Stochastic and variational characterization of a difference scheme for nonlinear PDEs

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1. Introduction

$$\begin{cases} u_t(x,t) + u_x(x,t) = h \ \ (h\text{:const.}), \\ u(x,0) = u_0(x) \ \text{on } \mathbb{R}. \end{cases}$$

$$\longrightarrow \qquad u(x,t) = \int_0^t h\, ds + u_0(x-t).$$

$$\begin{cases} u_t(x,t) = u_{xx}(x,t), \\ u(x,0) = u_0(x) \ \text{on } \mathbb{R}. \end{cases}$$

$$\longrightarrow \qquad u(x,t) = \int_{\mathbb{R}} \frac{1}{2\sqrt{\pi t}}\, e^{-\frac{(x-y)^2}{4t}} u_0(y) dy.$$

$$
(\mathrm{HJ})\begin{cases} v_t(x,t)+H(x,t,v_x(x,t))=h \quad (x\in\mathbb{R},\ t\in[0,T]), \\ v(x,0)=v_0(x) \quad (x\in\mathbb{R}). \end{cases}
$$

$$
(\mathrm{HJ})_\nu\begin{cases} v^\nu_t(x,t)+H(x,t,v^\nu_x(x,t))=h+\frac{\nu^2}{2}v^\nu_{xx}(x,t), \\ v^\nu(x,0)=v_0(x). \end{cases}
$$

$$
(\mathrm{HJ})_\Delta\begin{cases} \dfrac{v^{k+1}_m-\frac{v^k_{m-1}+v^k_{m+1}}{2}}{\Delta t}+H(x_m,t_k,\frac{v^k_{m+1}-v^k_{m-1}}{2\Delta x})=h, \\ v^0_m=v_0(x_m). \end{cases}
$$

$H(x,t,u)\in C^2(\mathbb{T}^2\times\mathbb{R})$, $H_{uu}>0$, $\lim_{|u|\to\infty}\frac{H(x,t,u)}{|u|}=+\infty$,

$v_0\in Lip(\mathbb{R})$, bdd. below.

Representation formula for sol. of (HJ)

$$v(x,t) = \inf_{\substack{\gamma\in AC\\ \gamma(t)=x}} \Big[\int_0^t \{L(\gamma(s),s,\gamma'(s)) + h\} ds + v_0(\gamma(0)) \Big],$$

$$L(x,t,\xi) := \sup_{u\in\mathbb{R}} \{\xi u - H(x,t,u)\}, \quad L \in C^2(\mathbb{T}^2 \times \mathbb{R}).$$

$v : \mathbb{R} \times [0,T] \to \mathbb{R}$ is the viscosity sol. of (HJ), i.e.

v: Lipschitz, semiconcave, satisfies equation a.e.

There exists a minimizer $\gamma_*(s)$, which is a C^2-sol. of

$$\frac{d}{dt}\{L_\xi(\gamma(s),s,\gamma'(s))\} = L_x(\gamma(s),s,\gamma'(s)).$$

$(x_*(s),u_*(s)) := (\gamma_*(s), L_\xi(\gamma_*(s),s,\gamma_*(s)))$ is a sol. of

$$x'(s) = H_u(x(s),s,u(s)), \qquad u'(s) = -H_x(x(s),s,u(s)).$$

Representation formula for sol. of (HJ)$_\nu$

$$v^\nu(x,t) = \inf_{\xi \in C^1} E\Big[\int_0^t \{L(\gamma(s), s, \xi(\gamma(s), s)) + h\} ds + v_0(\gamma(0)) \Big]$$

ξ : smooth function,

γ : $d\gamma(s) = \xi(\gamma(s), s) ds + \nu dw(s), \ \gamma(t) = x$,

w : Brownian motion,

$E[\ ]$: expectation w.r.t. the Wiener measure.

v^ν is the C^2-solution of (HJ)$_\nu$.

