Asymptotic Behavior of a Leray Solution around a Rotating Obstacle

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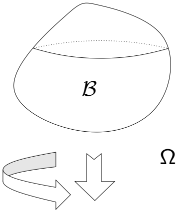
Tokyo, March 10, 2010

- Steady state flow of a Navier-Stokes liquid past a translating and rotating body $\mathcal{B}$ (in a frame attached to $\mathcal{B}$):

$$\begin{cases} \mu\Delta v - \nabla p - v\cdot\nabla v + \xi\cdot\nabla v + \omega\wedge x\cdot\nabla v - \omega\wedge v = f & \text{in } \Omega, \\ \operatorname{div}(v) = 0 & \text{in } \Omega, \\ v = v_* & \text{on } \partial\Omega, \\ \lim_{|x|\to\infty} v(x) = 0. & \end{cases}$$

- $\xi \in \mathbb{R}^3$ = translational velocity, $\omega \in \mathbb{R}^3$ = rotational velocity, $\Omega \subset \mathbb{R}^3$ exterior domain.

- Leray solution: $\nabla v \in L^2(\Omega)$, $v \in L^6(\Omega)$, $p \in L^2_{loc}(\Omega)$
- Question: Are Leray solutions Physically Reasonable? (in the sense of Finn)



- Answer:

  A) $\xi \neq 0$, $\omega = 0$: Yes! (Babenko, 1973)

  B) $\xi \neq 0$, $\omega \neq 0$: Yes! (Galdi, K., 2010)

  C) $\xi = 0$, $\omega = 0$: If (v, p) satisfies the Energy Inequality and the data is small, then Yes! (Galdi 1993)

  D) $\xi = 0$, $\omega \neq 0$: If (v, p) satisfies the Energy Inequality and the data is small, then Yes! (Galdi, K. 2010)

- The topic of this talk is the case D.

$$
(1) \quad \begin{cases} \Delta v - \nabla p - \mathcal{R} v \cdot \nabla v + \mathcal{T}\big(\mathrm{e}_1 \times x \cdot \nabla v - \mathrm{e}_1 \times v\big) = f & \text{in } \Omega, \\ \mathrm{div}(v) = 0 & \text{in } \Omega, \\ v = v_* & \text{on } \partial\Omega, \end{cases}
$$

Definition

A Leray solution (v, p) is *Physically Reasonable* iff:

I) Energy Equality:

$$
2\int_\Omega |\mathbf{D}(v)|^2 \,\mathrm{d}x = -\int_\Omega f \cdot v \,\mathrm{d}x + \int_{\partial\Omega} \big(\mathrm{T}(v,p)\cdot n\big)\cdot v_* \,\mathrm{d}S - \frac{\mathcal{R}}{2}\int_{\partial\Omega} |v_*|^2\, v_* \cdot n \,\mathrm{d}S + \frac{\mathcal{T}}{2}\int_{\partial\Omega} |v_*|^2\, \mathrm{e}_1 \times x \cdot n \,\mathrm{d}S\,.
$$

II) Decays like the Stokes fundamental solution as $|x| \to \infty$.

III) Solution is unique for small data.

Theorem (Uniqueness)

Let $f \in L^2(\Omega) \cap L^{\frac{6}{5}}(\Omega)$*, and* $v_* \in W^{\frac{3}{2},2}(\partial\Omega)$*. Moreover, let* (v, p) *be a Leray solution that satisfies the Energy Inequality*

$$2\int_\Omega |\mathbf{D}(v)|^2 \,\mathrm{d}x \le -\int_\Omega f \cdot v \,\mathrm{d}x + \int_{\partial\Omega} \big(\mathrm{T}(v,p)\cdot n\big) \cdot v_* \,\mathrm{d}S - \frac{\mathcal{R}}{2}\int_{\partial\Omega} |v_*|^2 \, v_* \cdot n \,\mathrm{d}S + \frac{\mathcal{T}}{2}\int_{\partial\Omega} |v_*|^2 \, \mathrm{e}_1 \times x \cdot n \,\mathrm{d}S \,.$$

