Steady-State Navier-Stokes Flow Past a Rotating Body: Leray Solutions are Physically Reasonable

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Waseda University, March 10 2010

Description of the Problem

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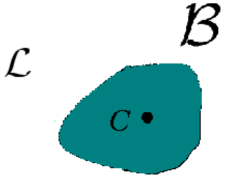
Consider the flow of a Navier-Stokes liquid, $\mathcal{L}$, past a rigid body $\mathcal{B}$, that is allowed to translate **and to rotate**



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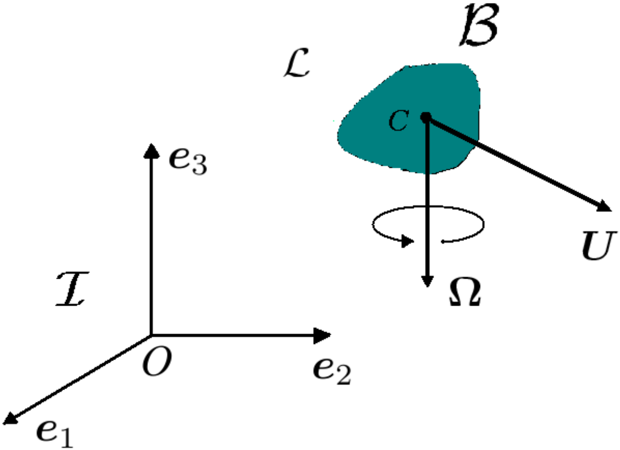

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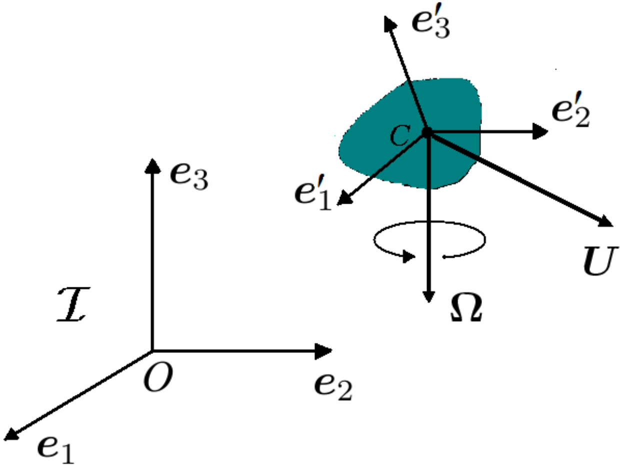
The motion of $\mathcal{L}$ is described from a frame, $\mathcal{S} = \{C, \boldsymbol{e}_i'\}$, attached to $\mathcal{B}$



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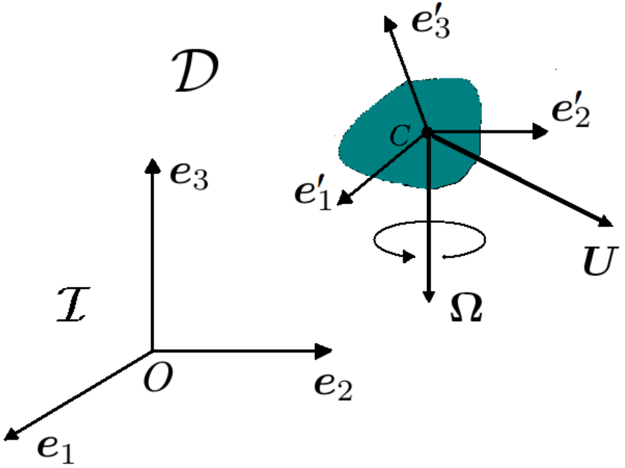

In this way, the region, $\mathcal{D}$, occupied by $\mathcal{L}$ becomes **time-independent**

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Therefore, the relevant equations become

$$\left.\begin{aligned} &(\boldsymbol{v} - \boldsymbol{\xi} - \boldsymbol{\omega} \times \boldsymbol{x}) \cdot \operatorname{grad} \boldsymbol{v} + \boldsymbol{\omega} \times \boldsymbol{v} = \nu \Delta \boldsymbol{v} - \operatorname{grad} p \\ &\operatorname{div} \boldsymbol{v} = 0 \end{aligned}\right\} \text{in } \mathcal{D}$$

