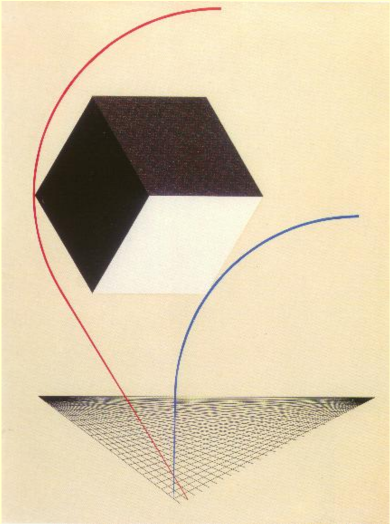


Oleg N. Kirillov

Technical University of Darmstadt
Darmstadt, Germany

Multiple eigenvalues and singularities in MHD:

Oscillatory dynamo and helical magnetorotational instability

Waseda University, March 10, 2010 – International Workshop on Mathematical Fluid Dynamics

1. Outline

The spherical MHD kinematic mean-field α^2-dynamo

Operator matrix, fundamental symmetry

Oscillatory dynamo and geomagnetic reversals

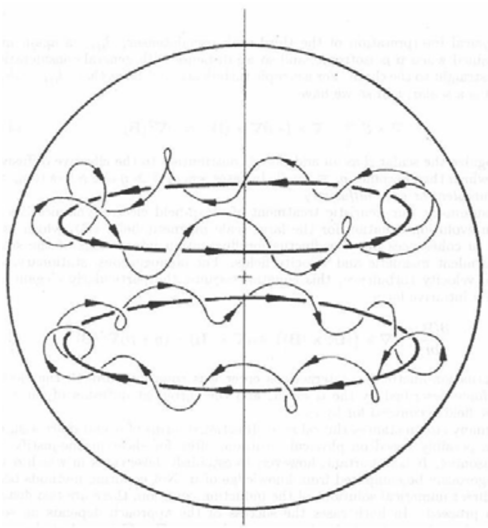
Exceptional points in the spectrum

Three dimensional Arnold tongues of oscillatory dynamo

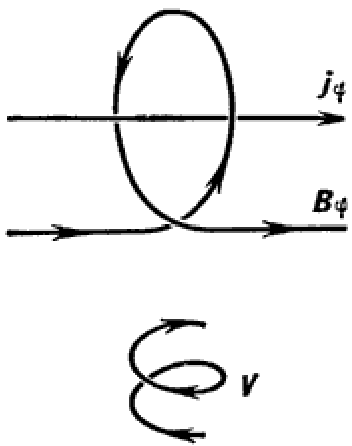
Role of Krein signature of eigenvalues

O.N.K., U. Günther, F. Stefani Physical Review E 79, 016205 2009

The spherical MHD kinematic mean-field α^2 – dynamo

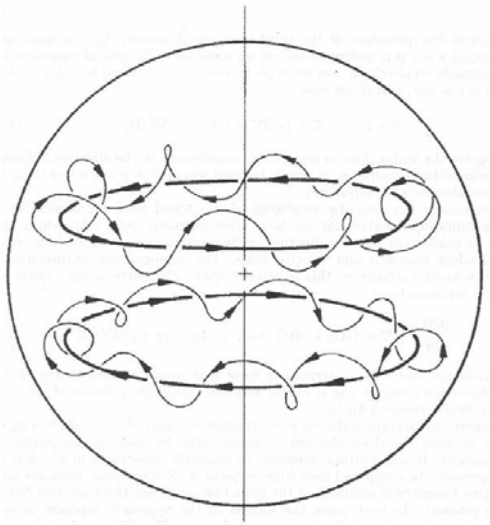


Rm, large: Magnetic field lines "frozen" in the electrically conducting fluid in a sphere.

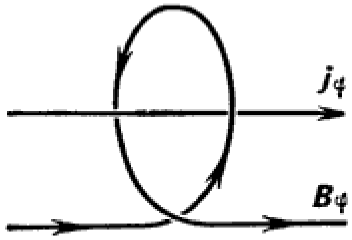


[1] E.N. Parker ApJ 122, 1955;

The spherical MHD kinematic mean-field α^2 – dynamo



Rm, large: Magnetic field lines "frozen" in the electrically conducting fluid in a sphere. Helical flow twists the field line of a seed toroidal magnetic field into poloidal planes. Diffusion detaches the loop



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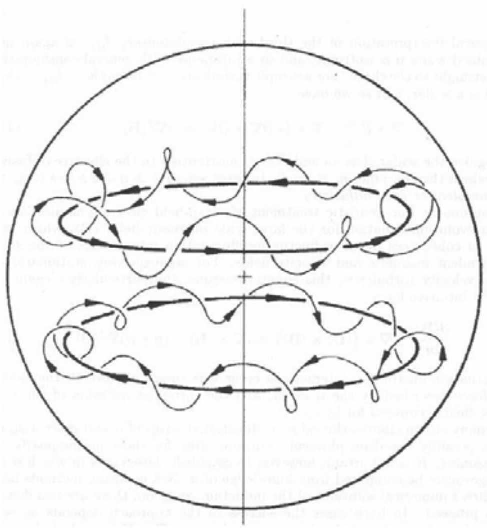

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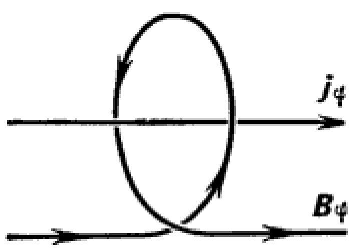

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Cyclonic turbulence: Many flux loops produce a large scale electrical current parallel to the toroidal field (inverse cascade). It induces poloidal field (α-effect) [1]

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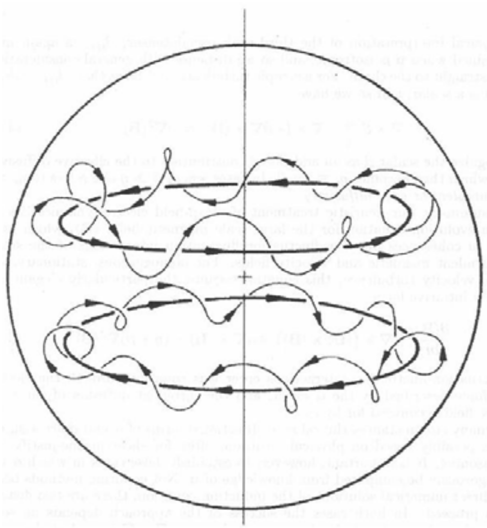

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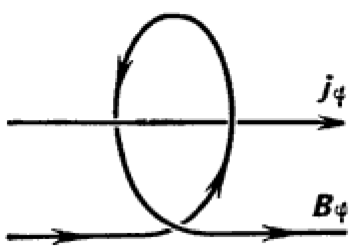

In α^2 - **dynamo** the α - effect is the source of both poloidal and toroidal mean field components [2]

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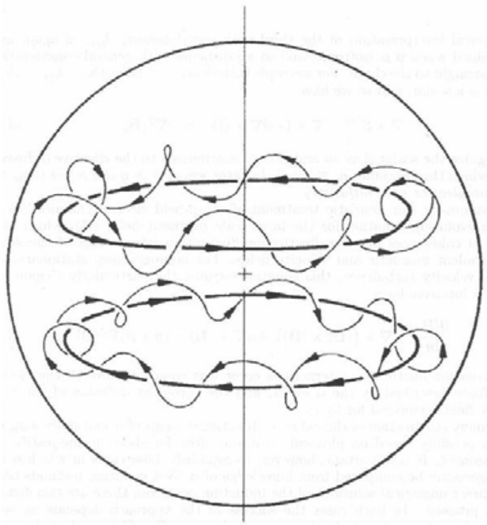

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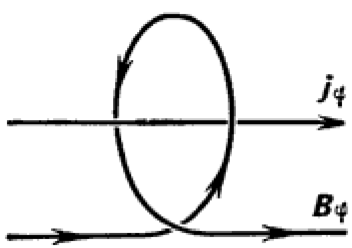

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It is widely accepted that magnetic fields of planets are generated by α^2 – **dynamo** [2]

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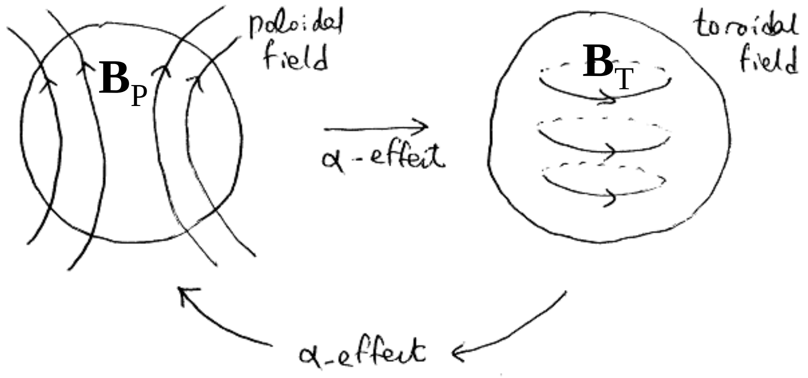
The spherical MHD kinematic mean-field α^2 – dynamo

Cyclonic turbulence lacks mirror symmetry: averaging yields large scale E.M.F. ~ $\alpha\mathbf{B}$



Self-amplification of magnetic field in α^2 - dynamo

The spherical MHD kinematic mean-field α^2 – dynamo

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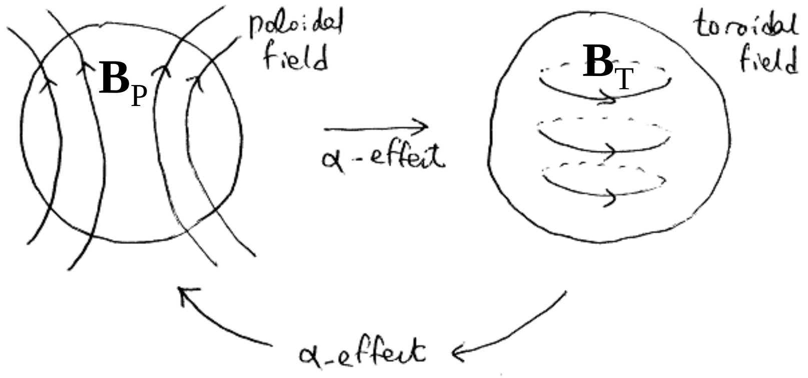
Mean field induction equation

$$\partial_t \mathbf{B} = \nabla \times (\alpha \mathbf{B}) + \nu_m \Delta \mathbf{B}, \quad \nabla \cdot \mathbf{B} = 0$$



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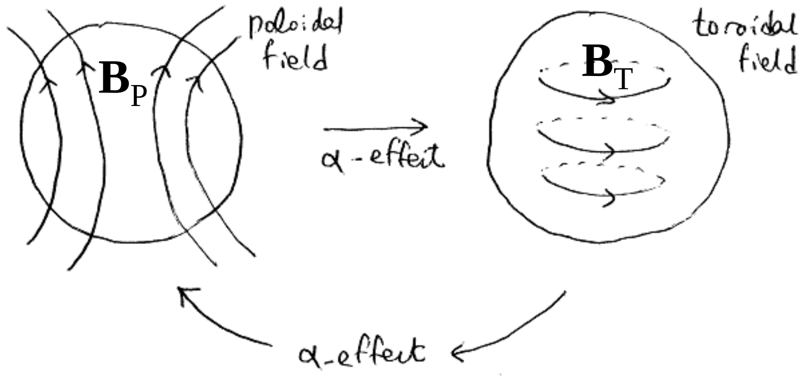
Helical turbulence function $\alpha = \alpha(r)$ couples toroidal and poloidal fields



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$$\mathbf{B} = \mathbf{B}_P + \mathbf{B}_T, \quad \mathbf{B}_P = \nabla \times \mathbf{A}_T$$

$$\mathbf{A}_T = -\mathbf{r} \times \nabla F_1, \quad \mathbf{B}_T = -\mathbf{r} \times \nabla F_2, \quad \int_{S^2} F_{1,2}\, d\omega = 0$$

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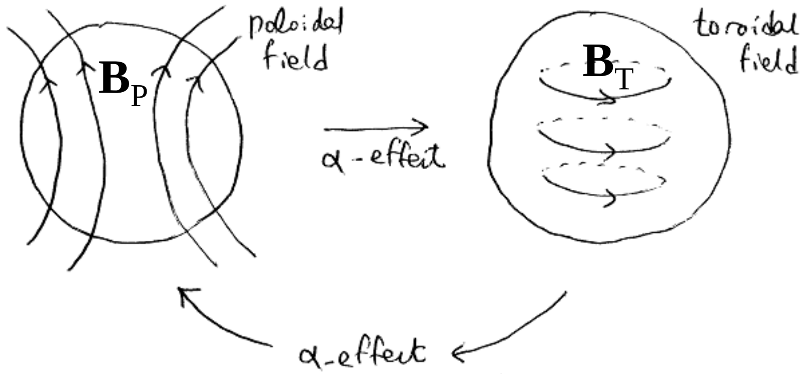

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α – coupled pair of induction PDEs [3]

$$\mathbf{r} \times \nabla \left[\nu_m \Delta F_1 + \alpha F_2 - \partial_t F_1 \right] = 0$$

$$\mathbf{r} \times \nabla \left[\nu_m \Delta F_2 - \frac{1}{r} (\partial_r \alpha)(\partial_r r F_1) - \alpha \Delta F_1 - \partial_t F_2 \right] = 0$$

[3] U. Günther, F. Stefani J. Math. Phys. 2003

The spherical MHD kinematic mean-field α^2 – dynamo

Re-scaling r and t: $\nu_m = 1$, boundary conditions at $r = 1$

A series expansion in spherical harmonics [3]

$$F_{1,2} = \sum_{l,m,n} e^{t\lambda_{l,n}} F_{1,2}^{(l,m,n)}(r) Y_l^m(\theta,\phi) \in L^2(\Omega, r^2 dr) \otimes L^2(S^2, d\omega), \quad \Omega = [0,1]$$

