Counterexamples of commutator estimates in the Besov and the Triebel-Lizorkin spaces related to the Euler equations

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Introduction

The Euler equations in $\mathbb{R}^n, n \geqslant 2$

$$
\text{(E)} \begin{cases} \dfrac{\partial u}{\partial t} + (u \cdot \nabla) u + \nabla p = 0, & (x,t) \in \mathbb{R}^n \times (0,\infty), \\ \operatorname{div} u = 0, & (x,t) \in \mathbb{R}^n \times (0,\infty), \\ u(x,0) = u_0(x), & x \in \mathbb{R}^n, \end{cases}
$$

where

$u = u(x,t) = (u^1(x,t), \cdots, u^n(x,t))$: velocity filed

$p = p(x,t)$: pressure

$u_0 = u_0(x) = (u_0^1(x), \cdots, u_0^n(x))$: given initial velocity field with $\operatorname{div} u_0 = 0$

Introduction

Known results : Existence of solutions

Kato ('72)

$$\forall u_0 \in [H^m(\mathbb{R}^n)]^n \ (m > n/2+1) \ \textit{with} \ \mathrm{div}\, u_0 = 0,$$
$$\exists T = T(\|u_0\|_{H^m}) > 0 \ s.t.$$
$$\exists 1 u \in C([0,T];[H^m(\mathbb{R}^n)]^n) : \textit{sol. of} \ \text{(E)}$$

- Kato-Ponce ('88) : $W^{s,p}(\mathbb{R}^n)$ with $s > n/p+1, 1 < p < \infty$
- Chae ('02), Chen-Miao-Zhang ('10) :

  $$F^s_{p,q}(\mathbb{R}^n) \text{ with } s > n/p+1, 1 < p,q < \infty$$
- Chae ('04) : $B^s_{p,1}(\mathbb{R}^n)$ with $s = n/p+1, 1 < p < \infty$
- Pak-Park ('04) : $B^1_{\infty,1}(\mathbb{R}^n)$ *i.e.* $s = n/p+1, p = \infty$

Notation

$\{\varphi_j\}_{j=0}^{\infty} \subset \mathscr{S}(\mathbb{R}^n)$ with

$$\operatorname{supp} \widehat{\varphi_0} \subset \{|\xi| \leqslant 3/2\},$$

$$\operatorname{supp} \widehat{\varphi}_j \subset \left\{2^{j-1} \leqslant |\xi| \leqslant 2^{j+1}\right\}, \quad j \in \mathbb{N},$$

$$\sum_{j=0}^{\infty} \widehat{\varphi}_j(\xi) = 1, \quad \forall \xi \in \mathbb{R}^n.$$

Definition (Littlewood-Paley Operators)

For $f \in \mathscr{S}'(\mathbb{R}^n)$,

$$\Delta_j f = \varphi_j * f, \quad j \in \mathbb{N} \cup \{0\},$$

$$S_j f = \sum_{k=0}^{j} \Delta_k f, \quad j \in \mathbb{N} \cup \{0\}, \quad (S_{-1} = S_{-2} = 0)$$

Notation

Definition (the Besov spaces)

For $s \in \mathbb{R}, 1 \leqslant p, q \leqslant \infty$,

$$B^s_{p,q}(\mathbb{R}^n) = \left\{ f \in \mathscr{S}'(\mathbb{R}^n) \;\middle|\; \|f\|_{B^s_{p,q}} < \infty \right\},$$

$$\|f\|_{B^s_{p,q}} = \left\| \left\{ 2^{sj} \left\|\Delta_j f\right\|_{L^p} \right\}_{j=0}^{\infty} \right\|_{\ell^q}$$

Definition (the Triebel-Lizorkin spaces)