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with the following side conditions

$$\boldsymbol{v}(\boldsymbol{y}) = \boldsymbol{\xi} + \boldsymbol{\omega} \times \boldsymbol{y}, \quad \boldsymbol{y} \in \Sigma \equiv \partial\mathcal{D}\,, \qquad \lim_{|\boldsymbol{x}| \to \infty} \boldsymbol{v}(\boldsymbol{x}) = \boldsymbol{0}$$

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Challenging feature:

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Unbounded coefficient $|\boldsymbol{\omega}\times\boldsymbol{x}|\to\infty$ **as** $|x|\to\infty$

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Let

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and set

$$\Omega^* := \left\{\boldsymbol{x}^* \in \mathbb{R}^3 : \ \boldsymbol{x}^* = \boldsymbol{x} - \lambda\,\boldsymbol{e}_1 \times \boldsymbol{e}, \text{ for some } \boldsymbol{x} \in \Omega\right\},$$

$$\boldsymbol{v}^*(x^*) := \boldsymbol{v}(x^* + \lambda\,\boldsymbol{e}_1 \times \boldsymbol{e}), \quad p^*(x^*) := p(x^* + \lambda\,\boldsymbol{e}_1 \times \boldsymbol{e}),$$

$$\boldsymbol{f}^*(x^*) := \boldsymbol{f}(x^* + \lambda\,\boldsymbol{e}_1 \times \boldsymbol{e}),$$

$$\mathrm{Re} := \left(\frac{|\xi|d}{\nu}\right)\boldsymbol{e}\cdot\boldsymbol{e}_1 \text{ (Reynolds number)}, \quad \mathrm{Ta} := \frac{|\omega|\,d^2}{\nu} \text{ (Taylor number)},$$

Equations in Dimensionless Form

The original problem becomes (stars omitted)

$$\left.\begin{aligned}&\Delta \boldsymbol{v} + \mathrm{Re}\,(\boldsymbol{e}_1 - \boldsymbol{v}) \cdot \mathrm{grad}\, \boldsymbol{v} + \mathrm{Ta}\,(\boldsymbol{e}_1 \times \boldsymbol{x} \cdot \mathrm{grad}\, \boldsymbol{v} - \boldsymbol{e}_1 \times \boldsymbol{v}) = \mathrm{grad}\, p \\ &\mathrm{div}\, \boldsymbol{v} = 0\end{aligned}\right\}\text{ in } \mathcal{D}$$

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Notice that ($\boldsymbol{\omega} \neq \mathbf{0}$)

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We suppose $\mathrm{Re} \neq 0$, and, to fix ideas, we shall take $\mathrm{Re} > 0$

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have at least one (smooth) solution?

Steady-State Flow: Existence of Solutions

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All properties listed in (A), (B) and (C) are related to the **asymptotic spatial behavior** of the velocity field $\boldsymbol{v}$ and corresponding pressure p.

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FINN (1965): Existence of PR solutions for data of **restricted** size

BABENKO (1972), GPG (1992): **Every Leray solution is PR**

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Unbounded Coefficient! $|\boldsymbol{e}_1 \times \boldsymbol{x}| \to \infty$ as $|x| \to \infty$

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Objective of this talk is to prove (or to give a flavor of the proof) that both Questions 1 and 2 are **positively answered.**

Rotation and Translation: Main Contributions

Over the past decades, both Questions 1 and 2 have been addressed, from different perspectives and by different approaches, by many authors:

T. HISHIDA (1999-2008), GPG (2003),
R. FARWIG, T. HISHIDA, & D. MÜLLER (2004-2009),
A.L. SILVESTRE, M. KEYD & GPG (2006-2009),
R. FARWIG & J. NEUSTUPA (2009),
P. DEURING, S. KRAČMAR, M. KRBEC, S. NEČASOVA & P. PENEL (2005-2009)
T. HISHIDA & Y. SHIBATA (2006-2009) ......

yet, without a definite answer.