Denote $\Delta_l := \frac{1}{r^2}\partial_r r^2 \partial_r - \frac{l(l+1)}{r^2}$

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Radial α-coupled equations for a spherical harmonic of degree l

$$\Delta_l F_1^{(l,m,n)} + \alpha F_2^{(l,m,n)} = \lambda_{l,n} F_1^{(l,m,n)}$$

$$\Delta_l F_2^{(l,m,n)} - \frac{1}{r}(\partial_r \alpha)(\partial_r r F_1^{(l.m.n)}) - \alpha \Delta_l F_1^{(l,m,n)} = \lambda_{l,n} F_2^{(l,m,n)}$$

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Operator matrix $A_\alpha := \begin{pmatrix} -A_l & \alpha(r) \\ A_{l,\alpha} & -A_l \end{pmatrix}$ with $A_l = -\partial_r^2 + \frac{l(l+1)}{r^2}$, $A_{l,\alpha} = \alpha(r) A_l - \alpha'(r)\partial_r$

Fundamental symmetry [3] $A_\alpha^* = J A_\alpha J$, $J = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$

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Fundamental symmetry [3] $A_\alpha^* = J A_\alpha J, \quad J = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$

Boundary eigenvalue problem

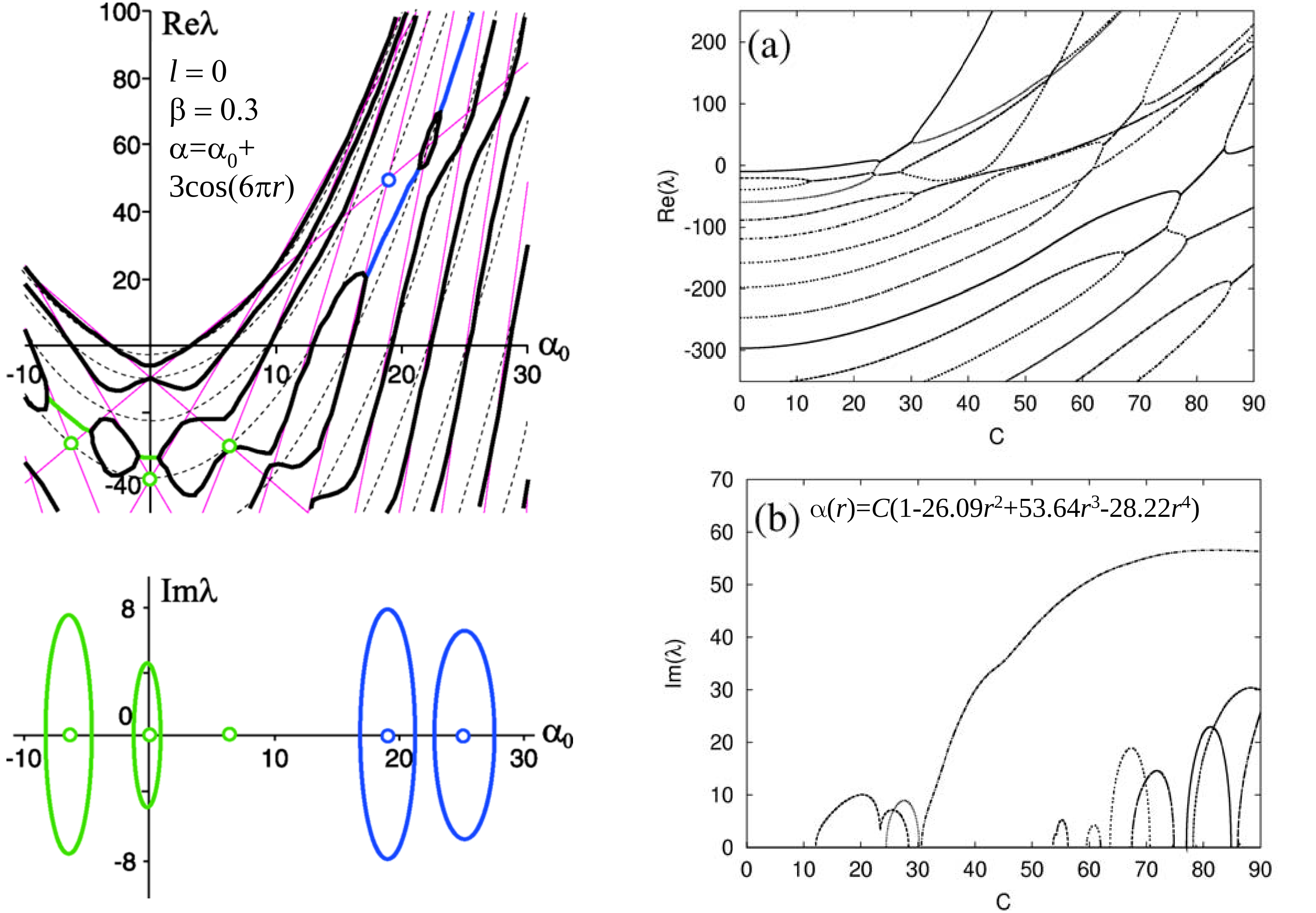
$$A_\alpha u = \lambda u, \quad D(A_\alpha) = \left\{ u \in \tilde{H} = L_2(0,1) \oplus L_2(0,1) \,|\, u(0) = 0, Bu\big|_{r=1} = 0 \right\}$$

with (indices (l,m,n) omitted) $u = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}, \quad B = \begin{pmatrix} \beta\partial_r + \beta l + 1 - \beta & 0 \\ 0 & 1 \end{pmatrix}, \quad 0 \le \beta \le 1$

In general $D(A_\alpha) \neq D(A_\alpha^*)$

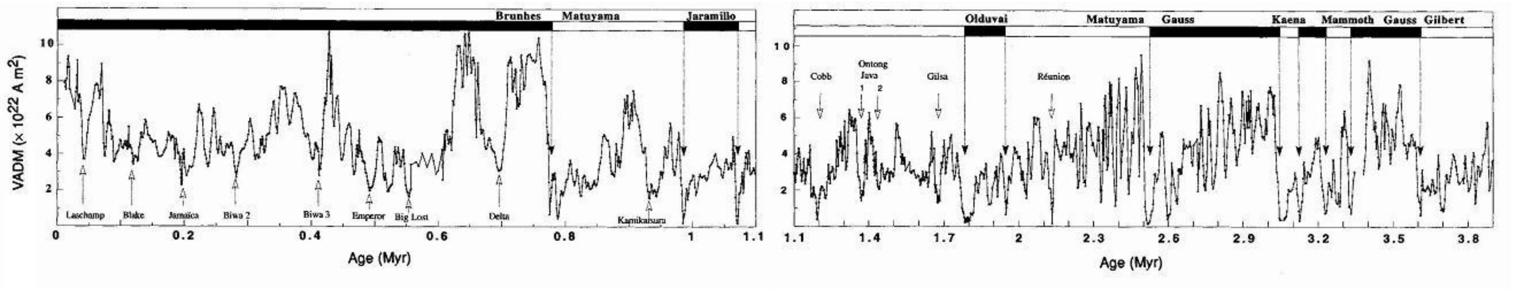
[3] U. Günther, F. Stefani J. Math. Phys. 2003

Typical spectra (eigencurves)



Oscillatory α^2-dynamo and geomagnetic reversals

Dipole field intensity from paleomagnetic data during past 4 Myr [4]



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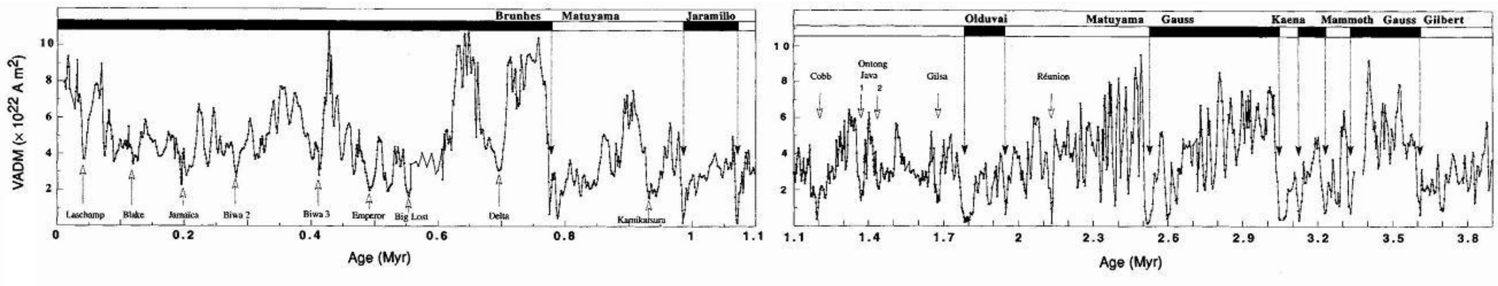
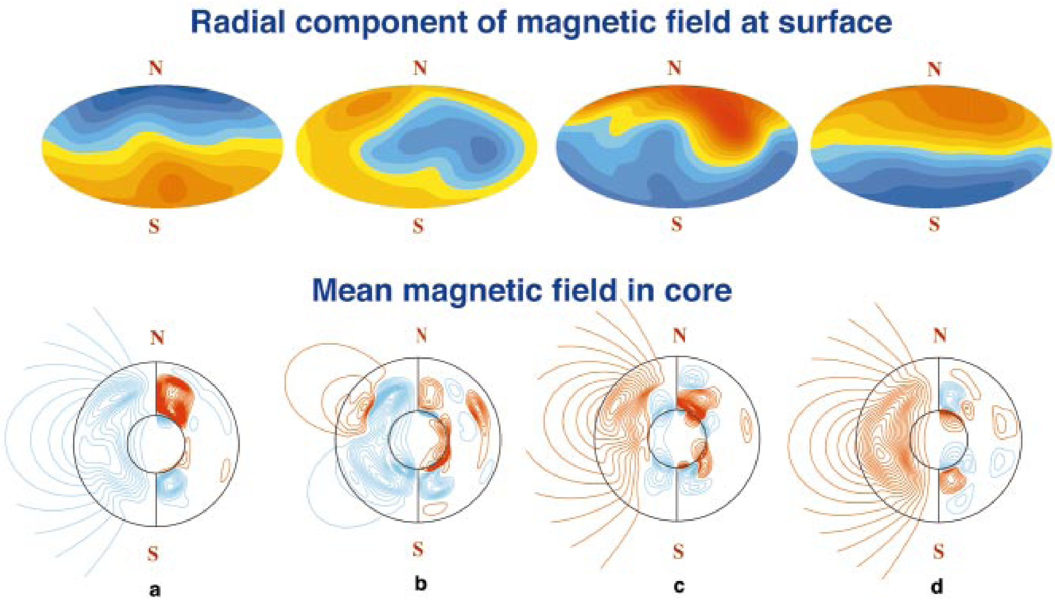


Geomagnetic field polarity reversals in numerical simulations [5]



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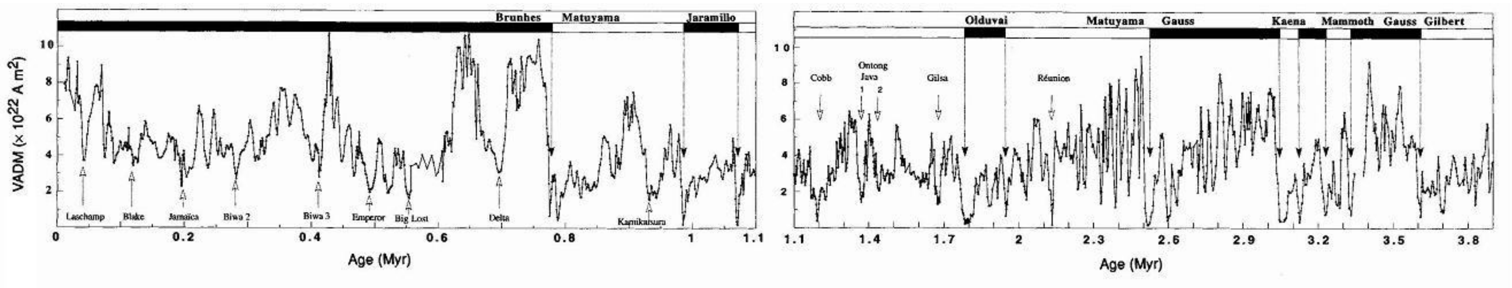

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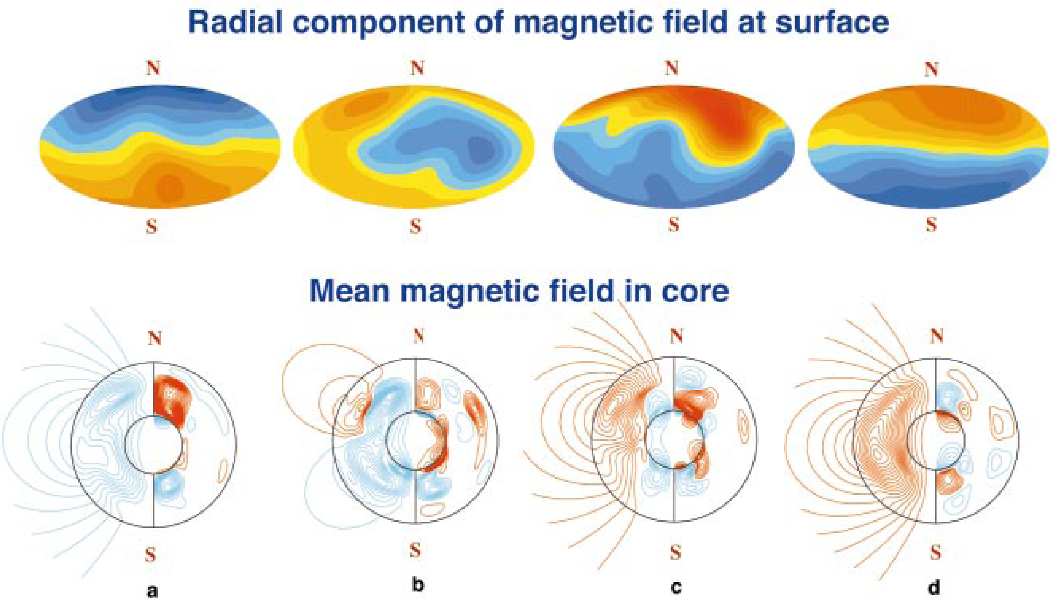

