Fingering patterns in rotating Hele-Shaw cells

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Introduction

Study the motion of an incompressible fluid in a rotating Hele-Shaw cell

An instationary moving-boundary problem.
local well-posedness for general initial data

a unique (for fixed volume) rotationally invariant equilibrium, which is unstable

bifurcation theory: global branches of steady fingering patterns (bifurcation parameter: surface tension coefficient or equivalently angular velocity)

when suitable bounded, the equilibria converge towards a circle along a global bifurcation branch

Governing equations

In the fluid domain $\Omega$:

\[ \nabla p = -K \vec{v} + \omega^2 \vec{x} \]

(Darcy’s law)

\[ \text{div} \vec{v} = 0 \]

(incompressibility)

$\vec{v}$: velocity, $p$: pressure, $\omega$: angular velocity,

$\vec{x} \in \mathbb{R}^2$: position vector,

$K = \mu/b^2$, with viscosity $\mu$ and size of the gap $b$. 
Governing equations

On the moving fluid boundary $\Gamma$:

\[ p = \gamma \kappa_{\Gamma} \]  
(Laplace-Young, dynamic)

\[ \langle \partial_t \Gamma - \vec{v}, \vec{\nu} \rangle = 0 \]  
(kinematic boundary condition)

$\gamma$: surface tension coefficient,
$\kappa_{\Gamma}$: curvature, $\vec{\nu}$: outward unit normal.

\[ \Omega(0) = \Omega_0 \]  
(initial condition)
Consider (small) perturbations $\rho$ of the initial domain

$$\Gamma_\rho = \Gamma_0 + \rho \vec{v}.$$

$\Lambda :$ signed distance to $\Gamma_0$,
$Y :$ projection onto $\Gamma_0$. 

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Then $\Gamma_\rho$ is the 0–level set of $N_\rho = \Lambda - \rho \circ Y$. Thus

$$\nu_\rho = \nabla N_\rho / |\nabla N_\rho|$$

and

$$\begin{align*}
\Delta p &= 2\omega^2 & \text{in } & \Omega_\rho \\
p &= \gamma \kappa_\rho & \text{on } & \Gamma_\rho \\
\partial_t N_\rho &= \frac{1}{K} \langle \nabla p - \omega^2 x | \nabla N_\rho \rangle & \text{on } & \Gamma_\rho \\
\rho(0) &= \rho_0 & \text{at } & t = 0.
\end{align*}$$

Theorem (local existence, analyticity)

*The Hele-Shaw problem is uniquely solvable, locally in time, for small initial data (of class $C^{4+\alpha}(\Gamma_0)$). The solution map is analytic with respect to the time and the initial data.*
Hanzawa transformation:

\[ \Theta_\rho = \text{id} + \varphi(\Lambda)\rho(Y)\nu(Y), \quad \Theta_\rho \in \text{Diff}^{4+\alpha}(\Omega_0, \Omega_\rho). \]

Pulled-back operators:

\[
A(\rho) = \Theta_\rho^* \circ \Delta \circ \Theta_\rho^*
\]

\[
B(\rho, \nu)(y) = -\frac{1}{K} \left< \nabla (\Theta_\rho^* \nu) - \omega^2 x \left| \nabla \nabla_\rho \right> (\Theta_\rho(y)) \right.
\]

Then the Hele-Shaw problem transforms into:

\[
\begin{align*}
A(\rho)\nu &= 2\omega^2 \quad \text{in} \quad \Omega \\
\nu &= \gamma \kappa_\rho \quad \text{on} \quad \Gamma \\
\partial_t \rho &= B(\rho, \nu) \quad \text{on} \quad \Gamma \\
\rho(0) &= \rho_0 \quad t = 0.
\end{align*}
\]
Let \( \nu = \mathcal{T}(\rho) \) be the solution operator to the elliptic bvp

\[
\mathcal{A}(\rho)\nu = 2\omega^2 \quad \text{in} \quad \Omega \\
\nu = \gamma\kappa_\rho \quad \text{on} \quad \Gamma.
\]

Then the transformed problem reduces to

\[
\partial_t \rho = B(\rho, \mathcal{T}(\rho)) \quad \text{on} \quad \Gamma \\
\rho(0) = \rho_0 \quad t = 0.
\]

\( B(\cdot, \mathcal{T}(\cdot)) : \mathcal{V} \subset C^{4+\alpha}(\Gamma) \longrightarrow C^{1+\alpha}(\Gamma). \)

A thorough investigation of the linearization of \( B(\cdot, \mathcal{T}(\cdot)) \), together with abstract theory for analytic semigroups yields local well-posedness.
The nonlinear operator

\[ \Phi(\rho) := B(\rho, T(\rho)) . \]

By the explicit form of \( \Theta_\rho : \)

\[ \rho \mapsto A(\rho), \quad B(\rho, v), \quad K(\rho) \in C^\omega , \]

thus \( \rho \mapsto T(\rho) \in C^\omega \) and so \( \Phi \in C^\omega \).

Furthermore

\[ D\Phi(0)[\cdot] := \frac{\gamma}{K} \partial_\nu (DT(0)[\cdot]) + F , \]

where \( F \) is a linear and bounded operator.

\( DT(0)[\rho] \) is the solution of the (linear) Dirichlet problem:
\[ \Delta \omega = -DA(0)[\rho]T(0) \quad \text{in} \quad \Omega \]
\[ \omega = -\gamma(\kappa^2 \rho + \rho'') \quad \text{on} \quad \Gamma \]

where \( \kappa \) denotes the curvature of \( \Gamma \).