As $\nu \to 0+$,

$$v^\nu \to v, \text{ locally uniformly.}$$

There exists a minimizer ξ_*

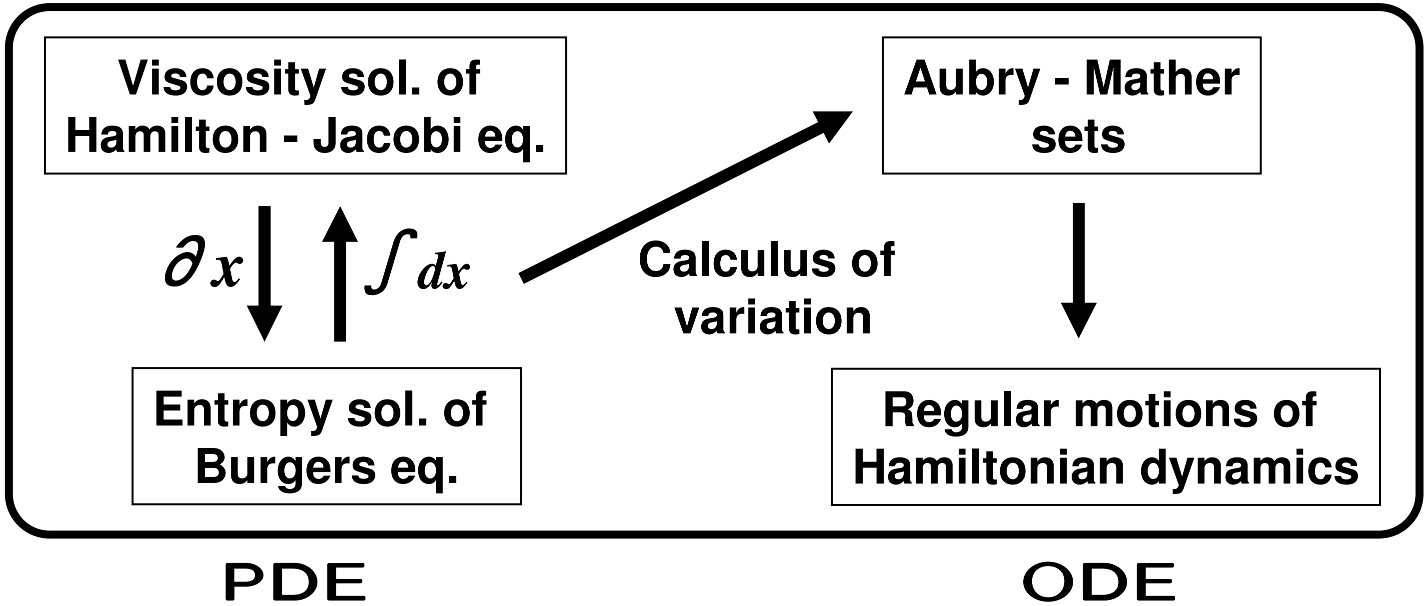
$$\xi_*(x,t) = H_u(x, t, v^\nu_x(x,t)).$$