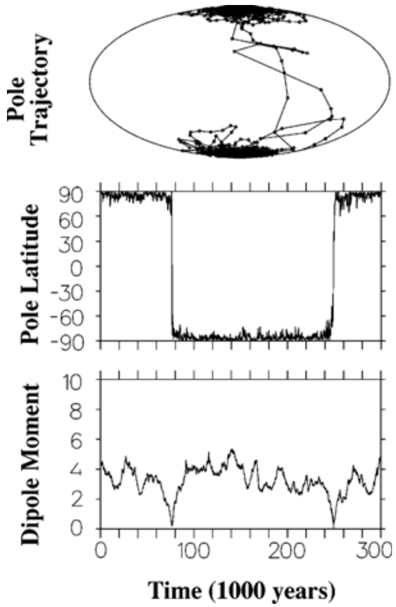

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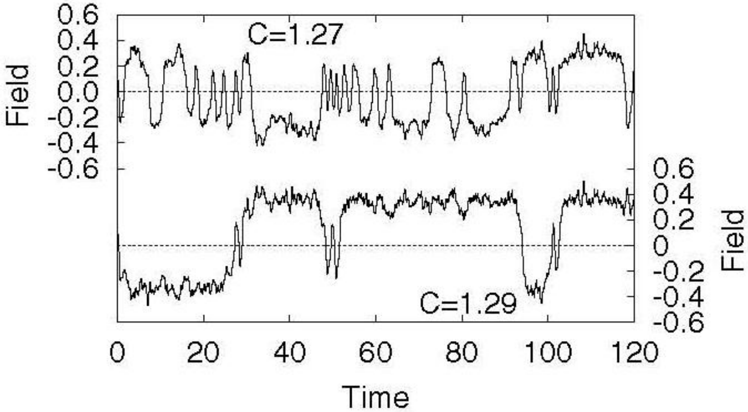

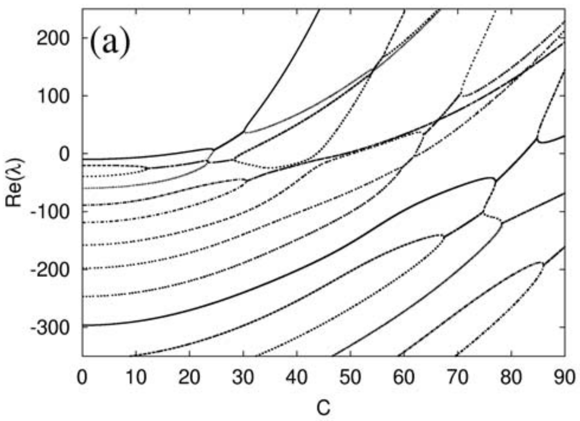

Field reversals are provoked by transitions to oscillatory solutions [6,7]

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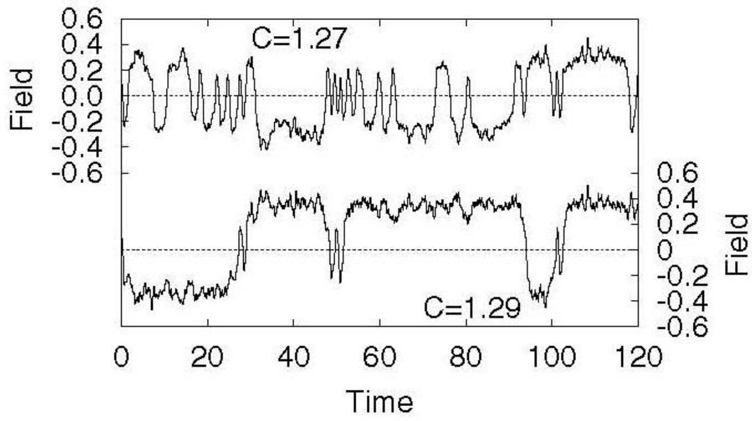

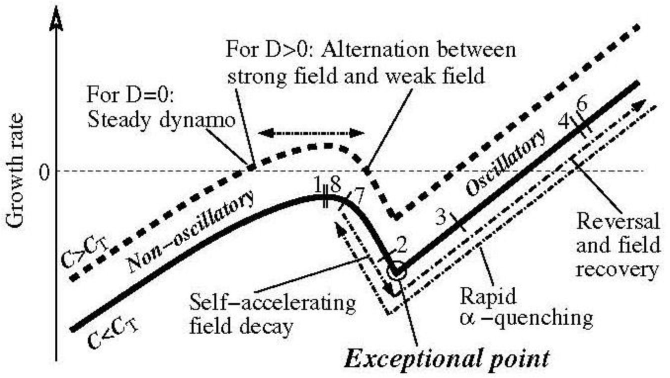

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Heuristic model [7]

$$\mathbf{r} \times \nabla\left[\nu_m \Delta F_1 + \alpha F_2 - \partial_t F_1\right] = 0$$

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Quenching α – effect (saturation)

$$\alpha(r,t) = C\frac{\alpha(r)}{1 + \left(\mathbf{B}(r,t)/B_0\right)^2} + noise$$

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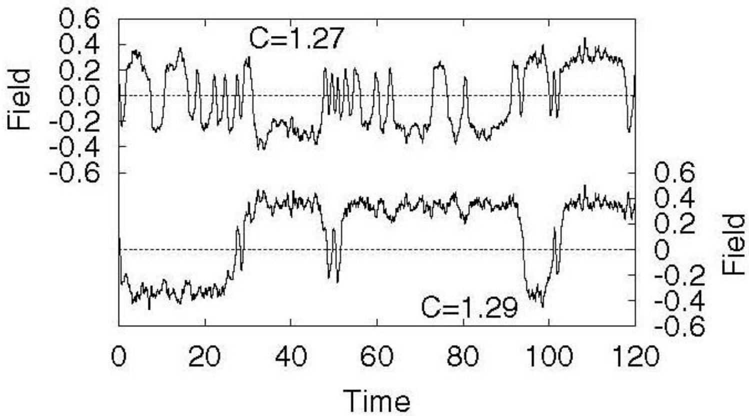

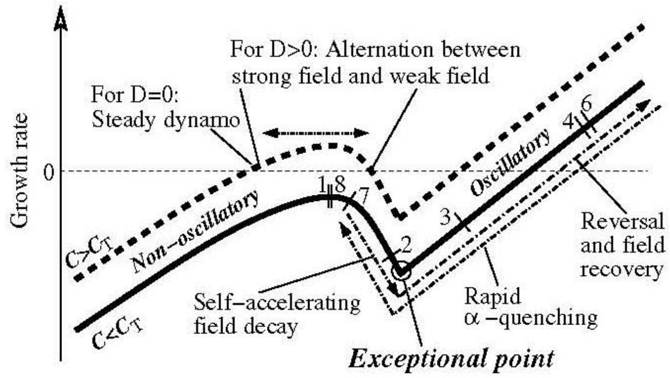

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Magnetic field dynamics in the vicinity of exceptional point

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How to excite the oscillatory α^2-dynamo?

Inverse problems for the dynamo operator [8]

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How to excite the oscillatory α^2–dynamo?

Inverse problems for the dynamo operator [8]

Given: Non-self-adjoint boundary eigenvalue problem

$$\begin{pmatrix} -A_l & \alpha(r) \\ A_{l,\alpha} & -A_l \end{pmatrix}\begin{pmatrix} F_1 \\ F_2 \end{pmatrix} = \lambda \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}$$

$$F_1(0) = F_2(0) = 0$$

$$(\beta l + 1 - \beta) F_1(1) + \beta F_1'(1) = 0, \quad F_2(1) = 0$$

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Find: Describe α-profiles that yield $\operatorname{Re}\lambda > 0, \quad \operatorname{Im}\lambda \neq 0$

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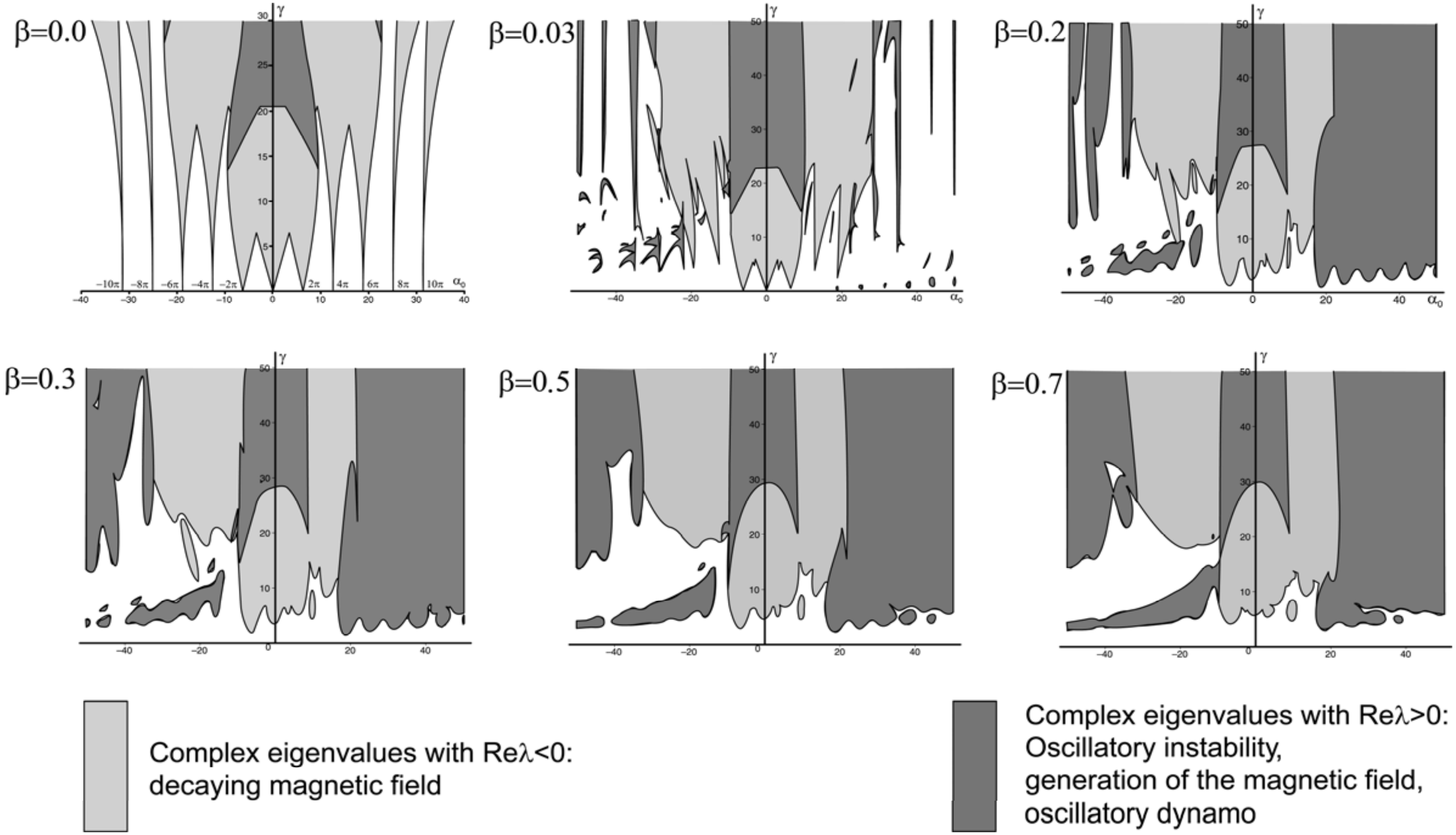
Note: Numerically possible (hard); without clear rule [8]

[8] F. Stefani and G. Gerbeth, Phys. Rev. E 2003

Why exciting the oscillatory dynamo is not easy?

We look for the α-profiles of the form $\alpha(r)=\alpha_0+\gamma\,\varphi(r)$

The problem reduces to description of Arnold's tongues in $(\alpha_0, \beta, \gamma)$-space



Pictured above: Numerical results for various boundary conditions

Eigencurves and spectral mesh

For $\alpha(r) = \alpha_0 = const$, eigenvalue problem can be solved exactly in 2 cases [10]

Isolating b. c. $(\beta = 1)$: $D(A_{\alpha_0}) \neq D(A^*_{\alpha_0})$

For $l = 0$: $\lambda_n(\alpha_0) = \frac{1}{4}(\alpha_0^2 - \pi^2 n^2), \quad n \in N$

Superconducting b. c. $(\beta = 0)$: $D(A_{\alpha_0}) = D(A^*_{\alpha_0})$

For $l = 0$: $\lambda_n^{\pm}(\alpha_0) = -(\pi n)^2 \pm \alpha_0 \pi n$

For $l \neq 0$: $\lambda_n^{\pm}(\alpha_0) = -\rho_n \pm \alpha_0 \sqrt{\rho_n}, \quad n \in Z^+$

Zeros of Bessel functions: $J_{l+1/2}\left(\sqrt{\rho_n}\right) = 0, \quad 0 < \sqrt{\rho_1} < \sqrt{\rho_2} < \cdots$

