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Local and Global Well-Posedness for a Hyperbolic Fluid Model

Joint work with Reinhard Racke (Konstanz)

Jürgen Saal

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Classical Navier-Stokes equations

Balance of momentum with flux q :

$$(NSG) \begin{cases} \partial_t u + (u \cdot \nabla) u + q + \nabla p = 0, \\ \operatorname{div} u = 0. \\ u|_{t=0} = u_0. \end{cases}$$

Classical flux-stress relation ("Fourier type law"):

$$q(t) = -\operatorname{div} S(t) = -\operatorname{div} (\nabla u + \nabla u^T) = -\Delta u$$

Consequences:

- (NSG) parabolic,
- infinite propagation speed.



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Exp. observed: $q(t + \tau) = -\operatorname{div} S(t)$ "delay with $\tau > 0$ small"

First Taylor-Approximation:

$$q(t) + \tau q_t(t) = -\operatorname{div} S(t) \quad \text{"Cattaneo's law"}$$



Hyperbolic Navier-Stokes-system

leads to “quasilinear” (HNSE):

$$\begin{aligned}\tau u_{tt} + \cancel{u_t} - \Delta u + (u \cdot \nabla) u + \nabla p + \tau \nabla p_t + \tau (\partial_t u \cdot \nabla) u &= -\tau (u \cdot \nabla) \partial_t u \\ \operatorname{div} u &= 0, \\ u|_{t=0} = u_0, \quad u_t|_{t=0} = u_1.\end{aligned}$$

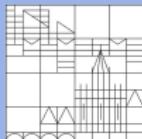


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Known: omit $\nabla p + \tau \nabla p_t$ and $\operatorname{div} u = 0 \Rightarrow$ “blow up” ($n \geq 2$)



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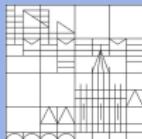
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Conjecture: blow up for (HNSE) (C)

Conclusions:

- classical fluid model is sensitive versus small perturbations
- first nonlinear example for which Fourier and Cattaneo produce opposite results
(linear: Timoshenko system, Fernández Sare, Racke '09).



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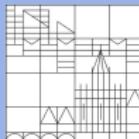
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- first nonlinear example for which Fourier and Cattaneo produce opposite results
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Problem concerning (C): finite propagation speed !?!

(open for (HNSE)).



Known results

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only for

$$(E) \begin{cases} \tau u_{tt} + u_t - \Delta u + (u \cdot \nabla)u + \nabla p = 0 \\ \operatorname{div} u = 0, \\ u|_{t=0} = u_0, \quad u_t|_{t=0} = u_1. \end{cases}$$

Paicu, Raugel '07: local wellposedness for $n = 3$, global for $n = 2$,
convergence for $\tau \rightarrow 0$.

Observe: (E) semilinear, i.e., not comparable to (HNSE)



Main results

Theorem (Racke, S. '09, local wellposedness)

Let $n = 2, 3$ and $m > n/2$. For each

$$(u_0, u_1) \in (H^{m+2}(\mathbb{R}^n) \cap L_\sigma^2(\mathbb{R}^n)) \times (H^{m+1}(\mathbb{R}^n) \cap L_\sigma^2(\mathbb{R}^n))$$

there exists $T_* > 0$ and a unique solution (u, p) of (HNSE) satisfying

$$u \in C^2([0, T_*], H^m(\mathbb{R}^n)) \cap C^1([0, T_*], H^{m+1}(\mathbb{R}^n))$$

$$\cap C([0, T_*], H^{m+2}(\mathbb{R}^n) \cap L_\sigma^2(\mathbb{R}^n)),$$

$$(1 + \tau \partial_t) \nabla p \in C([0, T_*], H^m(\mathbb{R}^n)).$$



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$$(1 + \tau \partial_t) \nabla p \in C([0, T_*], H^m(\mathbb{R}^n)).$$

Theorem (Racke, S. '09, global wellposedness)

Let $m_1 \geq 3$, $m \geq m_1 + 9$, $1 < p < 4/3$. $\exists \varepsilon > 0$ s.t., if

$$\|u_0\|_{m+2,2} + \|u_0\|_{m+1,2} + \|u_0\|_1 + \|u_1\|_1 + \|u_0\|_{m_1+6,p} + \|u_0\|_{m_1+5,p} < \varepsilon,$$

then the solution $(u, \nabla p)$ is global, i.e. $T_* = \infty$.



