Fingering patterns in rotating Hele-Shaw cells

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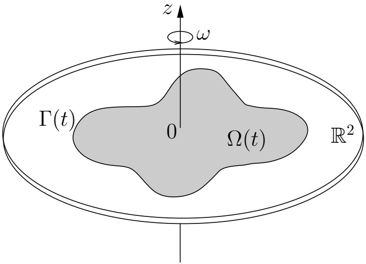
Introduction

Study the motion of an incompressible fluid in a rotating Hele-Shaw cell



An instationary moving-boundary problem.

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Summary of main results

- local well-posedness for general initial data
- a unique (for fixed volume) rotationally invariant equilibrium, which is unstable
- bifurcation theory: global branches of steady fingering patterns (bifurcation parameter: surface tension coefficient or equivalently angular velocity)
- when suitable bounded, the equilibria converge towards a circle along a global bifurcation branch

Joint work with **M. Ehrnström** and **B. Matioc**, to appear in J. Math. Fluid Mech., 2010.

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Governing equations

In the fluid domain Ω :

$$\nabla p = -K\vec{v} + \omega^2\vec{x}$$

(Darcy's law)

$$div\,\vec{v} = 0$$

(incompressibility)

$\vec{v}$: velocity, p : pressure, ω : angular velocity,

$\vec{x} \in \mathbb{R}^2$: position vector,

$K = \mu/b^2$, with viscosity μ and size of the gap b.

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Governing equations

On the moving fluid boundary Γ :

$$p = \gamma \kappa_\Gamma$$

(Laplace-Young, dynamic)

$$\langle \partial_t \Gamma - \vec{v}, \vec{\nu} \rangle = 0$$

(kinematic boundary condition)

γ : surface tension coefficient,

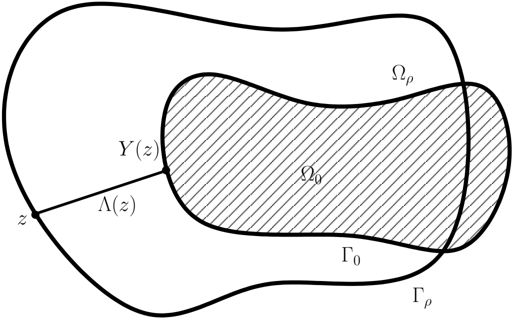
κ_Γ : curvature, $\vec{\nu}$: outward unit normal.

$$\Omega(0) = \Omega_0$$

(initial condition)

Consider (small) perturbations ρ of the initial domain

$$\Gamma_\rho = \Gamma_0 + \rho\vec{\nu}.$$



Λ : signed distance to Γ_0,

Y : projection onto Γ_0.

Then Γ_ρ is the 0−level set of $N_\rho = \Lambda - \rho \circ Y$. Thus $\nu_\rho = \nabla N_\rho / |\nabla N_\rho|$ and

$$\begin{aligned}
\Delta p &= 2\omega^2 && \text{in} \quad \Omega_\rho \\
p &= \gamma \kappa_\rho && \text{on} \quad \Gamma_\rho \\
\partial_t N_\rho &= \tfrac{1}{K} \langle \nabla p - \omega^2 x \big| \nabla N_\rho \rangle && \text{on} \quad \Gamma_\rho \\
\rho(0) &= \rho_0 && t = 0.
\end{aligned}$$

Theorem (local existence, analyticity)

The Hele-Shaw problem is uniquely solvable, locally in time, for small initial data (of class $C^{4+\alpha}(\Gamma_0)$). The solution map is analytic with respect to the time and the initial data.

Hanzawa transformation:

$$\Theta_\rho = \mathrm{id} + \varphi(\Lambda)\rho(Y)\nu(Y), \quad \Theta_\rho \in \mathit{Diff}^{4+\alpha}(\Omega_0, \Omega_\rho).$$

Pulled-back operators:

$$\mathcal{A}(\rho) = \Theta_\rho^* \circ \Delta \circ \Theta_*^\rho$$

$$\mathcal{B}(\rho, v)(y) = -\frac{1}{K} \left\langle \nabla(\Theta_*^\rho v) - \omega^2 x \middle| \nabla N_\rho \right\rangle (\Theta_\rho(y)).$$