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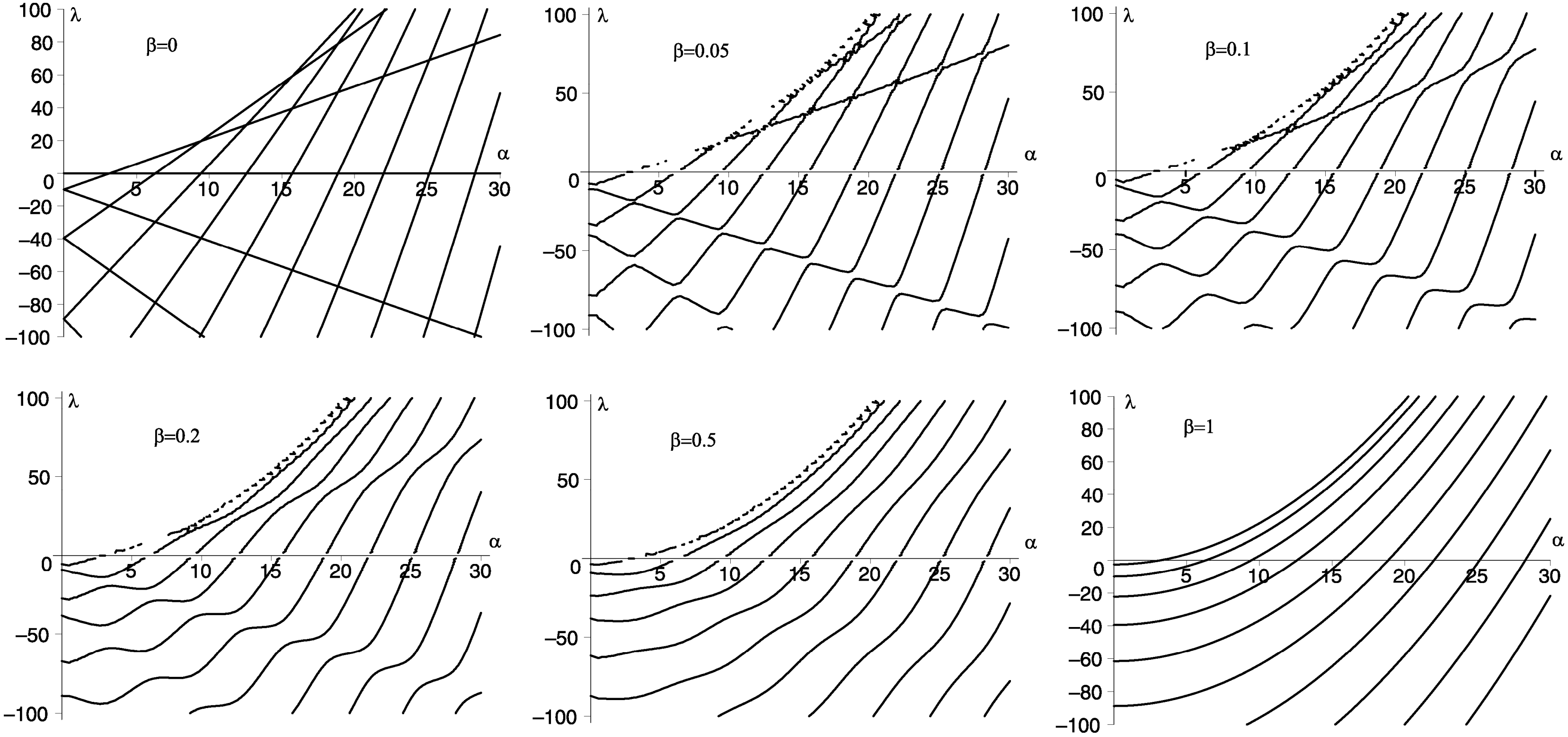
[10] U. Günther, O.N. Kirillov J. Phys. A 2006

Eigencurves and spectral mesh

Transition from superconducting to isolating boundary conditions [9]



Spectral mesh deforms to parabolic eigencurves when $\beta \in [0,1], \gamma = 0$

[9] O.N. Kirillov, U. Günther, F. Stefani Phys. Rev. E 2009

Eigencurves and spectral mesh

Superconducting b. c. (β=0): $D(A_\alpha) = D(A_\alpha^*)$

A_α is self-adjoint in a Krein space with indefinite inner product

$$[A_\alpha u, v] = [u, A_\alpha v], \quad u, v \in (K, [.,.]), \quad [.,.] = (J.,.), \quad J = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$

 U. Günther, O.N. Kirillov J. Phys. A 2006

Eigencurves and spectral mesh

Superconducting b. c. (β=0): $D(A_\alpha) = D(A_\alpha^*)$

A_α is self-adjoint in a Krein space with indefinite inner product

$$[A_\alpha u, v] = [u, A_\alpha v], \quad u, v \in (K, [.,.]), \quad [.,.] = (J.,.), \quad J = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$

$$\lambda_n^{\sigma_n} = -\rho_n + \sigma_n \alpha_0 \sqrt{\rho_n}, \quad u_n^{\sigma_n} = \begin{pmatrix} 1 \\ \sigma_n \sqrt{\rho_n} \end{pmatrix} N_n r^{\frac{1}{2}} J_{l+\frac{1}{2}}(\sqrt{\rho_n} r), \quad N_n = \frac{\sqrt{2}}{J_{l+\frac{3}{2}}(\sqrt{\rho_n})}$$

[10] U. Günther, O.N. Kirillov J. Phys. A 2006

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Krein signature $\sigma_n = \operatorname{sgn}[u_n, u_n] = \operatorname{sgn}(J u_n, u_n)$

Spectral mesh: σ_n= sign of the slope of an eigenline $\lambda(\alpha_0)$

[10] U. Günther, O.N. Kirillov J. Phys. A 2006

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Spectral mesh: σ_n= sign of the slope of an eigenline $\lambda(\alpha_0)$

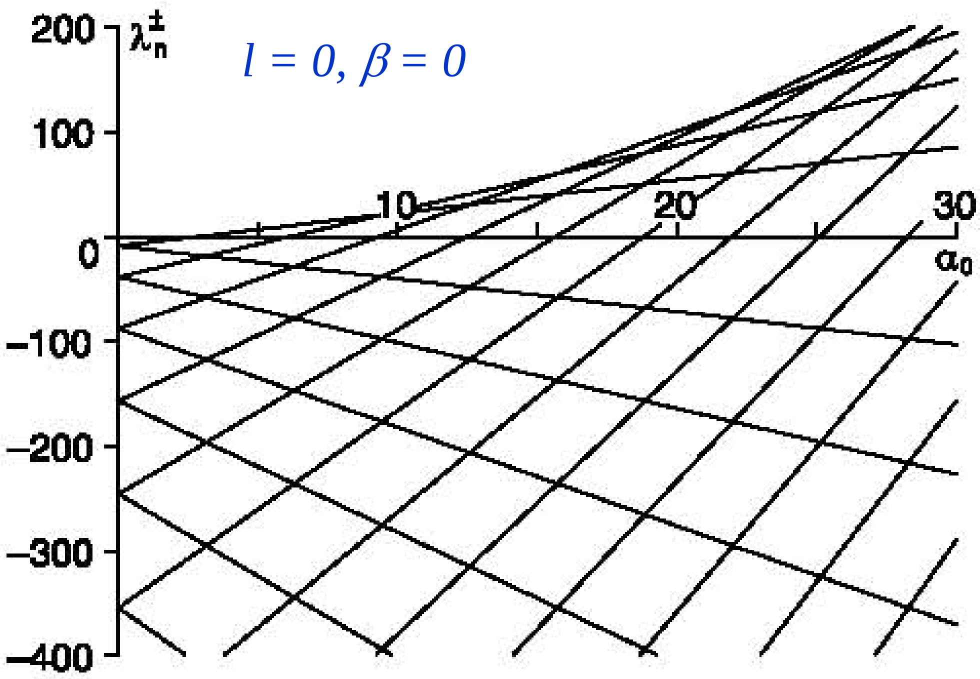
Double eigenvalue $\lambda_1 = \lambda_2$ →

- $\sigma_1 = \sigma_2$ definite Krein signature
- $\sigma_1 \neq \sigma_2$ mixed Krein signature

[10] U. Günther, O.N. Kirillov J. Phys. A 2006

Eigencurves and spectral mesh

Crossing of eigenlines: $\lambda_n^{\varepsilon} = \lambda_m^{\delta}, \quad \lambda_n^{\varepsilon} = -\rho_n + \varepsilon \alpha_0 \sqrt{\rho_n}, \quad \varepsilon, \delta = \pm$



Nodal point: $(\alpha_0^{\nu}, \lambda_0^{\nu})$

$$\alpha_0^{\nu} = \varepsilon \sqrt{\rho_n} + \delta \sqrt{\rho_m}$$

Doublet (semi-simple):

$$\lambda_0^{\nu} = \varepsilon \delta \sqrt{\rho_n \rho_m}$$

Two eigenvectors:

$$u_n^{\varepsilon}, \quad u_m^{\delta}$$

 U. Günther, O.N. Kirillov J. Phys. A 2006

Eigencurves and spectral mesh

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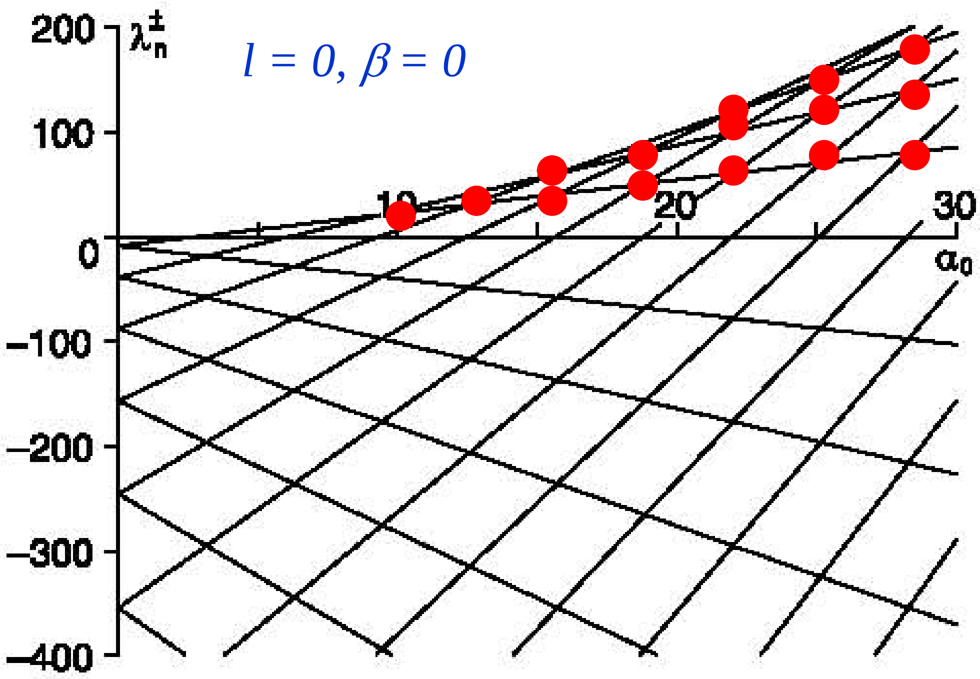

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Doublets at the crossings with λ>0 have **definite Krein signature**

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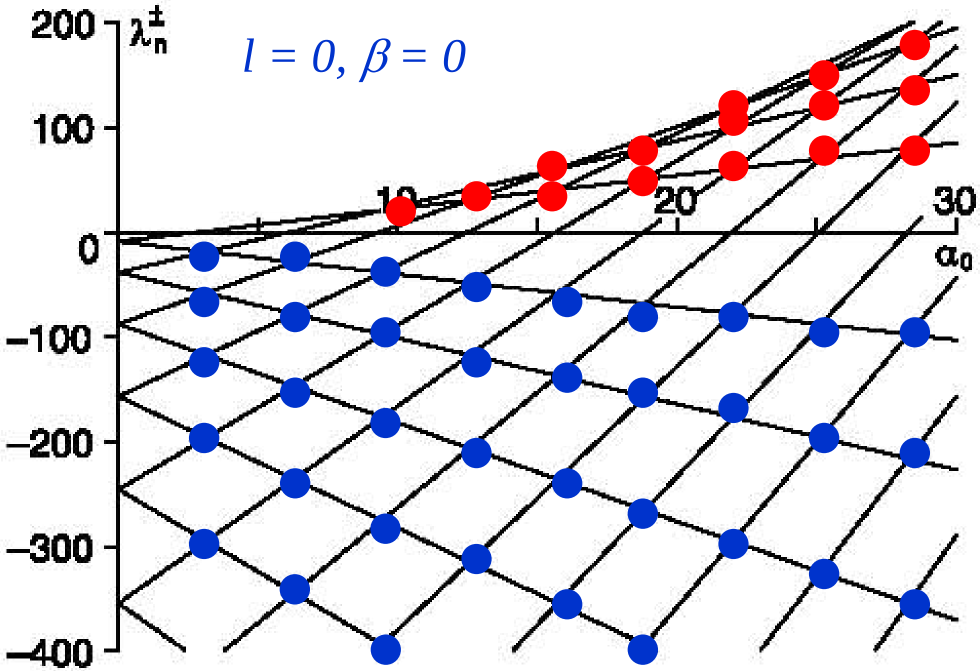

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Doublet (semi-simple):

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Two eigenvectors:

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Doublets at the crossings with $\lambda>0$ have **definite Krein signature**

Doublets at the crossings with $\lambda<0$ have **mixed Krein signature**

[10] U. Günther, O.N. Kirillov J. Phys. A 2006

Eigencurves and spectral mesh

Unfolding doublets by perturbation $\alpha(r)=\alpha_0^\nu+\gamma\varphi(r),\ \beta=0$

Indefinite inner product in the Krein space: $[.,.]=(J.,.)$

Define: $a_1=\varepsilon\frac{[Ku_n^\varepsilon,u_n^\varepsilon]}{\sqrt{\rho_n}},\quad a_2=\delta\frac{[Ku_m^\delta,u_m^\delta]}{\sqrt{\rho_m}},\quad b^2=\frac{[Ku_n^\varepsilon,u_n^\varepsilon][Ku_m^\delta,u_m^\delta]}{\sqrt{\rho_n\rho_m}}$

$$[Ku_n^\varepsilon,u_m^\delta]=\gamma\int_0^1\varphi\left[\left(\varepsilon\delta\sqrt{\rho_n\rho_m}+\frac{l(l+1)}{r^2}\right)u_m u_n+u_m' u_n'\right]dr$$

Splitting: $\lambda=\lambda_0^\nu+\Delta\lambda,\ \Delta\lambda=[(a_1+a_2)\pm\sqrt{(a_1-a_2)^2+\varepsilon\delta\, b^2}]/2$

$\varepsilon=\delta\ (\lambda_0^\nu>0)$: $\Delta\lambda$ is real; $\varepsilon=-\delta\ (\lambda_0^\nu<0)$: $\Delta\lambda$ can be complex