It can be shown that
\[ D\Phi(0)[\rho] = \frac{\gamma}{K} \partial_\nu \circ (\Delta, \text{tr})^{-1}(0, \rho'') + \tilde{F} \rho, \]

and that the principal part
\[ DN = \partial_\nu \circ ((\Delta, \text{tr})^{-1}(0, \cdot)) \]

is a Dirichlet-Neumann operator on \( \rho'' \):
\[ [\rho \mapsto DN \rho''] \in \mathcal{H}(h^{3+\alpha}(\Gamma), h^{\alpha}(\Gamma)), \]

The full problem

\[ \Delta p = 2\omega^2 \quad \text{in} \quad \Omega_\rho \]

\[ p = \gamma \kappa_\rho \quad \text{on} \quad \Gamma_\rho \]

\[ \partial_t N_\rho = \frac{1}{K} \langle \nabla p - \omega^2 x | \nabla N_\rho \rangle \quad \text{on} \quad \Gamma_\rho \]

\[ \rho(0) = \rho_0 \quad t = 0. \]

Steady states are thus solutions of

\[ \Delta p = 2\omega^2 \quad \text{in} \quad \Omega_\rho \]

\[ p = \gamma \kappa_\rho \quad \text{on} \quad \Gamma_\rho \]

\[ 0 = \langle \nabla p - \omega^2 x | \nabla N_\rho \rangle \quad \text{on} \quad \Gamma_\rho. \]
Observe: If \((p, \rho)\) is a solution to the above system then

\[ p(x) = \frac{\omega^2}{2} |x|^2 + c, \quad x \in \overline{\Omega}_\rho. \]

Particularly:

\[ \Omega_\rho \in C^{2+\alpha} \Rightarrow \Omega_\rho \in C^\infty, \]

invoking the bc \(p = \gamma \kappa_\rho\).

Theorem

*The trivial solution \(\rho \equiv 0\) is unstable.*

Key result: representation of \(D\Phi(0)\):

\[
D\Phi(0) \left[ \sum_{k \in \mathbb{Z}} \hat{\rho}(k) \exp(iks) \right] = \sum_{k \in \mathbb{Z}} \lambda_k \hat{\rho}(k) \exp(iks)
\]

with

\[
\lambda_k = \frac{|k|}{K} \left[ (\gamma + \omega^2) - \gamma k^2 \right].
\]

Observe: \(\lambda_1 = \frac{\omega^2}{K} > 0\), for all \(\gamma \geq 0\).
Bifurcating solutions

View stationary solutions as pairs \((\gamma, \rho)\). For any surface tension \(\gamma > 0\), \(\rho \equiv 0\) is a solution. There are other equilibria bifurcating from the curve \(\gamma \mapsto (\gamma, 0)\).

Bifurcation relation:

\[
\omega^2 = (l^2 - 1)\gamma, \quad l \geq 2.
\]

Local form of \(\varepsilon\)–analytic solutions

\[
\rho_l(\varepsilon) = \varepsilon \cos(l \cdot) + \delta(\varepsilon^2).
\]
\[ \mathcal{V} \subset C^{4+\alpha}(S) \] admissible functions for Hanzawa transformation

\[ \mathcal{V}_{0,e} := \left\{ \rho \in \mathcal{V}; \rho \text{ even with } \int_S \rho \, ds = 0 \right\} \]

Then, given \( l \geq 2 \), there exist a global continuation of the local bifurcation branch

\[ \Sigma_l : [0, \infty) \ni \varepsilon \mapsto (\gamma, \rho) \in \mathcal{V}_{0,e} \times (0, \infty). \]
To derive a global bifurcation result, let

\[ \Sigma := \{ (\gamma, \rho) \in (0, \infty) \times V_{0,e} ; \Phi(\gamma, \rho) = \} \]

denote the set of all steady states. Then

- \( \Sigma \subset (0, \infty) \times C^\infty(S^1) \), as noticed earlier.
- Assume that \( K \subset \Sigma \) is bounded, closed and bounded away from the boundary of \( (0, \infty) \times V_{0,e} \). Then \( K \) is compact.
- Given any \( (\gamma_0, \rho_0) \in \Sigma \), the derivative \( \partial_\rho \Phi(\gamma_0, \rho_0) \) is Fredholm of index 0.

These facts allow to apply results from the theory of global bifurcation for analytic mappings to conclude that:
unless $\Sigma_i$ is a closed loop, either

(i) $(\gamma, \rho)$ blows up in $\mathbb{R} \times C^{4+\alpha}(S)$

(ii) $(\gamma, \rho)$ approaches the boundary of $(0, \infty) \times V_{0,e}$.

If $\|\rho\|_{\infty} < 1/4$ and $\rho''$ is bounded then

$$\rho \to 0 \quad \text{in} \quad C^\infty \quad \text{if} \quad \gamma \to \infty \quad \text{as} \quad \varepsilon \to \infty.$$ 

Physically expected and fits the bifurcation relation.
Conclusions

- The problem is locally well-posed for initial data in a neighbourhood of a simply connected smooth domain.

- There is a volume-unique rotationally invariant steady state.

- At a countable number of small enough surface tension values there are fingering stationary solutions.

- The continuous bifurcation branches can be globally extended.

- If the surface tension tends to infinity along such a curve and free boundary is close to a circle, it approaches a circle.
Thank you for your attention!