Application to the Aubry-Mather theory

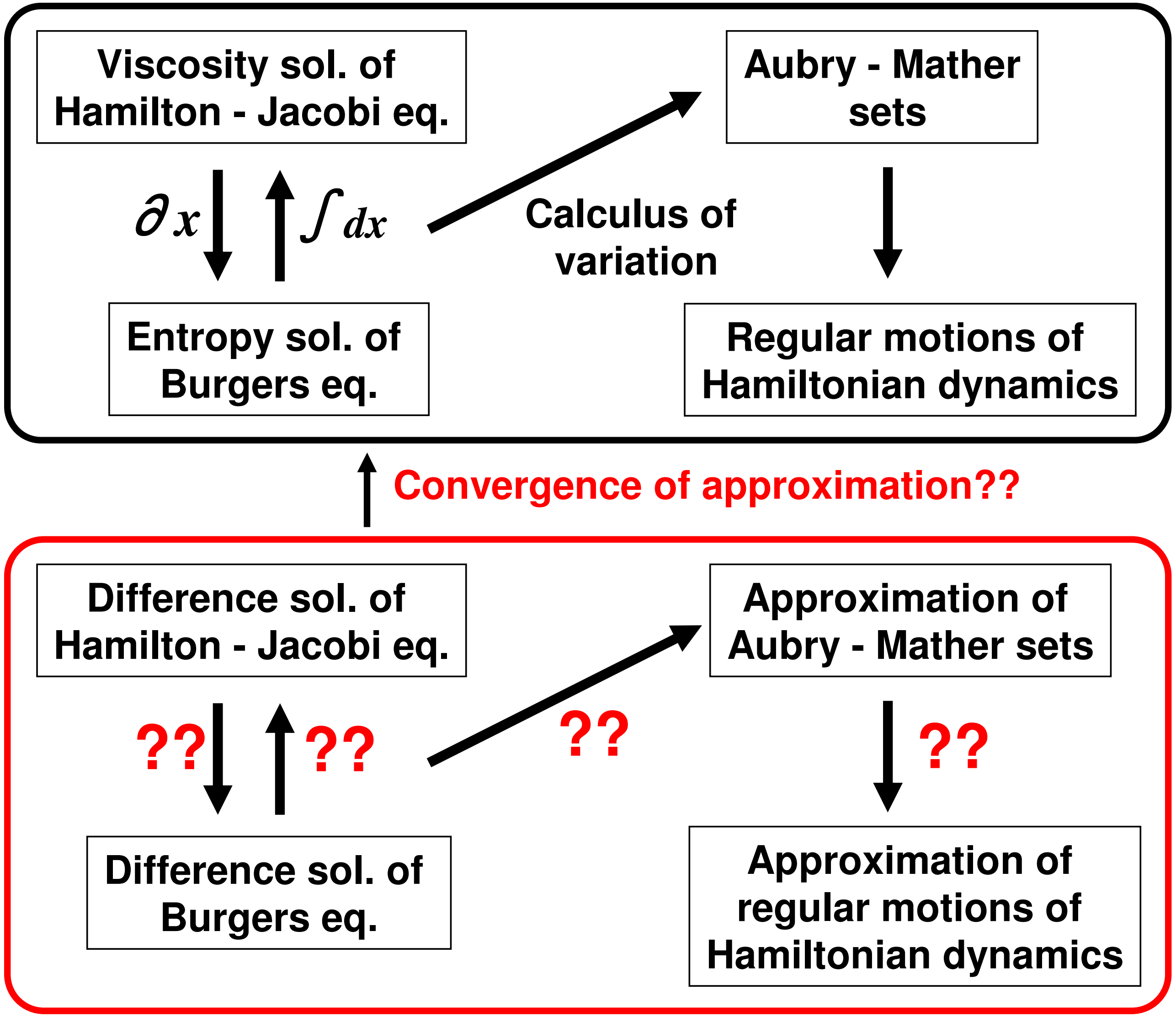
(BE) $$u_t + H(x,t,u)_x = 0 \quad \text{in } \mathbb{T}^2, \qquad \int_{\mathbb{T}} u(x,t)\,dx \equiv C.$$

(HJ) $$v_t + H(x,t,C+v_x) = \bar{H}(C) \quad \text{in } \mathbb{T}^2,$$

$$\bar{H}(C) = \iint_{\mathbb{T}^2} H(x,t,u(x,t))\,dxdt.$$



2. Problem and Results



Set $\lambda > 0$, $N, K \in \mathbb{N}$, $\Delta x := (2N)^{-1}, \Delta t := (2K)^{-1}$, $\frac{\Delta t}{\Delta x} = \lambda$. ($\Delta x, \Delta t$ are changed to 0_+ with the fixed λ.)