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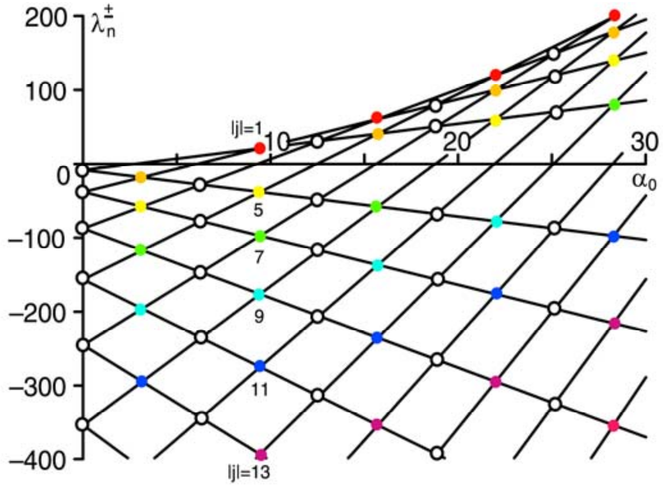
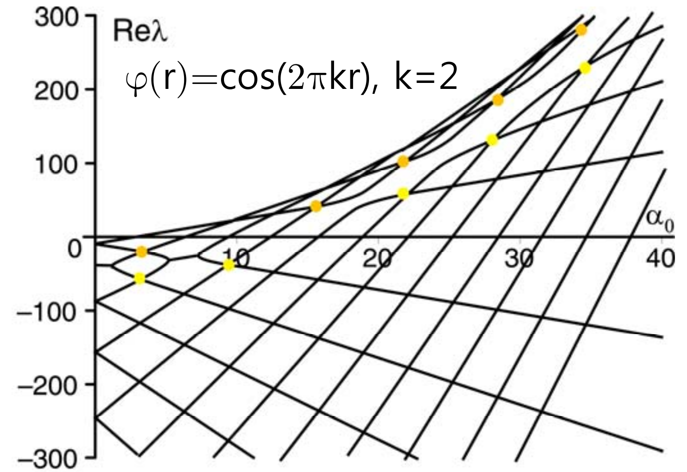
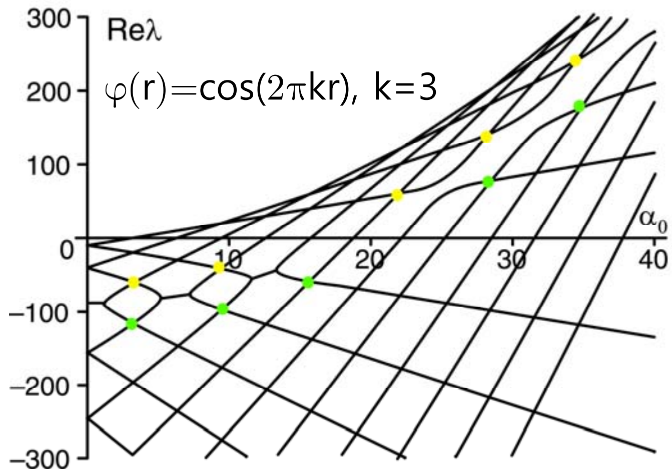
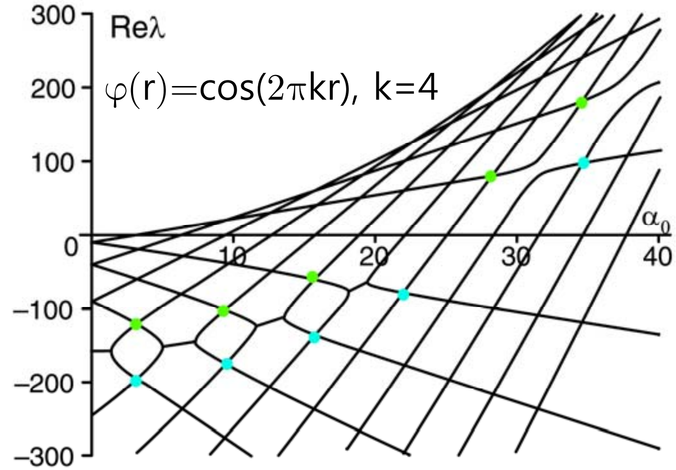
Two branches with the slopes of the same sign – real splitting





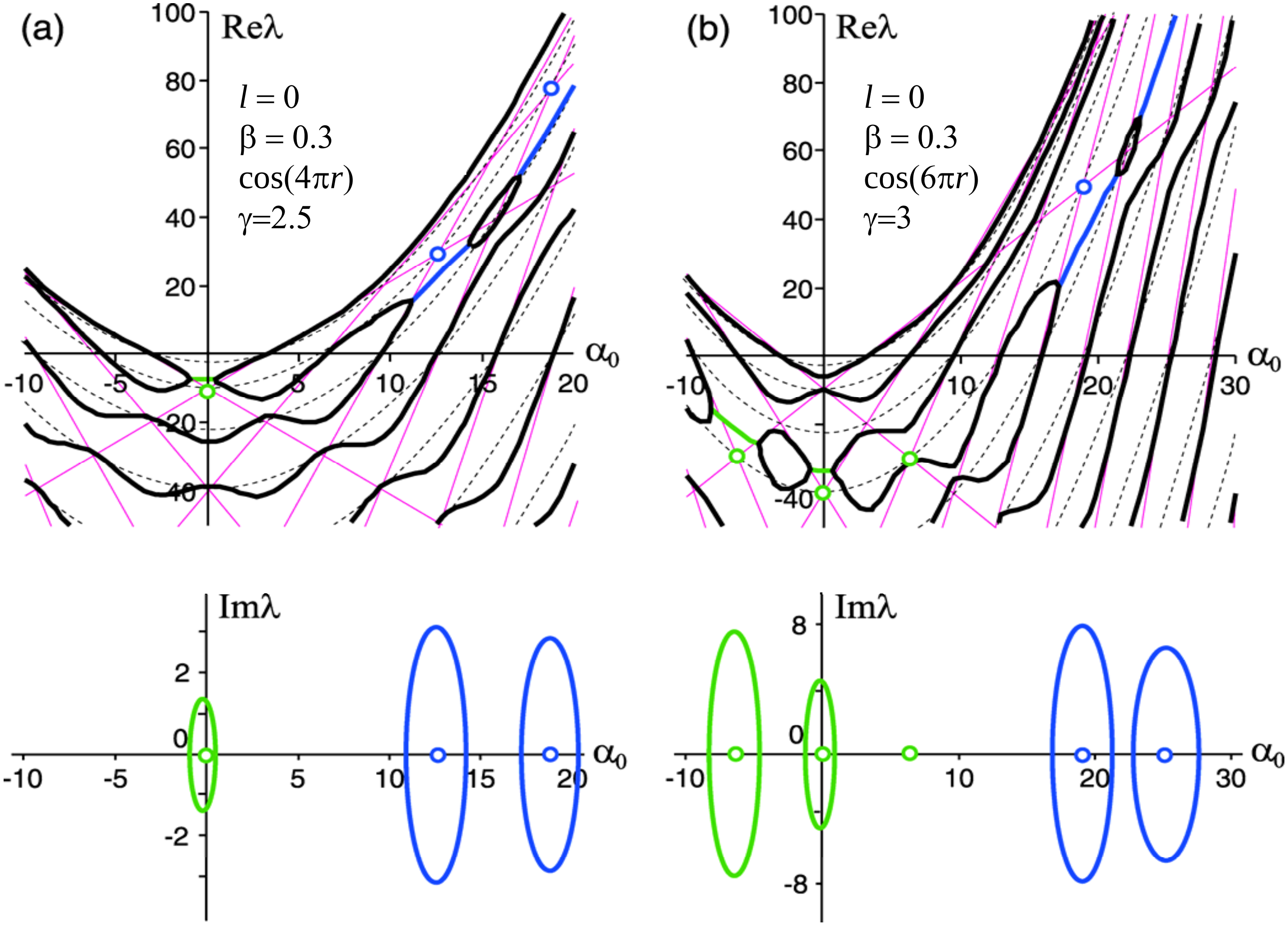
Two branches with the slopes of different signs – complex splitting





[10] U. Günther, O.N. Kirillov J. Phys. A 2006

Fourier coefficients select sequences of resonant crossings



[9] O.N. Kirillov, U. Günther, F. Stefani Phys. Rev. E 2009

α, β, γ – unfolding of the crossings

Node: $(\alpha_0^\nu, \lambda_0^\nu), \quad \lambda_0^\nu = \varepsilon\delta\pi^2 nm, \quad \alpha_0^\nu = \varepsilon\pi n + \delta\pi m, \quad \varepsilon, \delta = \pm$

After the splitting *:

$$\lambda(\alpha_0, \beta, \gamma) = \lambda_0^\nu - \varepsilon\delta\pi^2 nm\beta + \frac{\pi}{2}(\delta m + \varepsilon n)\Delta\alpha_0$$
$$\pm \frac{\pi}{2}\sqrt{((\delta m - \varepsilon n)\Delta\alpha_0)^2 + 4mn\left(\varepsilon\gamma\Delta\alpha - (-1)^{n+m}\pi n\beta\right)\left(\delta\gamma\Delta\alpha - (-1)^{n+m}\pi m\beta\right)}$$

Condition for existence of the complex eigenvalues:

$$((\varepsilon n - \delta m)\Delta\alpha_0)^2 +$$
$$mn((\varepsilon + \delta)\gamma\Delta\alpha - (-1)^{n+m}(n+m)\beta\pi)^2 - mn((\varepsilon - \delta)\gamma\Delta\alpha - (-1)^{n+m}(n-m)\beta\pi)^2 < 0$$

*: $\Delta\alpha_0 := \alpha_0 - \alpha_0^\nu, \quad \Delta\alpha := \int_0^1 \varphi(r)\cos((\varepsilon n - \delta m)\pi r)dr, \quad \int_0^1 \varphi(r)dr = 0$

[9] O.N. Kirillov, U. Günther, F. Stefani Phys. Rev. E 2009

Three dimensional Arnold tongues of oscillatory dynamo

$$\varphi(r) = \cos(2\pi k r), \quad \Delta\alpha = \int_0^1 \varphi(r)\cos((\varepsilon n - \delta m)r)dr = \begin{cases} 1/2, & 2k = \pm(\varepsilon n - \delta m) \\ 0, & 2k \neq \pm(\varepsilon n - \delta m) \end{cases}$$

Two sorts of the resonance domains in the plane (α_0, γ)

Hyperbolic regions for $\varepsilon = -\delta$ (decaying magnetic field):

$$mn(\varepsilon\gamma - (n-m)\beta\pi)^2 - 4k^2(\alpha_0 - \alpha_0^\nu)^2 > mn((n+m)\beta\pi)^2$$

$$\operatorname{Re}\lambda = -\pi^2 mn(1-\beta) + \varepsilon\frac{\pi}{2}(n-m)(\alpha_0 - \alpha_0^\nu) < 0$$

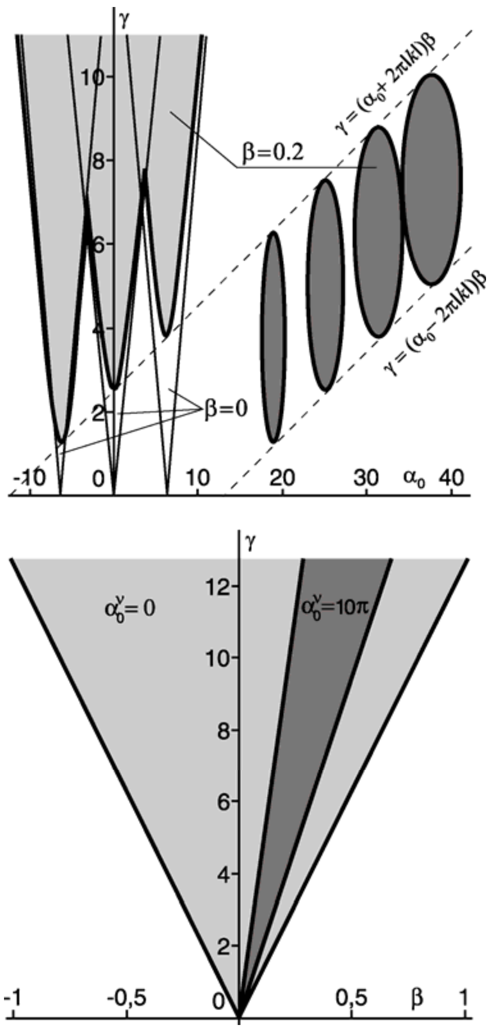
Elliptic regions for $\varepsilon = \delta$ (growing magnetic field):

$$4k^2(\alpha_0 - \alpha_0^\nu)^2 + mn(\varepsilon\gamma - (n+m)\beta\pi)^2 < mn((n-m)\beta\pi)^2$$

$$\operatorname{Re}\lambda = \pi^2 mn(1-\beta) + \varepsilon\frac{\pi}{2}(n+m)(\alpha_0 - \alpha_0^\nu) > 0$$

[9] O.N. Kirillov, U. Günther, F. Stefani Phys. Rev. E 2009

Resonance tongues and islands ($\varphi(r)=\cos(2\pi kr)$, $k = 2$)



- $\beta = 0$: only resonant tongues with $\mathrm{Re}\lambda < 0$ are visible in the plane (α_0, γ)
- $\beta > 0$: resonance “islands” of oscillatory dynamo with $\mathrm{Re}\lambda > 0$ appear in the “prohibited” zone in the plane (α_0, γ)
- Separate variation of γ or β does not yield the oscillatory dynamo
- Simultaneous changing of γ and β easily produces the non-trivial oscillatory dynamo regions
- γ is bounded from above (in a corridor)

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Oscillatory dynamo domains (Arnold tongues)

