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Local and Global Well-Posedness for a Hyperbolic Fluid Model

Joint work with Reinhard Racke (Konstanz)

Jürgen Saal

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Classical Navier-Stokes equations

Balance of momentum with flux q:

$$(NSG)\begin{cases} \partial_t u + (u\cdot\nabla)u + q + \nabla p = 0, \\ \operatorname{div} u = 0. \\ u\big|_{t=0} = u_0. \end{cases}$$

Classical flux-stress relation ("Fourier type law"):

$$q(t) = -\operatorname{div} S(t) = -\operatorname{div}(\nabla u + \nabla u^T) = -\Delta u$$

Consequences:
- (NSG) parabolic,
- infinite propagation speed.

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Exp. observed: $q(t+\tau) = -\operatorname{div} S(t)$ "delay with $\tau > 0$ small"

First Taylor-Approximation:

$$q(t) + \tau q_t(t) = -\operatorname{div} S(t) \qquad \text{"Cattaneo's law"}$$

Hyperbolic Navier-Stokes-system

leads to "quasilinear" (HNSE):

$$\tau u_{tt} + u_t - \Delta u + (u \cdot \nabla) u + \nabla p + \tau \nabla p_t + \tau(\partial_t u \cdot \nabla) u = -\tau (u \cdot \nabla) \partial_t u$$
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Conjecture: blow up for (HNSE) (C)

Conclusions:

- classical fluid model is sensitive versus small perturbations
- first nonlinear example for which Fourier and Cattaneo produce opposite results

  (linear: Timoshenko system, Fernandeź Sare, Racke '09).

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Problem concerning (C): finite propagation speed !?!

(open for (HNSE)).

Known results

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only for

$$(E)\begin{cases} \tau u_{tt} + u_t - \Delta u + (u\cdot\nabla)u + \nabla p &= 0 \\ \operatorname{div} u &= 0, \\ u\big|_{t=0} = u_0, \quad u_t\big|_{t=0} &= u_1. \end{cases}$$

Paicu, Raugel '07: local wellposedness for $n = 3$, global for $n = 2$, convergence for $\tau \to 0$.

Observe: (E) semilinear, i.e., not comparable to (HNSE)

Main results

Theorem (Racke, S. '09, local wellposedness)

Let $n = 2, 3$ *and* $m > n/2$. *For each*

$$(u_0, u_1) \in \big(H^{m+2}(\mathbb{R}^n) \cap L^2_\sigma(\mathbb{R}^n)\big) \times \big(H^{m+1}(\mathbb{R}^n) \cap L^2_\sigma(\mathbb{R}^n)\big)$$

there exists $T_* > 0$ *and a unique solution* (u, p) *of (HNSE) satisfying*

$$\begin{aligned} u \;&\in\; C^2([0,T_*), H^m(\mathbb{R}^n)) \cap C^1([0,T_*), H^{m+1}(\mathbb{R}^n)) \\ &\quad \cap C([0,T_*), H^{m+2}(\mathbb{R}^n) \cap L^2_\sigma(\mathbb{R}^n)), \\ (1+\tau\partial_t)\nabla p \;&\in\; C([0,T_*), H^m(\mathbb{R}^n)). \end{aligned}$$

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Theorem (Racke, S. '09, global wellposedness)

Let $m_1 \geq 3$, $m \geq m_1 + 9$, $1 < p < 4/3$. $\exists \varepsilon > 0$ s.t., if

$$\|u_0\|_{m+2,2} + \|u_0\|_{m+1,2} + \|u_0\|_1 + \|u_1\|_1 + \|u_0\|_{m_1+6,p} + \|u_0\|_{m_1+5,p} < \varepsilon,$$

then the solution $(u, \nabla p)$ is global, i.e. $T_* = \infty$.

Transformation into a first order system

Set $V := (v, \partial_1 v, \ldots, \partial_n v, \partial_t v)^T \in \mathbb{R}^{n(n+2)}$. $\Rightarrow$