Objective of this talk is to prove (or to give a flavor of the proof) that both Questions 1 and 2 are **positively answered.**

In other words, **for data of arbitrary size, there is always a corresponding, smooth PR solution.**

Rotation and Translation: Leray Solutions

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Existence of Leray Solution (Data of arbitrary size)

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The "rotational term" satisfies the fundamental property:

$$\int_{\mathcal{D}} (\boldsymbol{e}_1 \times \boldsymbol{x} \cdot \operatorname{grad} \boldsymbol{u} - \boldsymbol{e}_1 \times \boldsymbol{u}) \cdot \boldsymbol{u} = 0\,, \quad \text{for all } \boldsymbol{u} \in C_0^\infty(\mathcal{D}),\ \operatorname{div} \boldsymbol{u} = 0\,.$$

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Thanks to this property, the above problem (1) admits the formal *a priori* **global** estimate

$$\int_{\mathcal{D}} |\operatorname{grad} \boldsymbol{v}|^2 \leq C(\mathcal{D}, \mathrm{Re}, \mathrm{Ta})$$

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Theorem (Weinberger, 1982; Serre, 1987; Borchers, 1992)

Let $\mathcal{D}$ be an exterior domain in $\mathbb{R}^3$. For **any** $\mathrm{Re} > 0$ *and* $\mathrm{Ta} \geq 0$*, there exists at least one* $(\boldsymbol{v}, p) \in C^\infty(\mathcal{D}) \times C^\infty(\mathcal{D})$ (Leray solution) *to problem* (1) *.*

Main Result: Leray Solutions are PR Solutions

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Let $\mathrm{Re}, \mathrm{Ta} > 0$, *that is,* $\boldsymbol{\xi} \cdot \boldsymbol{\omega} \neq 0$, *be given. Let* $(\boldsymbol{v}, p)$ *be a smooth pair satisfying the following equations*

$$\left.\begin{aligned} &\Delta \boldsymbol{v} + \mathrm{Re}\,(\boldsymbol{e}_1 - \boldsymbol{v}) \cdot \operatorname{grad} \boldsymbol{v} + \mathrm{Ta}\,(\boldsymbol{e}_1 \times \boldsymbol{x} \cdot \operatorname{grad} \boldsymbol{v} - \boldsymbol{e}_1 \times \boldsymbol{v}) = \operatorname{grad} p \\ &\operatorname{div} \boldsymbol{v} = 0 \end{aligned}\right\} \text{ in } \mathcal{D}$$

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Then, if for some $R > 0$,

$$\operatorname{grad} \boldsymbol{v} \in L^2(\mathcal{D} \cap \{|x| > R\})\,, \quad \boldsymbol{v} \in L^6(\mathcal{D} \cap \{|x| > R\})\,,$$

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$$\mathrm{grad}\,\boldsymbol{v} \in L^2(\mathcal{D} \cap \{|x| > R\}), \quad \boldsymbol{v} \in L^6(\mathcal{D} \cap \{|x| > R\}),$$

for all sufficiently large $|x|$ *we have*

$$|\boldsymbol{v}(x)| \leq \mathcal{V}_1(x) + \mathcal{V}_2(x)$$

where

$$\mathcal{V}_1(x) = O([(1+|x|)(1+\mathrm{Re}\,s(x))]^{-1}), \quad \mathcal{V}_2(x) = O(|x|^{-3/2+\delta}), \text{ arbitrary } \delta > 0\,.$$

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Remark 3

The result includes the case of nonzero body forces decaying "sufficiently fast" at large distances.

Leray Solutions are PR: Sketch of the Proof

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For a fixed and sufficiently large $\rho > 0$, take a smooth "cut-off" function $\psi_\rho = \psi_\rho(x)$ that is 0 if $|x| < R$ and is 1 if $|x| > 2\rho$, and set

$$\boldsymbol{u} := \psi_\rho \boldsymbol{v} - \boldsymbol{z}\,, \quad \mathrm{div}\, \boldsymbol{z} = \boldsymbol{v} \cdot \mathrm{grad}\, \psi_\rho\,, \quad \mathsf{p} := \psi_\rho\, p$$

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Then, the original problem for $(\boldsymbol{v}, p)$ goes into the following one:

$$\left.\begin{aligned} &\Delta \boldsymbol{u} + \frac{\partial \boldsymbol{u}}{\partial x_1} + (\boldsymbol{e}_1 \times \boldsymbol{x} \cdot \operatorname{grad} \boldsymbol{u} - \boldsymbol{e}_1 \times \boldsymbol{u}) \\ &\qquad\qquad = \operatorname{div}\left[(\psi_\rho \boldsymbol{v}) \otimes (\psi_\rho \boldsymbol{v})\right] + \operatorname{grad} \mathsf{p} + \boldsymbol{f}_c \\ &\operatorname{div} \boldsymbol{u} = 0 \end{aligned}\right\} \text{in } \mathbb{R}^3$$

where $\boldsymbol{f}_c \in C_0^\infty(\mathbb{R}^3)$.