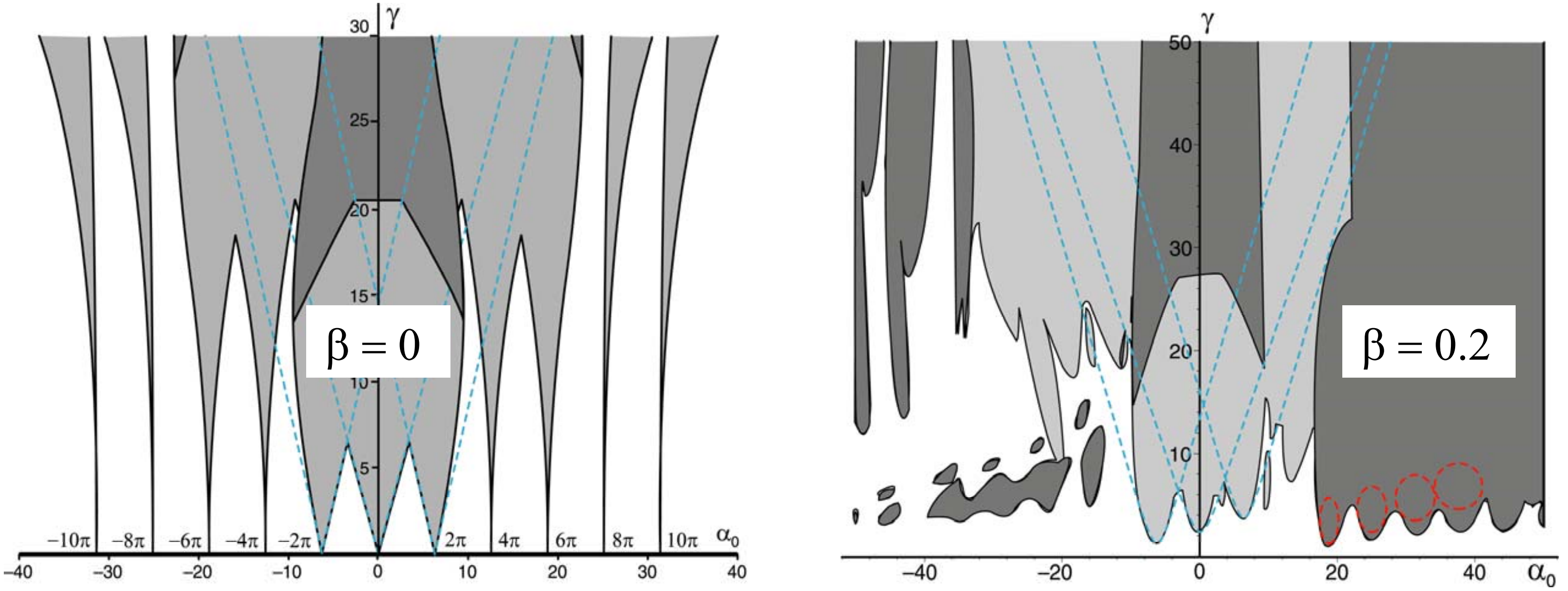
Numerical vs. perturbative (dashed)

3 principal tongues:

$$\gamma^2 - 4\alpha_0^2 > 16\pi^2\beta^2, \quad 16(\alpha_0 \pm 2\pi)^2 + (\gamma \pm 10\pi\beta)^2 < 4(\gamma \pm 4\pi\beta)^2$$

Infinitely many islands: $n = 1,2,\ldots$

$$16(\alpha_0 - (4+2n)\pi)^2 + n(4+n)(\gamma + 2(n+2)\beta\pi)^2 < 4n(4+n)(k\beta\pi)^2$$



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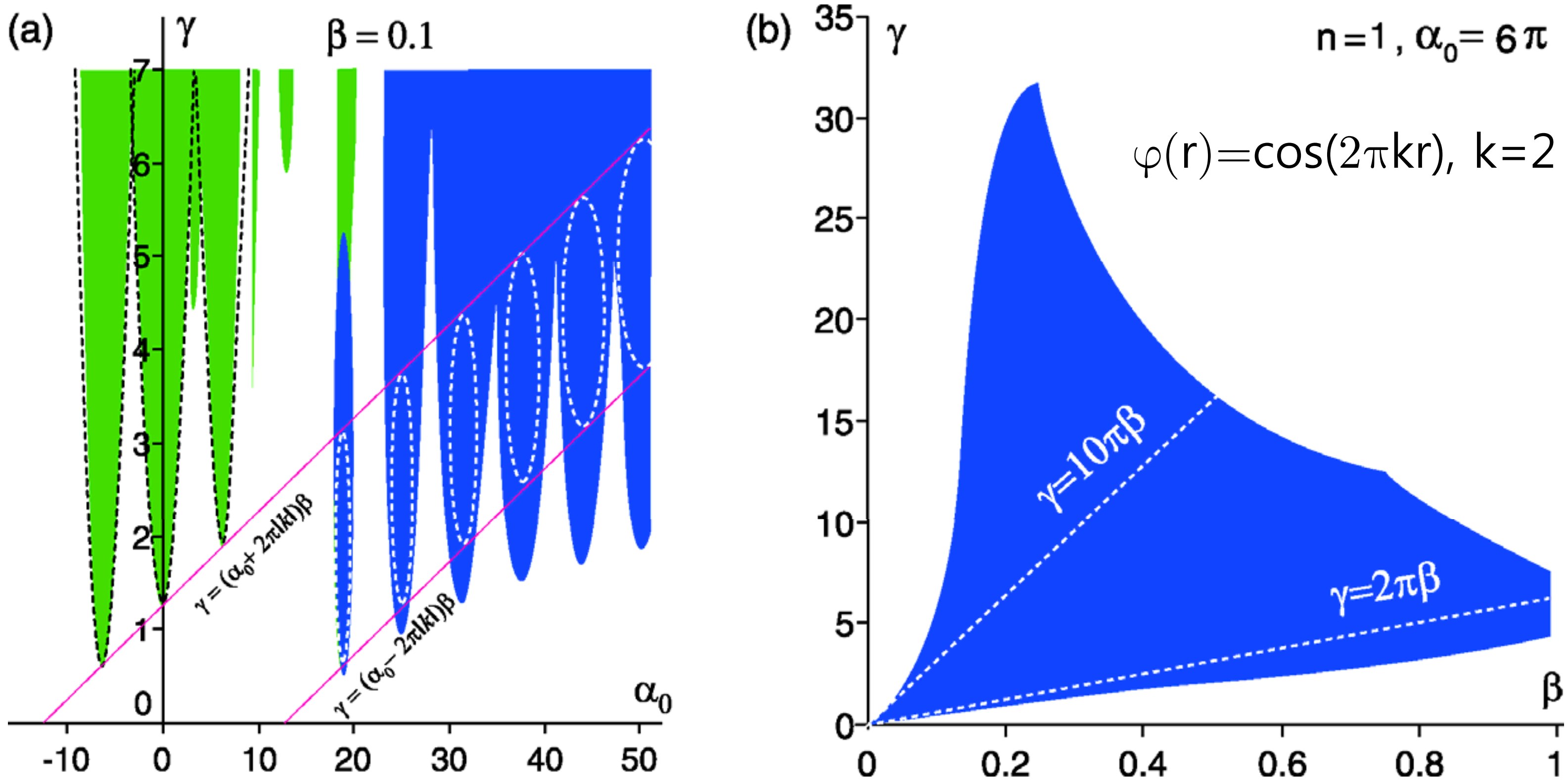

green (blue) – decaying (growing) oscillatory modes

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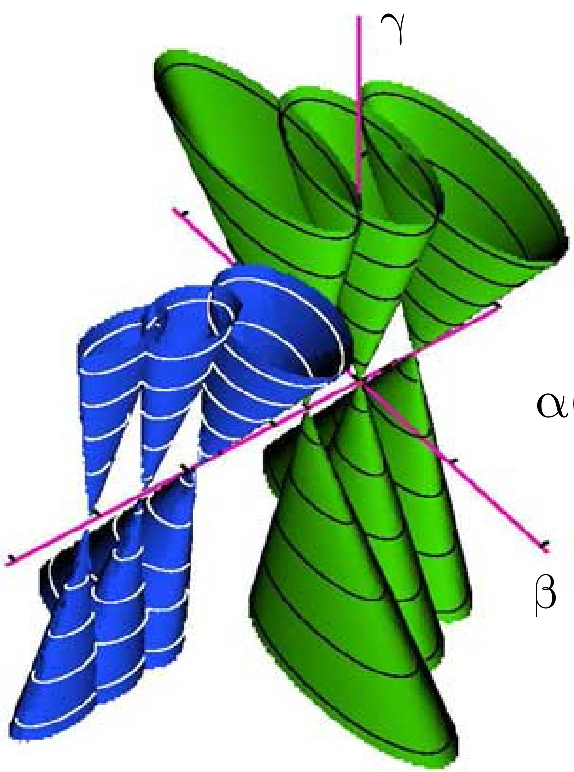
Krein signature (KS) determines orientation of the resonance zones

Definite KS (blue) – no instability at $\beta=0$

Large domains of oscillatory dynamo for $\beta > 0$

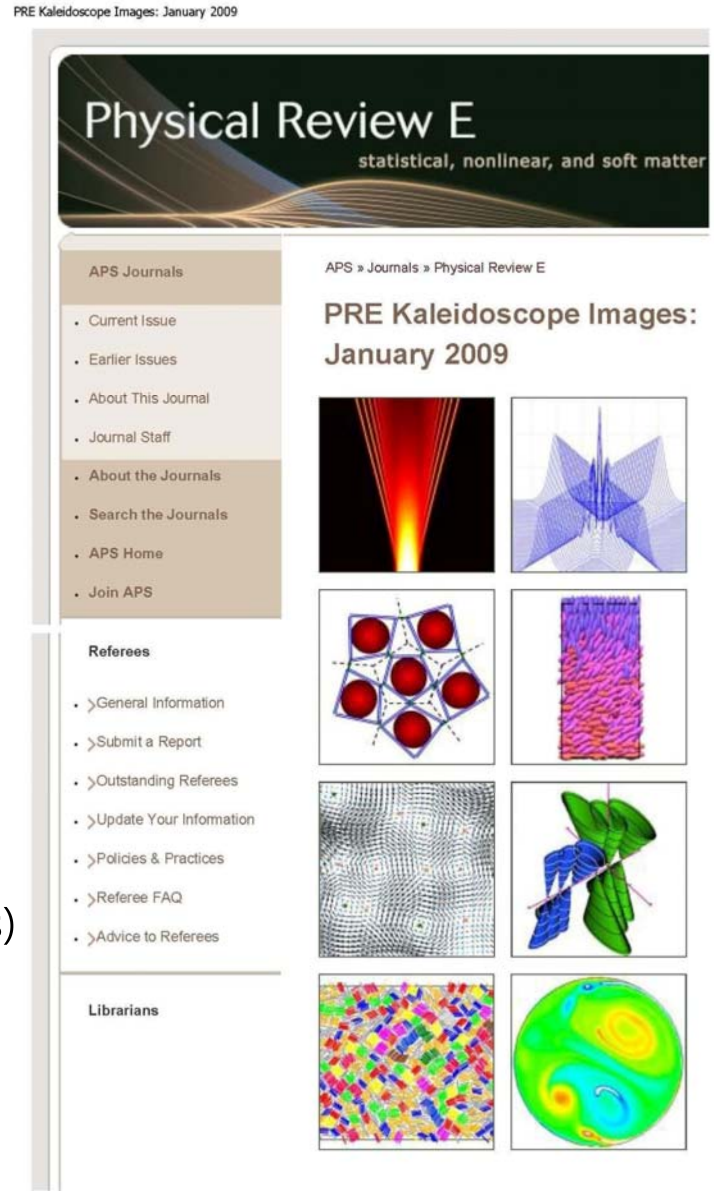


α_0

$$\alpha(r) = \alpha_0 + \gamma\,\varphi(r)$$

β (boundary conditions)

3D Arnold tongues in $(\alpha_0, \beta, \gamma)$–space



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2. Outline

Magnetized Taylor-Couette flow

Axisymmetric perturbations

Local stability analysis

Islands of helical magnetorotational instability

Standard and helical magnetorotational instability

Transition through spectral exceptional points

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Mathematical setting

Navier-Stokes equation for the fluid velocity ***u***

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\frac{1}{\rho}\nabla\left(p + \frac{B^2}{2\mu_0}\right) + \frac{1}{\mu_0\rho}(\mathbf{B}\cdot\nabla)\mathbf{B} + \nu\nabla^2\mathbf{u}$$

Induction equation for the magnetic field ***B***

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla\times(\mathbf{u}\times\mathbf{B}) + \eta\nabla^2\mathbf{B}$$

Mass continuity for incompressible flows and the solenoidal condition

$$\nabla\cdot\mathbf{u} = 0, \quad \nabla\cdot\mathbf{B} = 0$$

p : pressure, $\rho = const$: density, $\nu = const$: kinematic viscosity

$\eta = (\mu_0\sigma)^{-1}$: magnetic diffusivity, σ : conductivity of the fluid

μ_0 : magnetic permeability of free space

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Linearization w. r. t. axi-symmetric perturbations, cf. [Liu et al. 2006]

Steady state: A magnetized Taylor-Couette flow

$$\mathbf{u}_0 = R\Omega(R)\mathbf{e}_\phi, \quad p = p_0(R), \quad \mathbf{B}_0 = B_\phi^0(R)\mathbf{e}_\phi + B_z^0\mathbf{e}_z$$

$$B_\phi^0(R) = \frac{\mu_0 I}{2\pi R}, \qquad \Omega(R) = a + \frac{b}{R^2}, \qquad R\Omega^2 = \frac{1}{\rho}\frac{\partial p_0}{\partial R}, \qquad \kappa^2 = 2\Omega\left(2\Omega + R\frac{d\Omega}{dR}\right)$$

Axi-symmetric perturbation: $\mathbf{u}' = \mathbf{u}'(R,z) \quad \mathbf{B}' = \mathbf{B}'(R,z) \quad p' = p'(R,z)$

Operators: $D_1 = \partial_R \partial_R^\dagger + \partial_z^2, \quad \partial_t = \frac{\partial}{\partial t}, \quad \partial_R = \frac{\partial}{\partial R}, \quad \partial_z = \frac{\partial}{\partial z}, \quad \partial_R^\dagger = \partial_R + \frac{1}{R}, \quad \tilde{E} = \mathrm{diag}(D_1, 1, 1, 1)$