For $s \in \mathbb{R}, 1 \leqslant p < \infty, 1 \leqslant q \leqslant \infty$ or $p = q = \infty$,

$$F^s_{p,q}(\mathbb{R}^n) = \left\{ f \in \mathscr{S}'(\mathbb{R}^n) \;\middle|\; \|f\|_{F^s_{p,q}} < \infty \right\},$$

$$\|f\|_{F^s_{p,q}} = \left\| \left\| \left\{ 2^{sj} |\Delta_j f(\cdot)| \right\}_{j=0}^{\infty} \right\|_{\ell^q} \right\|_{L^p}$$

Main Results

Commutator (type) estimates in the Besov and the Triebel-Lizorkin spaces

$$\left\| \left\{ 2^{sj} \left\| S_{j-2} f \cdot \nabla \Delta_j g - \Delta_j (f \cdot \nabla g) \right\|_{L^p} \right\}_{j=0}^{\infty} \right\|_{\ell^q} \lesssim \|f\|_{B^s_{p,q}} \|g\|_{B^s_{p,q}} \qquad (1)$$

for $(f, g) \in [B^s_{p,q}(\mathbb{R}^n)]^n \times B^s_{p,q}(\mathbb{R}^n)$ with div $f = 0$.

$$\left\| \left\| \left\{ 2^{sj} \left| f \cdot \nabla \Delta_j g - \Delta_j (f \cdot \nabla g) \right| \right\}_{j=0}^{\infty} \right\|_{\ell^q} \right\|_{L^p} \lesssim \|f\|_{F^s_{p,q}} \|g\|_{F^s_{p,q}} \qquad (2)$$

for $(f, g) \in [F^s_{p,q}(\mathbb{R}^n)]^n \times F^s_{p,q}(\mathbb{R}^n)$ with div $f = 0$.

Kato-Ponce's commutator estimates

$$\| f (1-\Delta)^{\frac{s}{2}} g - (1-\Delta)^{\frac{s}{2}} (fg) \|_{L^p} \lesssim \|f\|_{W^{s,p}} \|g\|_{W^{s-1,p}}$$

for $(f, g) \in W^{s,p}(\mathbb{R}^n) \times W^{s-1,p}(\mathbb{R}^n)$.

Main Results

Theorem

(i) (I) $s < n/p + 1, 1 \leqslant p, q \leqslant \infty$

or

(II) $s = n/p + 1, 1 \leqslant p \leqslant \infty, 1 < q \leqslant \infty$

$\Longrightarrow$ *Commutator type estimates* (1) *fail.*

(ii) (I′) $s < n/p + 1, 1 \leqslant p < \infty, 1 \leqslant q \leqslant \infty$ *or* $p = q = \infty$

or

(II′) $s = n/p + 1, 1 < p < \infty, 1 \leqslant q \leqslant \infty$ *or* $p = q = \infty$

$\Longrightarrow$ *Commutator estimates* (2) *fail.*

Remark 1

$$(\mathrm{I}) \text{ or } (\mathrm{II}) \iff B^s_{p,q}(\mathbb{R}^n) \not\subseteq C^1(\mathbb{R}^n)$$

$$(\mathrm{I}') \text{ or } (\mathrm{II}') \iff F^s_{p,q}(\mathbb{R}^n) \not\subseteq C^1(\mathbb{R}^n)$$

Main Results

Remark 2 (the case of the Besov spaces $B^s_{p,q}(\mathbb{R}^n)$)

Chae ('04), Pak-Park ('04) :

Commutator type estimates (1) *are valid in* $B^s_{p,q}(\mathbb{R}^n)$ *with*

$$s > n/p+1,\ 1 < p < \infty,\ 1 \leqslant q \leqslant \infty$$

or

$$s = n/p+1,\ 1 < p \leqslant \infty,\ q = 1$$

In particular, (1) *is valid in* $B^1_{\infty,1}(\mathbb{R}^n)$.