If $(w, \pi) \in D^{1,2}(\Omega) \cap L^6(\Omega)^3 \times L^2(\Omega)$ *is another solution and*

$$[\![w]\!]_1 := \operatorname*{ess\,sup}_{x\in\Omega} \Big[(1+|x|)|w(x)|\Big] < \frac{1}{8\mathcal{R}},$$

then $(w, \pi) = (v, p)$*. In this case,* (v, p) *satisfies the Energy Equality.*

Theorem (Main Theorem)

Let $\mathcal{R}, \mathcal{T} \in (0, B]$, for some $B > 0$. There is a constant $M(\Omega, B) > 0$ such that if $f = \text{div}(F)$ an v_* satisfy

$$\mathcal{R}\left(\|F\| + \|v_*\|_{W^{\frac{3}{2},2}(\partial\Omega)}\right) < M,$$

then a Leray solution (v, p) that satisfies the Energy Inequality is Physically Reasonable.

Proof.

Under the given assumptions, a P.R. solution (w, π) exists with $[\![w]\!]_1 < \frac{1}{8\mathcal{R}}$ (Galdi, 2003). □

- Testing the equation

$$\Delta v - \nabla p - \mathcal{R} v \cdot \nabla v + \mathcal{T}\big(\mathrm{e}_1 \times x \cdot \nabla v - \mathrm{e}_1 \times v\big) = f$$

with $-\nabla \times (\psi_R^2 \nabla \times v)$, where $\psi_R = 0$ in B_r, $\psi_R = 1$ in $\mathrm{B}_{R,2r}$, $\psi_R = 0$ in B^{2R}, and $|\mathrm{D}^\alpha \psi_R| \le \frac{C}{|x|^{|\alpha|}}$, yields, since

$$\Big|\int_\Omega f \cdot \big(-\nabla \times (\psi_R^2 \nabla \times v)\big)\,\mathrm{d}x\Big| \le C\,(\|\nabla v\|_2^2 + \|f\|_2^2) + \varepsilon \|\psi_R \Delta v\|_2^2,$$

$$\Big|\int_\Omega (\mathrm{e}_1 \times v) \cdot \big(-\nabla \times (\psi_R^2 \nabla \times v)\big)\,\mathrm{d}x\Big| \le C\,\|\nabla v\|_2^2,$$

$$\Big|\int_\Omega (\mathrm{e}_1 \times x \cdot \nabla v) \cdot \big(-\nabla \times (\psi_R^2 \nabla \times v)\big)\,\mathrm{d}x\Big| \le C\,\|\nabla v\|_2^2,$$

$$\Big|\int_\Omega (v \cdot \nabla v) \cdot \big(-\nabla \times (\psi_R^2 \nabla \times v)\big)\,\mathrm{d}x\Big| \le C\,(\|\nabla v\|_2^2 + \|\nabla v\|_2^6) + \varepsilon \|\psi_R \Delta v\|_2^2,$$

$$\int_\Omega \nabla p \cdot \big(-\nabla \times (\psi_R^2 \nabla \times v)\big)\,\mathrm{d}x = 0,$$

$$\int_\Omega \Delta v \cdot \big(-\nabla \times (\psi_R^2 \nabla \times v)\big)\,\mathrm{d}x = \|\psi_R \Delta v\|_2^2 + \int_\Omega \Delta v \cdot (\nabla \times v) \times \nabla[\psi_R^2]\,\mathrm{d}x,$$

$$|\int_\Omega \Delta v \cdot (\nabla \times v) \times \nabla[\psi_R^2]\,\mathrm{d}x| \le C\,\|\nabla v\|_2^2 + \varepsilon \|\psi_R \Delta v\|_2^2,$$

we have

$$\|\psi_R \Delta v\|_2^2 \le C\left(\|\nabla v\|_2^2 + \|\nabla v\|_2^6 + \|f\|_2^2\right).$$

Lemma

Let $f \in L^2(\Omega)$, $v_* \in W^{\frac{3}{2},2}(\partial\Omega)$. A Leray solution (v,p) satisfies $\nabla^2 v \in L^2(\Omega)$.