Linearized equations:

$$\partial_t \tilde{E}\xi' = \tilde{H}\xi' \qquad \tilde{H} = \begin{pmatrix} \nu D_1^2 & 2\Omega\partial_z^2 & \frac{B_z^0}{\mu_0\rho}D_1\partial_z & -\frac{2B_\phi^0}{\mu_0\rho R}\partial_z^2 \\ -\frac{\kappa^2}{2\Omega} & \nu D_1 & 0 & \frac{B_z^0}{\mu_0\rho}\partial_z \\ B_z^0\partial_z & 0 & \eta D_1 & 0 \\ \frac{2B_\phi^0}{R} & B_z^0\partial_z & R\partial_R\Omega & \eta D_1 \end{pmatrix} \qquad \xi' = \begin{pmatrix} u_R' \\ u_\phi' \\ B_R' \\ B_\phi' \end{pmatrix}$$

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Local stability analysis [Pessah & Psaltis 2005]

Local stability analysis around a fiducial point (R_0, z_0)

Local coordinates $\tilde{R} = R - R_0 \quad \tilde{z} = z - z_0$

PDEs with constant coefficients $\partial_t \tilde{E}_0 \xi' = \tilde{H}_0 \xi'$

$$\tilde{E}_0 = \begin{pmatrix} D_1^0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \tilde{H}_0 = \begin{pmatrix} \nu (D_1^0)^2 & 2\Omega_0 \partial_{\tilde{z}}^2 & \frac{B_z^0}{\mu_0 \rho} D_1^0 \partial_{\tilde{z}} & -\frac{2B_\phi^0}{\mu_0 \rho R_0} \partial_{\tilde{z}}^2 \\ -\frac{\kappa_0^2}{2\Omega_0} & \nu D_1^0 & 0 & \frac{B_z^0}{\mu_0 \rho} \partial_{\tilde{z}} \\ B_z^0 \partial_{\tilde{z}} & 0 & \eta D_1^0 & 0 \\ \frac{2B_\phi^0}{R_0} & B_z^0 \partial_{\tilde{z}} & \frac{\kappa_0^2}{2\Omega_0} - 2\Omega_0 & \eta D_1^0 \end{pmatrix}$$

$$\Omega_0 = \Omega(R_0), \quad \kappa_0^2 = 2\Omega_0 \left(2\Omega_0 + R_0 \left. \frac{d\Omega}{dR} \right|_{R=R_0} \right), \quad B_\phi^0 = B_\phi^0(R_0), \quad D_1^0 = \partial_{\tilde{R}}^2 + \partial_{\tilde{z}}^2 + \frac{\partial_{\tilde{R}}}{R_0} - \frac{1}{R_0^2}$$

WKB approximation, cf. [Lakhin & Velikhov 2007, Rüdiger et al. 2008]

A plane wave $\xi' = \tilde{\xi}\exp(\gamma t + ik_R\tilde{R} + ik_z\tilde{z}), \quad \tilde{\xi} = (\tilde{u}_R, \tilde{u}_\phi, \tilde{B}_R, \tilde{B}_\phi)^T$

Restriction to the modes with $k_R R_0 \gg 1$

Eigenvalue problem $(H - \gamma I)\tilde{\xi} = 0 \qquad H = -\mathrm{diag}(\omega_\nu, \omega_\nu, \omega_\eta, \omega_\eta) + H_1 + H_2$

$$H_1 = \frac{i\omega_A}{\sqrt{\mu_0\rho}}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \mu_0\rho & 0 & 0 & 0 \\ 0 & \mu_0\rho & 0 & 0 \end{pmatrix}$$

$$H_2 = \begin{pmatrix} 0 & 2\Omega_0\alpha^2 & 0 & -2\omega_{A_\phi}\dfrac{\alpha^2}{\sqrt{\mu_0\rho}} \\ -2\Omega_0 - R_0\left.\dfrac{d\Omega}{dR}\right|_{R=R_0} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -2\omega_{A_\phi}\sqrt{\mu_0\rho} & 0 & R_0\left.\dfrac{d\Omega}{dR}\right|_{R=R_0} & 0 \end{pmatrix}$$

Alfvén frequencies

$$\omega_A^2 = \frac{k_z^2 (B_z^0)^2}{\mu_0\rho}, \quad \omega_{A_\phi}^2 = \frac{(B_\phi^0)^2}{\mu_0\rho R_0^2}$$

Viscous and resistive frequencies

$$\omega_\nu = \nu k^2, \ \omega_\eta = \eta k^2 \qquad k^2 = k_z^2 + k_R^2 \qquad \alpha = \frac{k_z}{k}$$

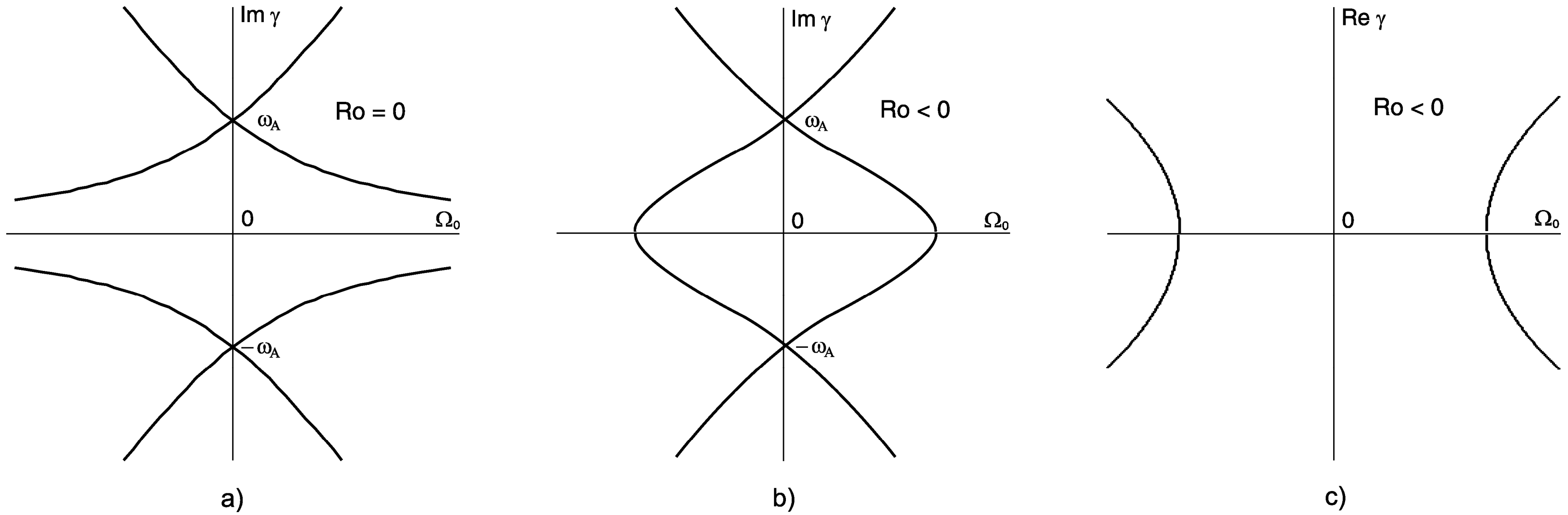
Splitting the Alfvén frequencies, cf. [Lehnert 1954, Nornberg et al. 2009]

Damped Alfvén modes ($H_2 = 0$)

$$\gamma_{1,2} = -\frac{\omega_\nu + \omega_\eta}{2} + \sqrt{\left(\frac{\omega_\nu - \omega_\eta}{2}\right)^2 - \omega_A^2}, \quad \gamma_{3,4} = -\frac{\omega_\nu + \omega_\eta}{2} - \sqrt{\left(\frac{\omega_\nu - \omega_\eta}{2}\right)^2 - \omega_A^2}$$

Fast and slow Magneto-Coriolis waves ($\omega_{A_\phi} = 0 \quad \left.\frac{d\Omega}{dR}\right|_{R=R_0} = 0$)

$$\gamma_{1,2} = i\sqrt{\omega_A^2 + \Omega_0^2\alpha^2} \pm i\alpha\Omega_0, \quad \gamma_{3,4} = -i\sqrt{\omega_A^2 + \Omega_0^2\alpha^2} \pm i\alpha\Omega_0$$



Dispersion relation $P(\gamma)=\det(H-\gamma I)=0$ **[LV07, Rüdiger & Schultz 2008]**

Dimensionless dispersion relation

$$P(\lambda)=\lambda^4+a_1\lambda^3+a_2\lambda^2+(a_3+ib_3)\lambda+a_4+ib_4=0 \qquad \gamma=\lambda\sqrt{\omega_\nu\omega_\eta}$$

Rossby, magnetic Prandtl, Reynolds, and Hartmann numbers

$$\mathrm{Ro}=\frac{1}{2}\frac{R_0}{\Omega_0}\frac{d\Omega}{dR}\bigg|_{R=R_0},\quad \mathrm{Pm}=\frac{\nu}{\eta}=\frac{\omega_\nu}{\omega_\eta},\quad \beta^*=\alpha\frac{\omega_{A_\phi}}{\omega_A},\quad \mathrm{Re}^*=\alpha\frac{\Omega_0}{\omega_\nu},\quad \mathrm{Ha}^*=\alpha\frac{B_z^0}{k\sqrt{\mu_0\rho\nu\eta}}$$

Coefficients

$$a_1=2\left(\sqrt{\mathrm{Pm}}+\frac{1}{\sqrt{\mathrm{Pm}}}\right) \qquad a_4=\left(1+\mathrm{Ha}^{*2}\right)^2+4\beta^{*2}\mathrm{Ha}^{*2}+4\mathrm{Re}^{*2}+4\mathrm{Re}^{*2}\mathrm{Ro}(\mathrm{Pm}\mathrm{Ha}^{*2}+1)$$

$$a_2=\frac{a_1^2}{4}+2(1+\mathrm{Ha}^{*2})+4\beta^{*2}\mathrm{Ha}^{*2}+4\mathrm{Re}^{*2}\mathrm{Pm}(1+\mathrm{Ro}) \qquad b_3=-8\beta^*\mathrm{Ha}^{*2}\mathrm{Re}^*\sqrt{\mathrm{Pm}}$$

$$a_3=a_1(1+\mathrm{Ha}^{*2})+2a_1\beta^{*2}\mathrm{Ha}^{*2}+8\mathrm{Re}^{*2}(1+\mathrm{Ro})\sqrt{\mathrm{Pm}} \qquad b_4=-4\beta^*\mathrm{Ha}^{*2}\mathrm{Re}^*(2+(1-\mathrm{Pm})\mathrm{Ro})$$

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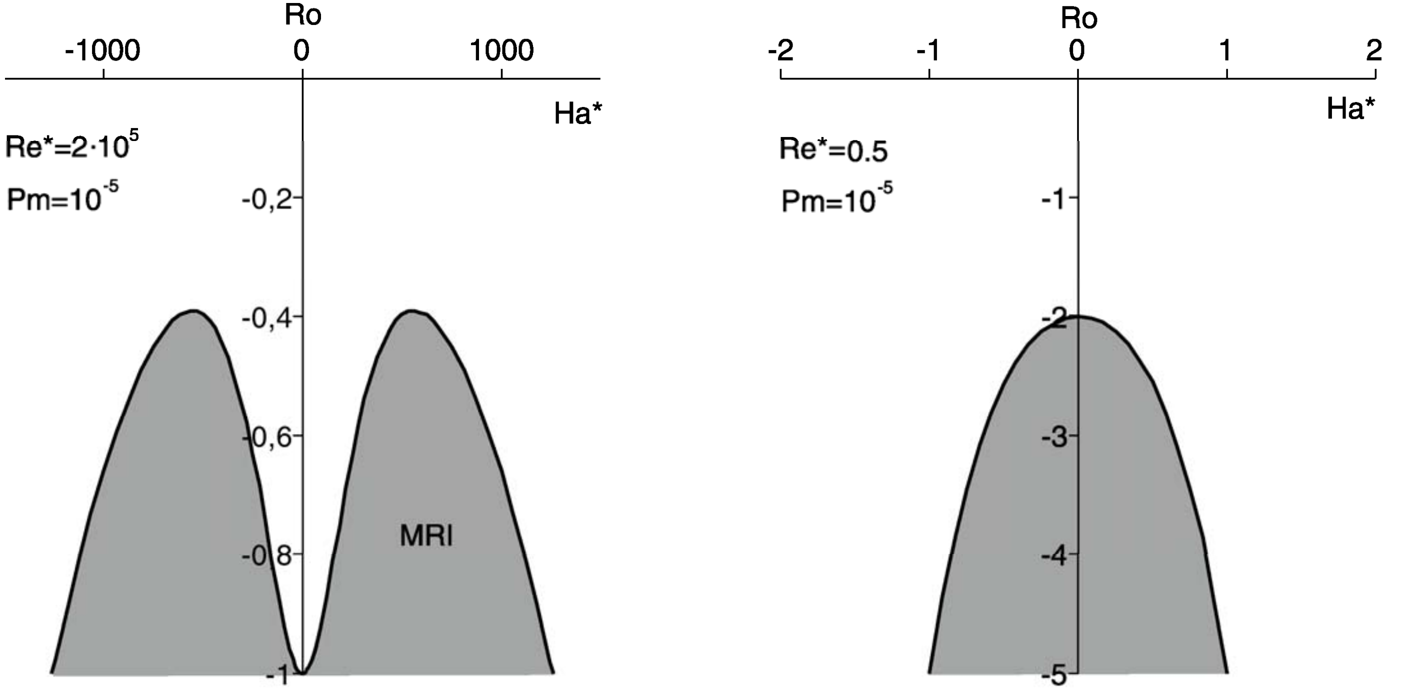
SMRI in the absence of the azimuthal magnetic field $(\beta^* = 0)$

$b_3=0$, $b_4=0$ Real coefficients: Routh-Hurwitz criterion

$$\mathrm{Ro} < \mathrm{Ro}^c := -\frac{\left(1+\mathrm{Ha}^{*2}\right)^2 + 4\mathrm{Re}^{*2}}{4\mathrm{Re}^{*2}(\mathrm{PmHa}^{*2}+1)}$$

Standard magneto-rotational instability (SMRI), cf. [Ji et al. 2001]