$$(\mathrm{BE})_\Delta \begin{cases} \dfrac{u_{m+1}^{k+1} - \frac{(u_m^k + u_{m+2}^k)}{2}}{\Delta t} + \dfrac{H(x_{m+2}, t_k, u_{m+2}^k) - H(x_m, t_k, u_m^k)}{2\Delta x} = 0, \\ u_{m\pm 2N}^k = u_m^k = u_m^{k\pm 2K}, \\ \displaystyle\sum_{0\le m<2N} u_m^k 2\Delta x \equiv C \quad (k+m = \text{even}). \end{cases}$$

$\forall\, C \in [C_0, C_1]$, $\exists/\ u_m^k = u_m^k(C)$ sol. of $(\mathrm{BE})_\Delta$ s.t.

$$u_m^k \in (-\lambda^{-1}, \lambda^{-1}),$$

which approximates a $\mathbb{Z}^2$-periodic entropy sol. in $C^0(\mathbb{T}; L^1(\mathbb{T}))$.

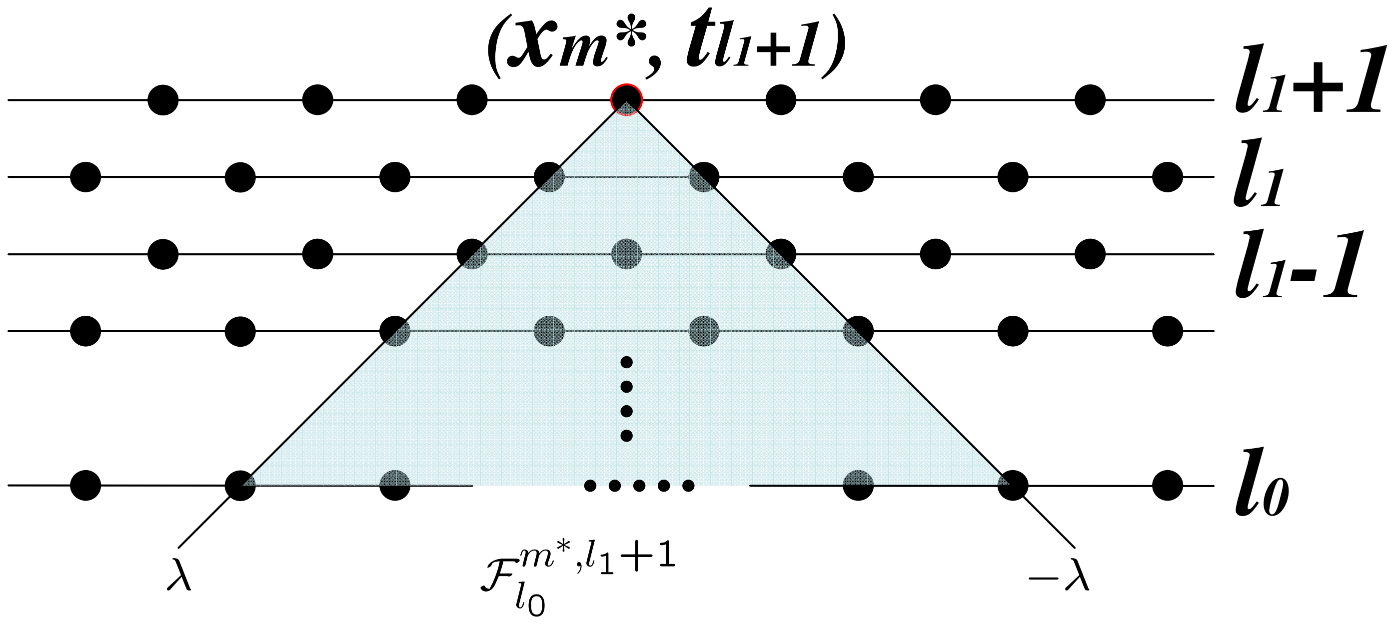
Set $\bar{H}_\triangle(C) := \sum_{\substack{0\le m<2N \\ 0\le k<2K}} H(x_m, t_k, u_m^k(C))$. Then $\bar{H}_\triangle \in C^1$.