(HNSE) turns into quasilinear first order system (QS):

$$\begin{cases} V_t + \mathcal{A}(V)V + \mathcal{B}(V)V = 0 & \text{in } (0,T) \times \mathbb{R}^n, \\ V|_{t=0} = V_0 = (u_0, \nabla u_0, u_1) & \text{in } \mathbb{R}^n, \end{cases}$$

$$\mathcal{A}(V)V = \mathbb{P} \sum_{j=1}^{n} A_j(V) \partial_j V \quad \text{includes} \quad -P\Delta,\ P(u \cdot \nabla)u\partial_t u$$

$$\mathcal{B}(V)V \quad \text{includes} \quad P((u_t + u) \cdot \nabla)u,\ u_t$$

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$$\begin{aligned} \mathbb{P} &:= \operatorname{diag}(I_n, \dots, I_n, P) \\ &: L^2(\mathbb{R}^n)^{n(n+2)} \to \mathcal{H} := L^2(\mathbb{R}^n)^{n(n+1)} \times L^2_\sigma(\mathbb{R}^n) \\ L^2_\sigma(\mathbb{R}^n) &= \{u \in L^2(\mathbb{R}^n)^n : \operatorname{div} u = 0\} \\ P &: L^2(\mathbb{R}^n) \to L^2_\sigma(\mathbb{R}^n) \quad \text{standard Leray-Helmholtz projection.} \end{aligned}$$

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Linearization

$$(LS)\begin{cases} V_t + \mathcal{A}(t)V + \mathcal{B}(t)V &= 0 \quad, t \in [0,T], \\ V(0) &= V_0. \end{cases}$$

with $\mathcal{A}(t) = \mathcal{A}(a(t))$, $\mathcal{B}(t) = \mathcal{B}(b(t))$

and $a, b : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$ fixed.

Lemma

Let $a, b \in LIP\,([0,T], BC(\mathbb{R}^n))^n$, $\operatorname{div} a = 0$. *Then*

$$\begin{aligned} \mathcal{A}(\cdot) + \mathcal{B}(\cdot) \quad &: \quad \mathscr{D}(\mathcal{A}) \to \mathcal{H} = (L^2)^{n(n+1)} \times L^2_\sigma \\ \mathscr{D}(\mathcal{A}) \quad &= \quad \left\{ V \in \mathcal{H} : \; V^{n+2} \in H^1(\mathbb{R}^n), \; P\Sigma_{j=1}^n \partial_j V^{j+1} \in L^2(\mathbb{R}^n) \right\} \end{aligned}$$

generates an evolution family $(\mathcal{U}(t,s))_{0 \le s \le t \le T} \subset \mathscr{L}(\mathcal{H})$.

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generates an evolution family $(\mathcal{U}(t,s))_{0 \le s \le t \le T} \subset \mathscr{L}(\mathcal{H})$.

Pf: $\forall\, t \in [0,T]$: $\mathcal{A}(t)$ skew-selfadj. $\overset{Stone}{\Rightarrow}$ $\mathcal{A}(t)$ gen. C_0-sg. of contr.

$\Rightarrow (\mathcal{A}(t))_{t \in [0,T]}$ is CD-system. $\mathcal{B}(\cdot)$ of lower order

$\Rightarrow (\mathcal{A}(t) + \mathcal{B}(t))_{t \in [0,T]}$ is CD-system. $\overset{Kato}{\Rightarrow}$ assertion.

Higher Sobolev regularity

Note: Standard approach fails by lack of finite propagation speed!

By formal calculation for $|\alpha| = m+1 > n/2+1$:

$$\begin{cases} \partial_t \partial^\alpha V + \mathcal{A}(t)\partial^\alpha V = F(V) & , t \in [0,T], \\ V(0) = V_0. \end{cases}$$

with

$$F(V) = -\, e_{n+2}\Big[P\sum_{j=1}^{n}\big(\partial^\alpha a_j \partial_j V^{n+2} - a_j \partial^\alpha \partial_j V^{n+2} + \partial^\alpha b_j V^{j+1}\big) + \partial^\alpha V^{n+2}/\sqrt{\tau}\Big] + e_1 \partial^\alpha V^{n+2}$$

By Energy methods:

$$\|\mathcal{U}(t,s)V_0\|_{m+1} \le C_1 \|V_0\|_{m+1} \exp\left(C_2 \int_s^t (\|(a(r),b(r))\|_{m+1} + 1)\,\mathrm{d}r\right)$$

$$\|\partial_t \mathcal{U}(t,0)V_0\|_m \le C_1 \|V_0\|_{m+1} (\|(a(t),b(t))\|_m + 1) \cdot \exp\left(C_2 \int_0^t (\|(a(r),b(r))\|_{m+1} + 1)\,\mathrm{d}r\right)$$

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Higher Sobolev regularity

Theorem

Let $a, b \in L^1((0,T), H^{m+1}) \cap C([0,T], H^m),\ V_0 \in H^{m+1} \cap \mathcal{H}.$

$\Rightarrow \quad V = \mathcal{U}(\cdot)V_0 \in C^1([0,T], H^m) \cap C([0,T], H^{m+1}).$

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Theorem

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$\Rightarrow \quad V = \mathcal{U}(\cdot)V_0 \in C^1([0,T], H^m) \cap C([0,T], H^{m+1})$.