Then the Hele-Shaw problem transforms into:

$$\begin{aligned} \mathcal{A}(\rho)v &= 2\omega^2 && \text{in} \quad \Omega \\ v &= \gamma\kappa_\rho && \text{on} \quad \Gamma \\ \partial_t \rho &= \mathcal{B}(\rho, v) && \text{on} \quad \Gamma \\ \rho(0) &= \rho_0 && t = 0. \end{aligned}$$

Solution operator

Let $v = \mathcal{T}(\rho)$ be the solution operator to the elliptic bvp

$$\begin{aligned} \mathcal{A}(\rho)v &= 2\omega^2 && \text{in} \quad \Omega \\ v &= \gamma\kappa_\rho && \text{on} \quad \Gamma. \end{aligned}$$

Then the transformed problem reduces to

$$\begin{aligned} \partial_t \rho &= \mathcal{B}(\rho, \mathcal{T}(\rho)) && \text{on} \quad \Gamma \\ \rho(0) &= \rho_0 && t = 0. \end{aligned}$$

$$\mathcal{B}(\cdot, \mathcal{T}(\cdot)) : \mathcal{V} \subset C^{4+\alpha}(\Gamma) \longrightarrow C^{1+\alpha}(\Gamma).$$

A thorough investigation of the linearization of $\mathcal{B}(\cdot, \mathcal{T}(\cdot))$, together with abstract theory for analytic semigroups yields **local well-posedness**.

The nonlinear operator

$$\Phi(\rho) := \mathcal{B}(\rho, \mathcal{T}(\rho)).$$

By the explicit form of Θ_ρ :

$$\rho \mapsto \mathcal{A}(\rho), \quad \mathcal{B}(\rho, v),\ K(\rho) \in C^\omega,$$

thus $\rho \mapsto \mathcal{T}(\rho) \in C^\omega$ and so $\Phi \in C^\omega$.

Furthermore

$$D\Phi(0)[\cdot] := \frac{\gamma}{K}\partial_\nu(D\mathcal{T}(0)[\cdot]) + F,$$

where F is a linear and bounded operator.

$D\mathcal{T}(0)[\rho]$ is the solution of the (linear) Dirichlet problem:

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$$\begin{aligned} \Delta\omega &= -D\mathcal{A}(0)[\rho]\mathcal{T}(0) && \text{in} \quad \Omega \\ \omega &= -\gamma(\kappa^2\rho + \rho'') && \text{on} \quad \Gamma \end{aligned}$$

where κ denotes the curvature of Γ.

It can be shown that

$$D\Phi(0)[\rho] = \frac{\gamma}{K}\partial_\nu \circ (\Delta, \mathrm{tr})^{-1}(0, \rho'') + \widetilde{F}\rho,$$

and that the principal part

$$D\mathcal{N} = \partial_\nu \circ ((\Delta, \mathrm{tr})^{-1}(0, \cdot))$$

is a Dirichlet-Neumann operator on ρ'' :

$$[\rho \mapsto D\mathcal{N}\rho''] \in \mathcal{H}(h^{3+\alpha}(\Gamma), h^\alpha(\Gamma)),$$

cf. **E.- Seiler, TAMS 2008**.

The full problem

$$\begin{aligned} \Delta p &= 2\omega^2 && \text{in} \quad \Omega_\rho \\ p &= \gamma\kappa_\rho && \text{on} \quad \Gamma_\rho \\ \partial_t N_\rho &= \frac{1}{K}\langle \nabla p - \omega^2 x \big| \nabla N_\rho \rangle && \text{on} \quad \Gamma_\rho \\ \rho(0) &= \rho_0 && t = 0. \end{aligned}$$

Steady states are thus solutions of

$$\begin{aligned} \Delta p &= 2\omega^2 && \text{in} \quad \Omega_\rho \\ p &= \gamma\kappa_\rho && \text{on} \quad \Gamma_\rho \\ 0 &= \langle \nabla p - \omega^2 x \big| \nabla N_\rho \rangle && \text{on} \quad \Gamma_\rho. \end{aligned}$$

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Observe: If (p, ρ) is a solution to the above system then

$$p(x) = \frac{\omega^2}{2}|x|^2 + c, \quad x \in \overline{\Omega}_\rho.$$

Particularly:

$$\Omega_\rho \in C^{2+\alpha} \Rightarrow \Omega_\rho \in C^\infty,$$

invoking the bc $p = \gamma\kappa_\rho$.