$$
(\mathrm{HJ})_\triangle \begin{cases} \dfrac{v_m^{k+1} - \frac{v_{m-1}^k + v_{m+1}^k}{2}}{\Delta t} + H(x_m, t_k, C + \frac{v_{m+1}^k - v_{m-1}^k}{2\Delta x}) = \bar{H}_\triangle(C), \\ v_{m+1\pm 2N}^k = v_{m+1}^k = v_{m+1}^{k\pm 2K}, \\ C + \dfrac{v_{m+1}^k - v_{m-1}^k}{2\Delta x} = u_m^k(C). \end{cases}
$$

$\exists\, v_{m+1}^k = v_{m+1}^k(C)$ sol. of $(\mathrm{HJ})_\triangle$, which approximates a $\mathbb{Z}^2$-periodic viscosity sol. in C^0.

Representation formula for sol. of $(\mathbf{HJ})_\triangle$

For each $m^*, l_1, l_0 \in \mathbb{Z}$ $(l_0 \le l_1)$, $v_{m^*}^{l_1+1}(C)$ includes the information on $\mathcal{F}_{l_0}^{m^*,l_1+1}$:



Define random walk on $\mathcal{F}_{l_0}^{m^*,l_1+1}$:

Take $\xi(x_m, t_{k+1}) = \xi_m^{k+1} : \mathcal{F}_{l_0}^{m^*,l_1+1} \to [-\lambda^{-1}, \lambda^{-1}]$.

$$\bar{\rho}_m^{k+1} := \frac{1}{2}(1 + \lambda\xi_m^{k+1}) \quad \text{for } x_m \to x_m - \Delta x \text{ at } t_{k+1},$$

$$\bar{\bar{\rho}}_m^{k+1} := \frac{1}{2}(1 - \lambda\xi_m^{k+1}) \quad \text{for } x_m \to x_m + \Delta x \text{ at } t_{k+1},$$

$$\mathcal{R}_{l_0}^{m^*,l_1+1} := \{\gamma \mid \gamma^k = \gamma^{k+1} \pm \Delta x,\ l_0 \le k \le l_1,\ \gamma^{l_1+1} = x_{m^*}\}.$$

$$\mu^{l_0}(\gamma) := \prod_{l_0<k\le l_1} \rho_{m(\gamma^{k+1})}^{k+1},\ \rho_{m(\gamma^{k+1})}^{k+1} = \bar{\rho}_{m(\gamma^{k+1})}^{k+1} \text{ or } \bar{\bar{\rho}}_{m(\gamma^{k+1})}^{k+1}.$$

μ^{l_0} yields a measure of $\mathcal{R}_{l_0}^{m^*,l_1+1}$.

- $$v_{m^*}^{l_1+1}(C) = \inf_{\xi} E_{\mu^{l_0}}\Big[\sum_{l_0<k\le l_1} \{L(\gamma^{k+1}, t_k, \xi^{k+1}_{m(\gamma^{k+1})}) - C\xi^{k+1}_{m(\gamma^{k+1})} + \bar{H}_{\Delta}(C))\}\Delta t + v^{l_0}_{m(\gamma^{l_0})}(C)\Big],$$

  where $E_{\mu^{l_0}}[\mathcal{L}(\gamma)] := \sum_{\gamma\in\mathcal{R}_{l_0}^{m^*,l_1+1}} \mu^{l_0}(\gamma)\mathcal{L}(\gamma)$,

- $\exists$ minimizer $\xi = \xi^*|_{\mathcal{F}_{l_0}^{m^*,l_1+1}}$, where

  $$\xi^*(x_m, t_{k+1}) = H_u(x_m, t_k, C + \frac{v^k_{m+1}(C) - v^k_{m-1}(C)}{2\Delta x}),$$

- With this ξ^*, $\bar{\gamma}^{l_0} := E_{\mu^{l_0}}[\gamma^{l_0}]$ is well-defined for $\forall\, l_0 \le l_1$,

- $$\lim_{l_0\to-\infty} \frac{\bar{\gamma}^{l_0}}{t_{l_0}} = \bar{H}'_{\Delta}(C).$$