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where $\boldsymbol{f}_c \in C_0^\infty(\mathbb{R}^3)$.

Since $\boldsymbol{z} \in C_0^\infty(\mathbb{R}^3)$, $\boldsymbol{u}(x) = \boldsymbol{v}(x)$ for *all* sufficiently large $|\boldsymbol{x}|$.

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The proof rests upon a result of R. Farwig (2006) and the following *uniqueness lemma*.

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Lemma

Let $\boldsymbol{u}, \mathsf{p}$*, with* $\boldsymbol{u} \in L^s(\mathbb{R}^3)$*, some* $s \in [1, \infty)$*, be a smooth solution to*

$$\left.\begin{array}{l} \Delta\boldsymbol{u} + \dfrac{\partial \boldsymbol{u}}{\partial x_1} + (\boldsymbol{e}_1 \times \boldsymbol{x} \cdot \mathrm{grad}\,\boldsymbol{u} - \boldsymbol{e}_1 \times \boldsymbol{u}) = \mathrm{grad}\,\mathsf{p} \\ \mathrm{div}\,\boldsymbol{u} = 0 \end{array}\right\} \text{in } \mathbb{R}^3\,.$$

Then $\boldsymbol{u} \equiv \mathrm{grad}\,\mathsf{p} \equiv \boldsymbol{0}$*. If* $s = \infty$*, then* $\boldsymbol{u} = \mathrm{const}$.

Leray Solutions are PR: Sketch of the Proof

Step 3: Using Summability Properties to get Pointwise Estimates.

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The "canonical" way of showing this is to represent the solution $\boldsymbol{u}$ to the problem

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where

$$\boldsymbol{F} := \operatorname{div}\left[(\psi_\rho \boldsymbol{v}) \otimes (\psi_\rho \boldsymbol{v})\right] + \boldsymbol{f}_c\,.$$

by means of the fundamental solution $\mathfrak{G} = \mathfrak{G}(x, y)$.

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(Recall that $\boldsymbol{v}(x) = \boldsymbol{u}(x)$ for all large $|\boldsymbol{x}|$).

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- The form of the fundamental tensor solution $\mathfrak{G}$ is very complicated ;
- Unlike the case $\mathrm{Ta} = 0$ (no rotation), the fundamental tensor $\mathfrak{G}$ does **not** satisfy the uniform estimate (that would be the starting point to establish asymptotic properties):

$$|\mathfrak{G}(x,y)| \le \frac{C}{|x-y|}\,, \quad \text{for all } x,y \in \mathbb{R}^3$$

for some C independent of x, y (Farwig, Hishida & Müller, 2004).

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Therefore, one would like to argue in a different way.

Leray Solutions are PR: Sketch of the Proof

GPG (2003) , Silvestre & GPG (2006,2007)

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Let

$$\boldsymbol{Q}(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\mathrm{Ta}\, t) & -\sin(\mathrm{Ta}\, t) \\ 0 & \sin(\mathrm{Ta}\, t) & \cos(\mathrm{Ta}\, t) \end{bmatrix}, \quad t \geq 0$$ (rotation matrix around $\boldsymbol{e}_1$)