Theorem

The trivial solution $\rho \equiv 0$ is unstable.

Key result: representation of $D\Phi(0)$:

$$D\Phi(0)\left[\sum_{k\in\mathbb{Z}} \hat{\rho}(k)\exp(iks)\right] = \sum_{k\in\mathbb{Z}} \lambda_k \hat{\rho}(k)\exp(iks)$$

with

$$\lambda_k = \frac{|k|}{K}[(\gamma + \omega^2) - \gamma k^2].$$

Observe: $\lambda_1 = \frac{\omega^2}{K} > 0$, for all $\gamma \geq 0$.

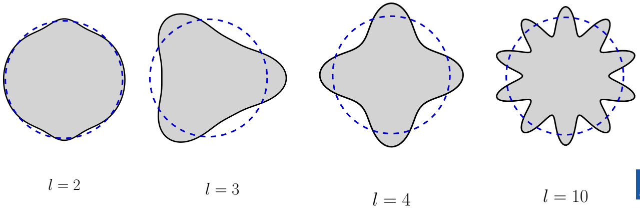
View stationary solutions as pairs (γ, ρ). For any surface tension $\gamma > 0$, $\rho \equiv 0$ is a solution. There are other equilibria bifurcating from the curve $\gamma \mapsto (\gamma, 0)$.

Bifurcation relation:

$$\omega^2 = (l^2 - 1)\gamma, \quad l \geq 2.$$

Local form of $\varepsilon-$ analytic solutions

$$\rho_l(\varepsilon) = \varepsilon \cos(l\cdot) + \delta(\varepsilon^2).$$



The setting for bifurcation

$\mathcal{V} \subset C^{4+\alpha}(\mathbb{S})$ admissible functions for Hanzawa transformation

$$\mathcal{V}_{0,e} := \left\{ \rho \in \mathcal{V};\ \rho \text{ even with } \int_{\mathbb{S}} \rho\, ds = 0 \right\}$$

Then, given $l \geq 2$, there exist a global continuation of the local bifurcation branch

$$\Sigma_l : [0, \infty) \ni \varepsilon \longmapsto (\gamma, \rho) \in \mathcal{V}_{0,e} \times (0, \infty).$$

To derive a global bifurcation result, let

$$\Sigma := \{(\gamma, \rho) \in (0, \infty) \times \mathcal{V}_{0,e}\, ;\ \Phi(\gamma, \rho) =\}$$

denote the set of all steady states. Then

- $\Sigma \subset (0, \infty) \times C^\infty(\mathbb{S}^1)$, as noticed earlier.
- Assume that $K \subset \Sigma$ is bounded, closed and bounded away from the boundary of $(0, \infty) \times \mathcal{V}_{0,e}$. Then K is compact.
- Given any $(\gamma_0, \rho_0) \in \Sigma$, the derivative $\partial_\rho \Phi(\gamma_0, \rho_0)$ is Fredholm of index 0.

These facts allow to apply results from the theory of global bifurcation for analytic mappings to conlude that:

unless Σ_l is a closed loop, either

(i) (γ, ρ) blows up in $\mathbb{R} \times C^{4+\alpha}(\mathbb{S})$

(ii) (γ, ρ) approaches the boundary of $(0, \infty) \times \mathcal{V}_{0,e}$.

If $\|\rho\|_\infty < 1/4$ and ρ'' is bounded then

$$\rho \to 0 \quad \text{in} \quad C^\infty \quad \text{if} \quad \gamma \to \infty \quad \text{as} \quad \varepsilon \to \infty.$$

Physically expected and fits the bifurcation relation.

Conclusions

- The problem is locally well-posed for initial data in a neighbourhood of a simply connected smooth domain.
- There is a volume-unique rotationally invariant steady state.
- At a countable number of small enough surface tension values there are fingering stationary solutions.
- The continuous bifurcation branches can be globally extended.
- If the surface tension tends to infinity along such a curve and free boundary is close to a circle, it approaches a circle.

Thank you for your attention!