- Let $\psi = 0$ on B_ρ and $\psi = 1$ on $\mathbb{R}^3 \setminus B_{2\rho}$ and put

$$z := \psi v + H, \quad q = \psi p,$$

where $\operatorname{div}(H) = \nabla\psi \cdot v$. Then

$$\begin{cases} \Delta z - \nabla q + \mathcal{T}\left(e_1 \times x \cdot \nabla z - e_1 \times z\right) = \psi f + G + \mathcal{R}\psi\, v \cdot \nabla v & \text{in } \mathbb{R}^3, \\ \operatorname{div}(z) = 0 & \text{in } \mathbb{R}^3. \end{cases}$$

- Taking divergence on both sides:

$$-\Delta q = \operatorname{div}\left(\psi f\right) + \operatorname{div}(G) + \mathcal{R} \operatorname{div}\left(\psi\, v \cdot \nabla v\right) \quad \text{in } \mathbb{R}^3.$$

- If $f = \operatorname{div}(F)$:

$$-\Delta q = \operatorname{div}\left(\operatorname{div}(\psi F)\right) + \operatorname{div}(\tilde{G}) + \mathcal{R} \operatorname{div}\left(\psi\, v \cdot \nabla v\right) \quad \text{in } \mathbb{R}^3.$$

- Another solution is given by $\bar{q} = q_1 + q_2 + q_3$ with

$$q_1 := \mathfrak{F}^{-1}\left(\frac{-\xi_j\xi_k}{|\xi|^2}\mathfrak{F}\big((\psi F)_{jk}\big)\right) \qquad q_2 := \mathfrak{F}^{-1}\left(\frac{i\xi_j}{|\xi|^2}\mathfrak{F}\big(\tilde{G}_j\big)\right)$$

$$q_3 := \mathcal{R}\mathfrak{F}^{-1}\left(\frac{i\xi_j}{|\xi|^2}\mathfrak{F}\big((\psi\, v\cdot\nabla v)_j\big)\right).$$

- If $f \in L^2(\Omega)\cap L^{\frac{3}{2}}(\Omega)$ we have $\bar{q} \in L^3(\mathbb{R}^3)$ and $\nabla\bar{q} \in L^{\frac{3}{2}}(\mathbb{R}^3)$.
- Since $\Delta(q-\bar{q}) = 0$ we have

$$\nabla(\bar{q}-q)(x) = \frac{C}{R^3}\int_{\mathrm{B}_R(x)}\nabla(\bar{q}-q)\,\mathrm{d}y$$

$$\leq \frac{C}{R^3}\left(\|\nabla\bar{q}\|_{\frac{3}{2}}\,|\mathrm{B}_R|^{\frac{1}{3}} + R\,\|\nabla q/(1+|y|)\|_2\,|\mathrm{B}_R|^{\frac{1}{2}}\right) \xrightarrow{R\to\infty} 0$$

- Note that $\|\nabla q/(1+|y|)\|_2 < \infty$ since $\|\Delta z\|_2 < \infty$ and

$$\Delta z - \nabla q + \mathcal{T}\big(\mathrm{e}_1\times y\cdot\nabla z - \mathrm{e}_1\times z\big) = \psi f + G + \mathcal{R}\psi\, v\cdot\nabla v \text{ in } \mathbb{R}^3.$$

Lemma

Let $f \in L^2(\Omega) \cap L^{\frac{3}{2}}(\Omega)$, $v_* \in W^{\frac{3}{2},2}(\partial\Omega)$. *A Leray solution* (v, p) *satisfies* $(p + c) \in L^3(\Omega)$ *for some constant* $c \in \mathbb{R}$.