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Set:

$$\boldsymbol{y} = \boldsymbol{Q}(t) \cdot \boldsymbol{x}\,, \quad t \geq 0\,.$$

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Set:

$$\boldsymbol{y} = \boldsymbol{Q}(t) \cdot \boldsymbol{x}\,, \quad t \geq 0\,.$$

Define

$$\boldsymbol{w}(y,t) := \boldsymbol{Q}(t) \cdot \boldsymbol{u}(\boldsymbol{Q}^\top(t) \cdot \boldsymbol{y})\,, \quad \pi(y,t) := \mathsf{p}(\boldsymbol{Q}^\top(t) \cdot \boldsymbol{y})$$

$$\boldsymbol{V}(y,t) := \boldsymbol{Q}(t) \cdot [\psi_\rho \boldsymbol{v}](\boldsymbol{Q}^\top(t) \cdot \boldsymbol{y})\,, \quad \boldsymbol{F}_c(y,t) := \boldsymbol{Q}(t) \cdot \boldsymbol{f}_c(\boldsymbol{Q}^\top(t) \cdot \boldsymbol{y})$$

Leray Solutions are PR: Sketch of the Proof

Then $(\boldsymbol{w}, \pi)$ satisfies the following Oseen-like IVP

$$\left.\begin{aligned} &\frac{\partial \boldsymbol{w}}{\partial t} = \Delta \boldsymbol{w} + \frac{\partial \boldsymbol{w}}{\partial x_1} - \operatorname{div}\left[\boldsymbol{V} \otimes \boldsymbol{V}\right] - \operatorname{grad} \pi - \boldsymbol{F}_c \\ &\operatorname{div} \boldsymbol{w} = 0 \end{aligned}\right\} \text{in } \mathbb{R}^3 \times (0, \infty)$$

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with $\boldsymbol{F}_c \in L^\infty(0, \infty; C_0^\infty(\mathbb{R}^3))$.

The solution $\boldsymbol{w}$ can be then represented by means of convolutions involving the well-known (and very well-studied) Oseen fundamental tensor $\boldsymbol{\Gamma} = \boldsymbol{\Gamma}(\xi, \tau)$.

Leray Solutions are PR Solutions: Sketch of the Proof

We thus have

$$\boldsymbol{w}(y,t) = \boldsymbol{w}_1(y,t) + \boldsymbol{w}_2(y,t) + \boldsymbol{w}_3(y,t)$$

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$$\boldsymbol{w}_3(y,t) = -\int_{\mathbb{R}^3}\int_0^t \mathrm{grad}\,\boldsymbol{\Gamma}(y-z,\tau) : [\boldsymbol{V}\otimes\boldsymbol{V}](z,t-\tau)d\tau\,dz$$

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Leray Solutions are PR Solutions: Sketch of the Proof

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$$|\boldsymbol{v}(x)| \le C_\theta \left[\frac{\|\boldsymbol{f}_c\|_r}{(1+|x|)(1+s(x))} + \left(\int_{|y|\ge R} |\mathrm{grad}\,\boldsymbol{v}|^2 \right)^{1-\theta} \right] .$$

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A crucial role in the proof of the lemma is played by the previously established summability properties:

$$\boldsymbol{v} \in L^{s_1}(\mathcal{D}^R)\,, \quad \mathrm{grad}\, \boldsymbol{v} \in L^{s_2}(\mathcal{D}^R)\,, \quad D^2\boldsymbol{v} \in L^{s_3}(\mathcal{D}^R)\,,$$

all $s_1 > 2$, $s_2 > 4/3$, $s_3 > 1\,,$

where $\mathcal{D}^R := \mathcal{D} \cap \{|y| \geq R\}\,.$

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$$|\boldsymbol{v}(x)| \leq C_\theta \left[\frac{\|\boldsymbol{f}_c\|_r}{(1+|x|)(1+s(x))} + \left(\int_{|y|\geq R} |\mathrm{grad}\,\boldsymbol{v}|^2 \right)^{1-\theta} \right] .$$

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Final Remark.

There is still one question that remains *open*, concerning the *leading term in the asymptotic expansion.*

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Precisely, it is expected that $\boldsymbol{v} = \boldsymbol{v}(x)$ can be expressed, for large $|x|$, as:

$$\boldsymbol{v}(x) = \boldsymbol{v}_1(x) + \boldsymbol{v}_2(x)$$

with

$$|\boldsymbol{v}_1(x)| = O([(1 + |x|)(1 + \mathrm{Re}\, s(x))]^{-1}) \,, \quad \boldsymbol{v}_2(x) = O(|x|^{-3/2+\delta}),$$

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THANK YOU!