Omega \phi = \int_{\partial\Omega} \tilde{u} \cdot \hat{n}$$

Existence of steady Brinkman solutions

Theorem

There exists $(u, p) \in H^1(\Omega) \times L^2_0(\Omega)$ *satisfying*

$$u|_{\partial\Omega} = \tilde{u} \in H^{1/2}(\partial\Omega), \qquad \nabla \cdot u = \phi \in L^2(\Omega)$$

under small data constraint on $\tilde{u}$ *and* ϕ.

Remark

- $\phi \in L^2_0(\Omega) \Rightarrow$ *no small data required on* $\tilde{u}$
- *a priori estimate* $+$ *Leray-Schauder* $\Rightarrow$ *existence*

Existence of steady Brinkman solutions

Navier-Stokes solutions, $\nabla \cdot u = 0$

$\partial\Omega$-**connected**:

- Existence in H^1 for any data
- Uniqueness for small data

$\partial\Omega$-**not connected**:

- Existence guaranteed only if $|\tilde{u}|_{\Gamma_i}$ is small
- Represents sources and/or sinks

Problem reformulation

Lemma

Fix $\varepsilon > 0$ and $\tilde{u} \in H^{1/2}(\partial\Omega)$, $\phi \in L^2(\Omega)$

- *There exists $\tilde{u}_0 \in H^1(\Omega)$ such that $\tilde{u}_0|_{\partial\Omega} = \tilde{u}$ and*

$$||\tilde{u}_0||_{L^4} < \varepsilon$$

Note: $\phi \equiv 0 \Rightarrow$ *there exists* $\nabla \cdot \tilde{u}_0 = 0$
(ref. Hopf (1941, 1957), Galdi (1994))

- *There exists $\tilde{u}_* \in H^1_0(\Omega)$ such that*

$$\nabla \cdot (\tilde{u}_{ext}) = \phi, \qquad \tilde{u}_{ext} := \tilde{u}_* + \tilde{u}_0$$

and

$$||\tilde{u}_*||_{H^1(\Omega)} \leq C(||\tilde{u}||_{H^{1/2}(\partial\Omega)} + ||\phi||)$$

Problem reformulation

Problem

(Reformulated Brinkman) Find $U = u - \tilde{u}_{ext} \in V$

$$
\begin{aligned}
&(\tilde{\nu}\nabla U, \nabla v) + (U \cdot \nabla U, v) + (\nu K^{-1} U, v) \\
&\quad = (f, v) - (\tilde{\nu}\nabla \tilde{u}_{ext}, \nabla v) - (\nu K^{-1} \tilde{u}_{ext}, v) \\
&\qquad - (\tilde{u}_{ext} \cdot \nabla \tilde{u}_{ext}, v) - (U \cdot \nabla \tilde{u}_{ext}, v) - (\tilde{u}_{ext} \cdot \nabla U, v)
\end{aligned}
$$

Existence of steady Brinkman solutions

Energy equation: $v = U \Rightarrow$

$$
\begin{aligned}
&||\tilde{\nu}^{1/2}\nabla U||^2 + \nu||K^{-1/2}U|| \\
&\quad = (f, U) - (\tilde{\nu}\nabla\tilde{u}_{ext}, \nabla U) - (\nu K^{-1}\tilde{u}_{ext}, U) \\
&\qquad - (U\cdot\nabla\tilde{u}_{ext}, U) - (\tilde{u}_{ext}\cdot\nabla\tilde{u}_{ext}, U) - (\tilde{u}_{ext}\cdot\nabla U, U)
\end{aligned}
$$

Existence of steady Brinkman solutions

Crucial estimate

$$\int_\Omega U \cdot \nabla \tilde{u}_{ext} \cdot U$$

$$= -\int_\Omega U \cdot \nabla U \cdot \tilde{u}_{ext} \leq C||\tilde{u}_{ext}||_{L^4}||\nabla U||^2$$

$$\int_\Omega \tilde{u}_{ext} \cdot \nabla U \cdot U$$

$$= -\int_\Omega \phi |U|^2 \leq C||\phi||||\nabla U||^2$$

Existence of steady Brinkman solutions

Conclusion Terms must be absorbed into LHS

- $\phi \equiv 0 \Rightarrow$

$$||\tilde{u}_{ext}||_{L^4} < \varepsilon, \qquad \text{(Hopf extension)}$$

- Otherwise, require

$$C(||\tilde{u}_{ext}||_{L^4} + ||\phi||) \leq \frac{\nu}{2}, \qquad \text{(smallness condition)}$$

THANK YOU FOR YOUR ATTENTION.
QUESTIONS?