- Let (v, p) be a Leray solution and (w, π) a P.R. solution to

$$\begin{cases} \Delta v - \nabla p - \mathcal{R} v \cdot \nabla v + \mathcal{T}\big(\mathrm{e}_1 \times x \cdot \nabla v - \mathrm{e}_1 \times v\big) = f & \text{in } \Omega, \\ \mathrm{div}(v) = 0 & \text{in } \Omega, \\ v = v_* & \text{on } \partial\Omega. \end{cases}$$

Then

$$\begin{aligned} &-\int_{\Omega_R} \nabla v : \nabla w \,\mathrm{d}x + \int_{\partial \mathrm{B}_R} (\nabla v \cdot n) \cdot w \,\mathrm{d}S - \int_{\partial \mathrm{B}_R} p\,(w \cdot n)\,\mathrm{d}S \\ &\quad - \mathcal{R} \int_{\Omega_R} (v \cdot \nabla v) \cdot w \,\mathrm{d}x + \mathcal{T} \int_{\Omega_R} \big(\mathrm{e}_1 \times x \cdot \nabla v - \mathrm{e}_1 \times v\big) \cdot w \,\mathrm{d}x \\ &\quad = -\int_{\partial\Omega} \big((\nabla v - pI) \cdot n\big) \cdot w \,\mathrm{d}S + \int_{\Omega_R} f \cdot w \,\mathrm{d}x. \end{aligned}$$

- $\left|\int_{\partial \mathrm{B}_R} p\,(w \cdot n)\,\mathrm{d}S\right| \leq \operatorname{ess\,sup}_{x \in \Omega} \big[(1 + |x|)|w(x)|\big] \left(R \int_{\partial \mathrm{B}_R} |p|^3 \,\mathrm{d}S\right)^{\frac{1}{3}} \xrightarrow{R \to \infty} 0.$

- Put $u = v - w$, then, using the Energy Inequality for v and E.E. for w:

$$\begin{aligned}\int_\Omega |\nabla u|^2 \,\mathrm{d}x &= \int_\Omega |\nabla v|^2 \,\mathrm{d}x + \int_\Omega |\nabla w|^2 \,\mathrm{d}x - 2\int_\Omega \nabla v : \nabla w \,\mathrm{d}x \\ &\le \mathcal{R}\left(\int_\Omega (u\cdot\nabla u)\cdot w \,\mathrm{d}x - \int_\Omega (w\cdot\nabla u)\cdot u \,\mathrm{d}x \right) \\ &\le 2\,\mathcal{R} \operatorname*{ess\,sup}_{x\in\Omega} \big[(1+|x|)|w(x)|\big] \left(\int_\Omega \frac{|u|}{1+|x|} |\nabla u| \,\mathrm{d}x \right) \\ &\le 8\,\mathcal{R}\, [\![w]\!]_1 \int_\Omega |\nabla u|^2 \,\mathrm{d}x.\end{aligned}$$

- We conclude that $u = 0$ when $8\,\mathcal{R}\,[\![w]\!]_1 < 1$.

Theorem (Uniqueness)

Let $f \in L^2(\Omega) \cap L^{\frac{6}{5}}(\Omega)$, and $v_* \in W^{\frac{3}{2},2}(\partial\Omega)$. A Leray solution (v,p) that satisfies the E.I. and and a P.R. solution (w,π) coincide when $8\,\mathcal{R}\,[\![w]\!]_1 < 1$.

$$\begin{cases} \mu\Delta v - \nabla p - v\cdot\nabla v + \xi\cdot\nabla v + \omega\wedge x\cdot\nabla v - \omega\wedge v = f & \text{in } \Omega, \\ \operatorname{div}(v) = 0 & \text{in } \Omega, \\ v = v_* & \text{on } \partial\Omega, \\ \lim_{|x|\to\infty} v(x) = 0. & \end{cases}$$

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