Transformation into a first order system

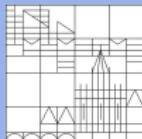
Set $V := (v, \partial_1 v, \dots, \partial_n v, \partial_t v)^T \in \mathbb{R}^{n(n+2)}$. \Rightarrow

(HNSE) turns into **quasilinear first order system (QS)**:

$$\begin{cases} V_t + \mathcal{A}(V)V + \mathcal{B}(V)V &= 0 & \text{in } (0, T) \times \mathbb{R}^n, \\ V|_{t=0} &= V_0 = (u_0, \nabla u_0, u_1) & \text{in } \mathbb{R}^n, \end{cases}$$

$$\mathcal{A}(V)V = \mathbb{P} \sum_{j=1}^n A_j(V) \partial_j V \quad \text{includes} \quad -P\Delta, \quad P(u \cdot \nabla)u \partial_t u$$

$$\mathcal{B}(V)V \quad \text{includes} \quad P((u_t + u) \cdot \nabla)u, \quad u_t$$



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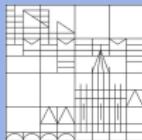
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$$\begin{aligned} \mathbb{P} &:= \text{diag}(I_n, \dots, I_n, P) \\ &: L^2(\mathbb{R}^n)^{n(n+2)} \rightarrow \mathcal{H} := L^2(\mathbb{R}^n)^{n(n+1)} \times L_\sigma^2(\mathbb{R}^n) \end{aligned}$$

$$L_\sigma^2(\mathbb{R}^n) = \{u \in L^2(\mathbb{R}^n)^n : \text{div } u = 0\}$$

$$P : L^2(\mathbb{R}^n) \rightarrow L_\sigma^2(\mathbb{R}^n) \quad \text{standard Leray-Helmholtz projection.}$$



Linearization

$$(LS) \begin{cases} V_t + \mathcal{A}(t)V + \mathcal{B}(t)V &= 0, \\ V(0) &= V_0. \end{cases}, t \in [0, T],$$

with $\mathcal{A}(t) = \mathcal{A}(a(t))$, $\mathcal{B}(t) = \mathcal{B}(b(t))$

and $a, b : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ fixed.

Lemma

Let $a, b \in LIP([0, T], BC(\mathbb{R}^n))^n$, $\operatorname{div} a = 0$. Then

$$\mathcal{A}(\cdot) + \mathcal{B}(\cdot) : \mathcal{D}(\mathcal{A}) \rightarrow \mathcal{H} = (L^2)^{n(n+1)} \times L_\sigma^2$$

$$\mathcal{D}(\mathcal{A}) = \{V \in \mathcal{H} : V^{n+2} \in H^1(\mathbb{R}^n), P\Sigma_{j=1}^n \partial_j V^{j+1} \in L^2(\mathbb{R}^n)\}$$

generates an evolution family $(\mathcal{U}(t, s))_{0 \leq s \leq t \leq T} \subset \mathcal{L}(\mathcal{H})$.



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generates an evolution family $(\mathcal{U}(t, s))_{0 \leq s \leq t \leq T} \subset \mathcal{L}(\mathcal{H})$.

Pf: $\forall t \in [0, T] : \mathcal{A}(t)$ skew-selfadj. $\xrightarrow{\text{Stone}}$ $\mathcal{A}(t)$ gen. C_0 -sg. of contr.

$\Rightarrow (\mathcal{A}(t))_{t \in [0, T]}$ is CD-system. $\mathcal{B}(\cdot)$ of lower order

$\Rightarrow (\mathcal{A}(t) + \mathcal{B}(t))_{t \in [0, T]}$ is CD-system. $\xrightarrow{\text{Kato}}$ assertion.



Higher Sobolev regularity

Note: Standard approach fails by lack of finite propagation speed!

By **formal** calculation for $|\alpha| = m + 1 > n/2 + 1$:

$$\begin{cases} \partial_t \partial^\alpha V + \mathcal{A}(t) \partial^\alpha V &= F(V) \quad , t \in [0, T], \\ V(0) &= V_0. \end{cases}$$

with

$$F(V) = -e_{n+2} \left[P \sum_{j=1}^n (\partial^\alpha a_j \partial_j V^{n+2} - a_j \partial^\alpha \partial_j V^{n+2} + \partial^\alpha b_j V^{j+1}) \right. \\ \left. + \partial^\alpha V^{n+2} / \sqrt{\tau} \right] + e_1 \partial^\alpha V^{n+2}$$

By Energy methods:

$$\|\mathcal{U}(t, s)V_0\|_{m+1} \leq C_1 \|V_0\|_{m+1} \exp \left(C_2 \int_s^t (\|(a(r), b(r))\|_{m+1} + 1) dr \right)$$

$$\|\partial_t \mathcal{U}(t, 0)V_0\|_m \leq C_1 \|V_0\|_{m+1} (\|(a(t), b(t))\|_m + 1)$$

$$\cdot \exp \left(C_2 \int_0^t (\|(a(r), b(r))\|_{m+1} + 1) dr \right)$$

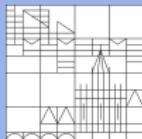


Higher Sobolev regularity

Theorem

Let $a, b \in L^1((0, T), H^{m+1}) \cap C([0, T], H^m)$, $V_0 \in H^{m+1} \cap \mathcal{H}$.