$s < 1 \implies$ (1) *fails for* $1 \leqslant \forall p, \forall q \leqslant \infty$

$s = 1 \implies$ (1) *is valid if and only if* $p = \infty, q = 1$

- $s = 1 \ \cdots$ optimal differential order for (1)
- $B^1_{\infty,1}(\mathbb{R}^n) \ \cdots$ optimal class for (1)

Outline of Proof

case (II) : $B^{n/p+1}_{p,q}(\mathbb{R}^n)$ with $1 \leqslant p \leqslant \infty$, $1 < q \leqslant \infty$

We construct a sequence $\{(f^N, g^N)\}_{N=1}^{\infty}$ such that

$$\sup_{N\in\mathbb{N}} \|f^N\|_{B^{n/p+1}_{p,q}} < \infty, \quad \sup_{N\in\mathbb{N}} \|g^N\|_{B^{n/p+1}_{p,q}} < \infty$$

and

$$\left\| \left\{ 2^{(n/p+1)j} \left\| S_{j-2} f^N \cdot \nabla \Delta_j g^N - \Delta_j (f^N \cdot \nabla g^N) \right\|_{L^p} \right\}_{j=0}^{\infty} \right\|_{\ell^q} \to \infty,$$

as $N \to \infty$.

Let $(f, g) \in [\mathscr{S}(\mathbb{R}^n)]^n \times \mathscr{S}(\mathbb{R}^n)$ satisfy

(i) $\operatorname{div} f = 0$,

(ii) $\operatorname{supp}(\widehat{f}, \widehat{g}) \subset \{|\xi| \sim 2\}$: sufficiently small,

(iii) $f(x) \cdot \nabla g(y) \geqslant c$, for $\forall x, \forall y \in B_r(0)$

Outline of Proof

We put

$$f^N(x) = \frac{1}{2^{N+2}} f(2^{N+2}x), \quad g^N(x) = \sum_{k=1}^{N} \frac{1}{k2^k} g(2^{k-1}x).$$

Then $\{(f^N, g^N)\}_{N=1}^{\infty}$ satisfy

$$\operatorname{supp} \widehat{f^N} \subset \{|\xi| \sim 2^{N+3}\}, \quad \operatorname{supp} \widehat{g^N} \subset \bigcup_{k=1}^{N} \left\{|\xi| \sim 2^k\right\}$$

Hence we have

$$\Delta_j f^N = \begin{cases} f^N, & j = N+3 \\ 0, & \text{otherwise} \end{cases}, \quad \Delta_j g^N = \begin{cases} \frac{1}{j2^j} g(2^{j-1}\cdot), & 1 \leqslant j \leqslant N \\ 0 & \text{otherwise}, \end{cases}$$

which yields

$$\|f^N\|_{B^{n/p+1}_{p,q}} = c\|f\|_{L^p}, \quad \|g^N\|_{B^{n/p+1}_{p,q}} = c\|g\|_{L^p} \left(\sum_{j=1}^{N} \frac{1}{j^q} \right)^{\frac{1}{q}}$$

Outline of Proof

On the other hand, we see that

$$S_{j-2} f^N \cdot \nabla \Delta_j g^N = 0 \ (\forall j), \quad \operatorname{supp} \mathscr{F}[f^N \cdot \nabla g^N] \subset \{|\xi| \sim 2^{N+3}\}$$

Hence

$$\begin{aligned}
&\left\| \left\{ 2^{(n/p+1)j} \left\| S_{j-2} f^N \cdot \nabla \Delta_j g^N - \Delta_j (f^N \cdot \nabla g^N) \right\|_{L^p} \right\}_{j=0}^{\infty} \right\|_{\ell^q} \\
&= 2^{(n/p+1)(N+3)} \| f^N \cdot \nabla g^N \|_{L^p} \\
&\geqslant c \left\| \sum_{k=1}^{N} \frac{1}{k} f(\cdot) \cdot \nabla g \left(\frac{\cdot}{2^{N+3-k}} \right) \right\|_{L^p(B_r(0))} \\
&\geqslant c |B_r(0)|^{1/p} \sum_{k=1}^{N} \frac{1}{k} \longrightarrow \infty, \quad \text{as } N \longrightarrow \infty.
\end{aligned}$$