Justification: by method of viscosity solutions, i.e., for $\varepsilon > 0$ consider:

$$V_t - \varepsilon \Delta V + \mathcal{A}(t)V + \mathcal{B}(t)V = 0.$$

$\Rightarrow$ smoothing

Then apply weak* convergence, variation of constant formula

Observation: regularity of a, b $<$ regularity of V !

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Quasilinear local existence (Thm 1)

Apply Majda's fixed point method: Consider

$$(AS)\begin{cases} \partial_t V_{k+1} + \mathcal{A}(V_k)V_{k+1} + \mathcal{B}(V_k)V_{k+1} = 0 & \text{in } (0,T), \\ V_{k+1}(0) = V_0. \end{cases}$$

with startig point $V_0(t,x) := V_0(x)$.

Thm(higher reg.) $\Rightarrow$ V_k well-defined $(k \in \mathbb{N}_0)$.

Energy estimates $\Rightarrow$

$\exists\, T > 0 : \quad \|V_k(t)\|_{m+1},\ \|\partial_t V_k(t)\|_m \leq M \quad (t \in [0,T],\ k \in \mathbb{N}_0)$.

$$\Rightarrow \quad V_k \overset{*}{\rightharpoonup} V \in W^{1,\infty}(H^m) \cap L^\infty(H^{m+1}) \hookrightarrow C(H^m) \cap L^1(H^{m+1}).$$

Thm(higher reg.) $\Rightarrow$ $V \in C^1(H^m) \cap C(H^{m+1})$.

$$\Rightarrow \quad u \in C^2(H^m) \cap C^1(H^{m+1}) \cap C(H^{m+2}).$$

Global existence

$$\begin{aligned} \tau u_{tt} - \Delta u + u_t &= -P\left[(u\cdot\nabla)u + \tau(u\cdot\nabla)\partial_t u + \tau(\partial_t u\cdot\nabla)u\right] =: N(u) \\ \operatorname{div} u &= 0 \\ u\big|_{t=0} &= u_0, \quad u_t\big|_{t=0} = u_1 \end{aligned}$$

Important:

- damping term u_t $\Rightarrow$ decay for linear solution $L(t)u_1 = L(t)(0, u_1)$ as
$$\|\partial^\alpha \partial_t^j L(t)u_1\|_2 \leq C(1+t)^{-(\frac{|\alpha|}{2}+j)}\|u_1\|_{|\alpha|+j-1,2}.$$
- nonlinearities contain always a "∇" $\Rightarrow$ extra decay
(i.p. important for $n = 2$)
- avoid L^1 and L^∞ due to presence of P
($\Rightarrow$ proof becomes even more technical)

Global existence

Step 1: High energy estimates, i.e., for $s > n/2 + 1$:

$$E_s(t) \leq CE_s(0) \exp\left(C \int_0^t (\|u\|_\infty^2 + \|u_t\|_{1,\infty} + \|\nabla u\|_\infty)(r) dr \right)$$

where

$$E_s(t) := \frac{1}{2} \sum_{|\alpha| \leq s} (\|\partial^\alpha u_t\|_2^2 + \|\partial^\alpha \nabla u\|_2^2 + \varepsilon_2 \|\partial^\alpha u\|_2^2)(t).$$

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Step 2: Use representation

$$u(t) = L(t)(u_1 + \frac{1}{\tau} u_0) + \partial_t L(t) u_0 + \frac{1}{\tau} \int_0^t L(t-r) N(u(r)) dr$$

and prove that $\exists\, M_0 > 0 \ \ \forall\, T > 0$:

$$\sup_{0 \leq t \leq T} \Big\{ (1+t)^{1-\frac{2}{q}} \|u(t)\|_{m_1,q} + (1+t)^{\frac{3}{2}-\frac{2}{q}} (\|u_t(t)\|_{m_1,q} + \|\nabla u(t)\|_{m_1,q})$$

$$+ (1+t)^{\frac{1}{2}} \|u(t)\|_{m,2} + (1+t)(\|u_t(t)\|_{m,2} + \|\nabla u(t)\|_{m,2}) \Big\} \leq M_0.$$

Global existence

Step 3: Uniform bound:

$$\|E_{m+1}(T)\| \leq CE_{m+1}(0)\exp(C_0(M_0^2 + M_0)) < \varepsilon,$$

if $E_{m+1}(0) < \varepsilon/C$.

$\Rightarrow$ local solution extends to a global one.

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Thank you!