$\Rightarrow V = \mathcal{U}(\cdot)V_0 \in C^1([0, T], H^m) \cap C([0, T], H^{m+1})$.



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Justification: by method of viscosity solutions, i.e., for $\varepsilon > 0$ consider:

$$V_t - \varepsilon \Delta V + \mathcal{A}(t)V + \mathcal{B}(t)V = 0.$$

\Rightarrow smoothing

Then apply weak* convergence, variation of constant formula

Observation: regularity of a, b < regularity of V !



Quasilinear local existence (Thm 1)

Apply Majda's fixed point method: Consider

$$(AS) \begin{cases} \partial_t V_{k+1} + \mathcal{A}(V_k)V_{k+1} + \mathcal{B}(V_k)V_{k+1} = 0 & \text{in } (0, T), \\ V_{k+1}(0) = V_0. \end{cases}$$

with startig point $V_0(t, x) := V_0(x)$.

Thm(higher reg.) $\Rightarrow V_k$ well-defined ($k \in \mathbb{N}_0$).

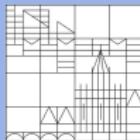
Energy estimates \Rightarrow

$\exists T > 0 : \|V_k(t)\|_{m+1}, \|\partial_t V_k(t)\|_m \leq M \quad (t \in [0, T], k \in \mathbb{N}_0)$.

$\Rightarrow V_k \xrightarrow{*} V \in W^{1,\infty}(H^m) \cap L^\infty(H^{m+1}) \hookrightarrow C(H^m) \cap L^1(H^{m+1})$.

Thm(higher reg.) $\Rightarrow V \in C^1(H^m) \cap C(H^{m+1})$.

$\Rightarrow u \in C^2(H^m) \cap C^1(H^{m+1}) \cap C(H^{m+2})$.



Global existence

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$$\begin{aligned}\tau u_{tt} - \Delta u + u_t &= -P[(u \cdot \nabla)u + \tau(u \cdot \nabla)\partial_t u + \tau(\partial_t u \cdot \nabla)u] =: N(u) \\ \operatorname{div} u &= 0 \\ u|_{t=0} &= u_0, \quad u_t|_{t=0} = u_1\end{aligned}$$

Important:

- damping term $u_t \Rightarrow$ decay for linear solution $L(t)u_1 = L(t)(0, u_1)$ as $\|\partial^\alpha \partial_t^j L(t)u_1\|_2 \leq C(1+t)^{-(\frac{|\alpha|}{2}+j)} \|u_1\|_{|\alpha|+j-1,2}$.
- nonlinearities contain always a “ ∇ ” \Rightarrow extra decay (i.p. important for $n = 2$)
- avoid L^1 and L^∞ due to presence of P
 $(\Rightarrow$ proof becomes even more technical)



Global existence

Step 1: High energy estimates, i.e., for $s > n/2 + 1$:

$$E_s(t) \leq CE_s(0) \exp \left(C \int_0^t (\|u\|_\infty^2 + \|u_t\|_{1,\infty} + \|\nabla u\|_\infty)(r) dr \right)$$

where

$$E_s(t) := \frac{1}{2} \sum_{|\alpha| \leq s} (\|\partial^\alpha u_t\|_2^2 + \|\partial^\alpha \nabla u\|_2^2 + \varepsilon_2 \|\partial^\alpha u\|_2^2)(t).$$



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Step 2: Use representation

$$u(t) = L(t)(u_1 + \frac{1}{\tau}u_0) + \partial_t L(t)u_0 + \frac{1}{\tau} \int_0^t L(t-r)N(u(r))dr$$

and prove that $\exists M_0 > 0 \quad \forall T > 0$:

$$\begin{aligned} & \sup_{0 \leq t \leq T} \left\{ (1+t)^{1-\frac{2}{q}} \|u(t)\|_{m_1,q} + (1+t)^{\frac{3}{2}-\frac{2}{q}} (\|u_t(t)\|_{m_1,q} + \|\nabla u(t)\|_{m_1,q}) \right. \\ & \left. + (1+t)^{\frac{1}{2}} \|u(t)\|_{m,2} + (1+t)(\|u_t(t)\|_{m,2} + \|\nabla u(t)\|_{m,2}) \right\} \leq M_0. \end{aligned}$$



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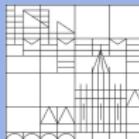
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Step 3: Uniform bound:

$$\|E_{m+1}(T)\| \leq CE_{m+1}(0) \exp(C_0(M_0^2 + M_0)) < \varepsilon,$$

if $E_{m+1}(0) < \varepsilon/C$.

⇒ local solution extends to a global one.



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Thank you!