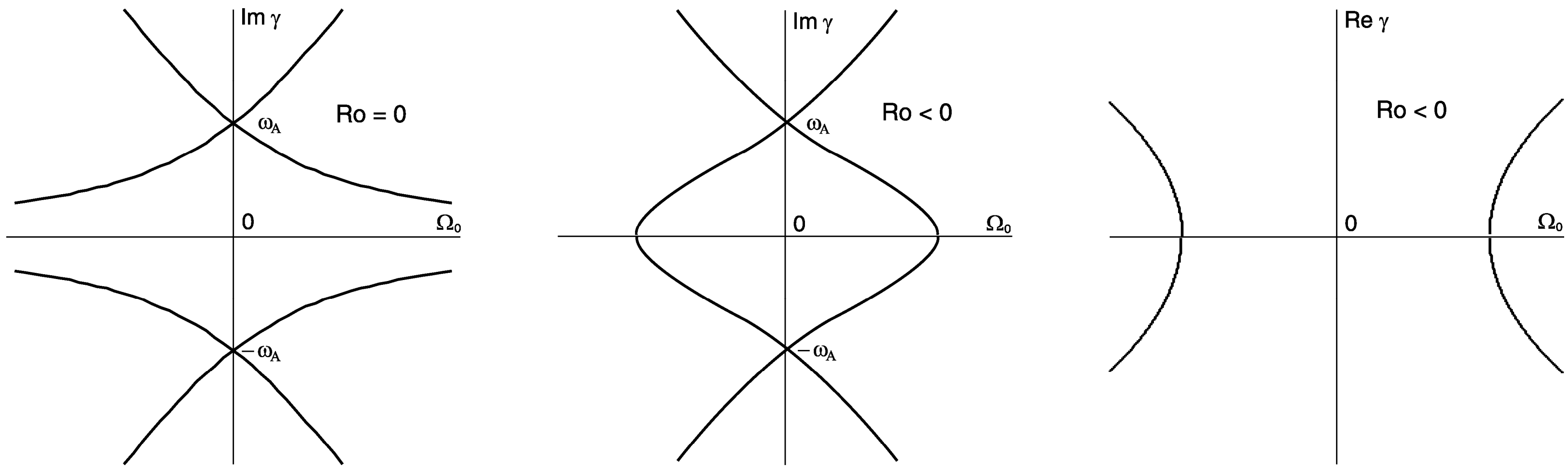
SMRI as destabilization of slow Magneto-Coriolis waves [Nornberg 2008]

No dissipation ($\omega_\nu = 0,\ \omega_\nu = 0$)

$$\gamma = \pm\sqrt{-2\Omega_0^2\alpha^2(1+\mathrm{Ro}) - \omega_A^2 \pm 2\Omega_0\alpha\sqrt{\Omega_0^2\alpha^2(1+\mathrm{Ro})^2 + \omega_A^2}}$$

Threshold of SMRI

$$\Omega_0^c = -\frac{\omega_A^2}{2\alpha^2 R_0 \partial_R \Omega} \qquad \mathrm{Ro} = -\frac{\mathrm{Ha}^{*2}}{4\mathrm{Re}^{*2}\mathrm{Pm}}$$



a) b) c)

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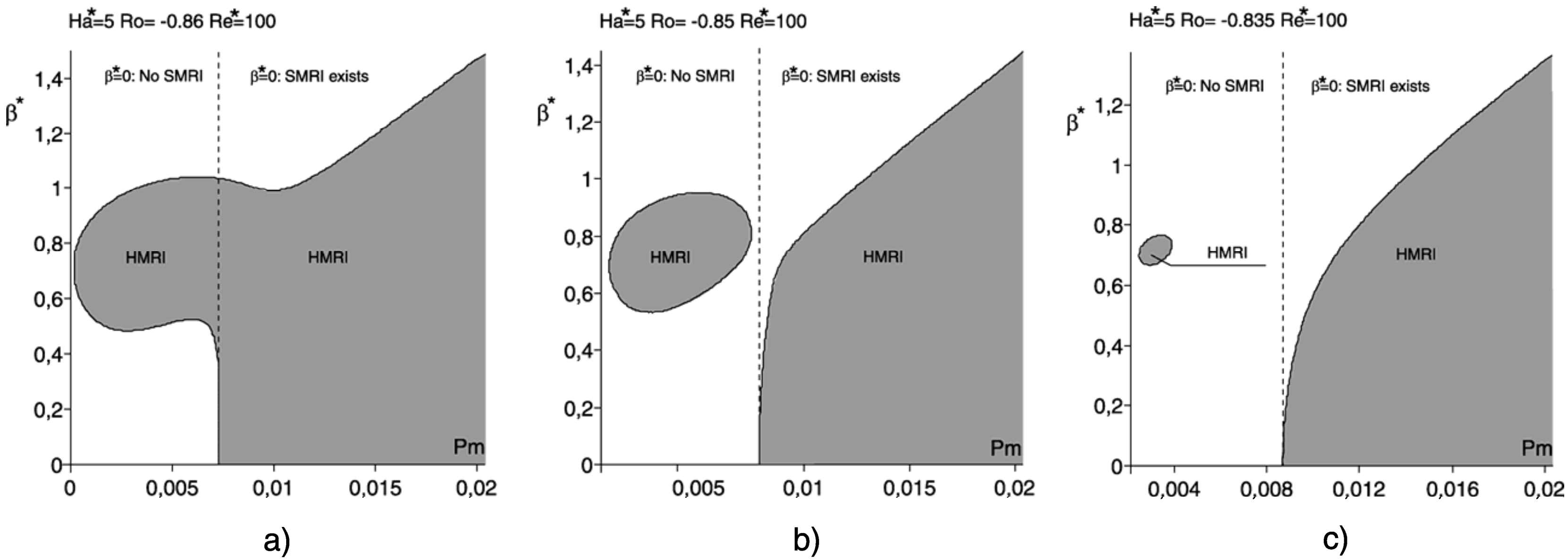
Helical magnetorotational instability (HMRI)



Dashed line: SMRI – threshold for $\beta^* = 0$

$$\mathrm{Pm} = \mathrm{Pm}^c := -\frac{\left(1+\mathrm{Ha}^{*2}\right)^2 + 4\mathrm{Re}^{*2}(1+\mathrm{Ro})}{4\,\mathrm{Ro}\,\mathrm{Re}^{*2}\mathrm{Ha}^{*2}}$$

Left to the dashed line – a (semi-) „island“ of the *essential HMRI*

Right to the dashed line – a „continent“ of the *helically modified SMRI*

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Essential HMRI as a weakly destabilized inertial oscillation

Series expansions of the roots in the vicinity of $\beta^* = 0,\ \mathrm{Pm} = 0$

$$\lambda_{2,4} = -\frac{1}{\sqrt{\mathrm{Pm}}} + \mathrm{Ha}^{*2}\sqrt{\mathrm{Pm}} + o(\mathrm{Pm}^{1/2})$$

$$\lambda_{1,3} = \left[-1 - \mathrm{Ha}^{*2} \pm 2\mathrm{Re}^{*}\sqrt{-(1+\mathrm{Ro})}\right]\sqrt{\mathrm{Pm}} + o(\mathrm{Pm}^{1/2})$$

The roots $\lambda_{1,3}$ are complex for Ro > -1

Frequency of the inertial wave

$$\omega = 2\Omega_0 \frac{k_z}{k}\sqrt{\mathrm{Ro}+1}$$

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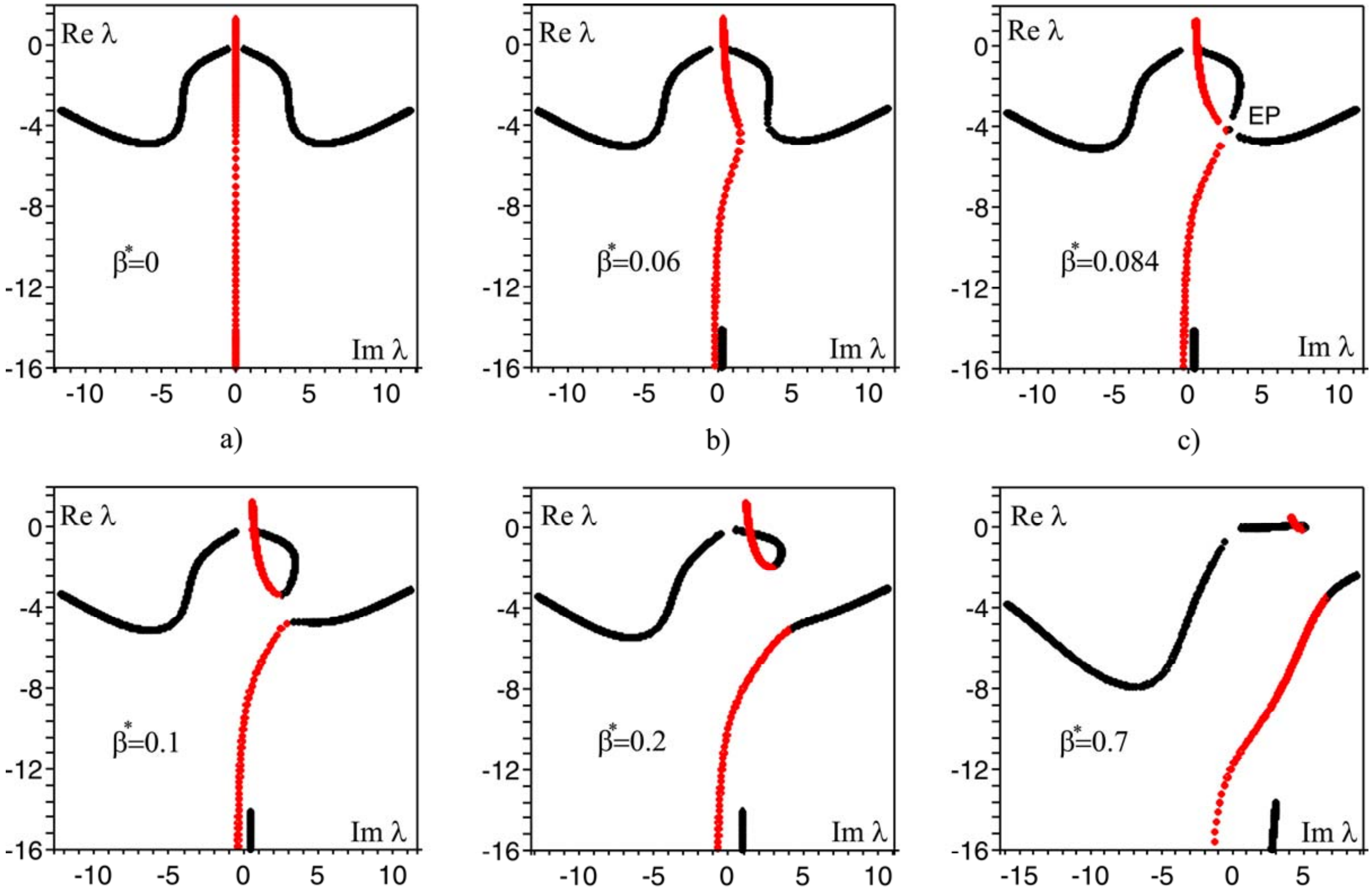
The mechanism of continuous transition from SMRI to HMRI



Inertial wave (black) gets destabilized through an exceptional point

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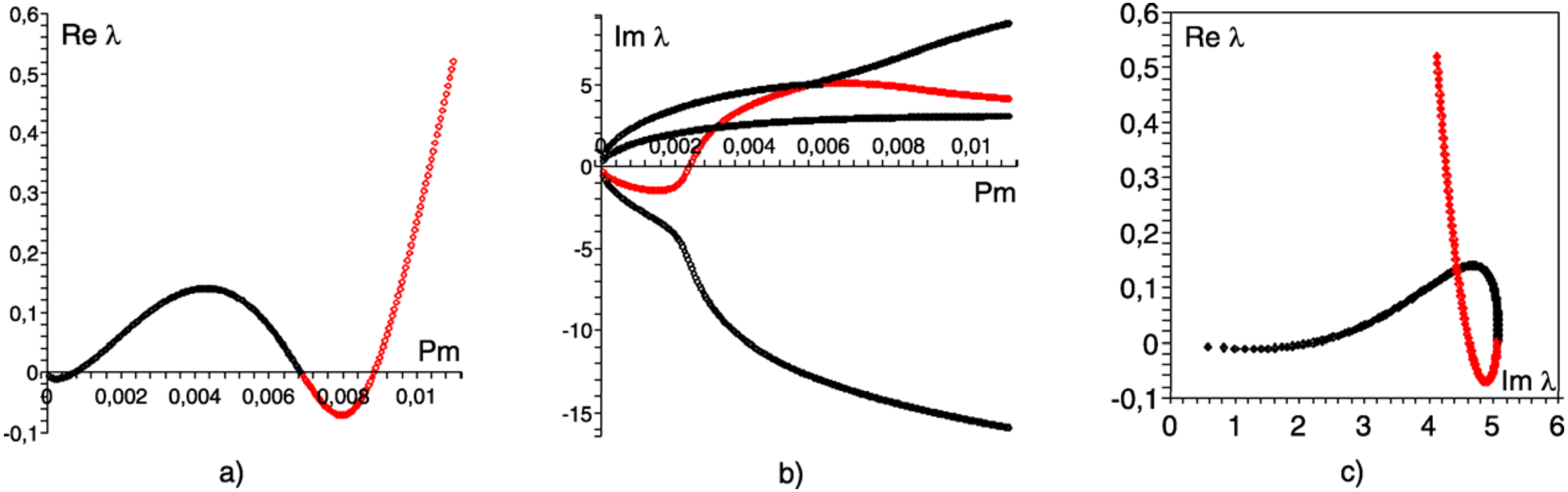
Continuous connection between SMRI and HMRI – a paradox?

[Hollerbach & Rüdiger 2005]: There exist continuous and monotonic connection between SMRI (a destabilized slow magneto-Coriolis wave) and HMRI (a weakly destabilized inertial oscillation, [Liu et al. 2006])



The hidden exceptional point governs transfer of instability between the branch of (helically modified) SMRI and a complex branch of the inertial wave and thus reconciles both findings !

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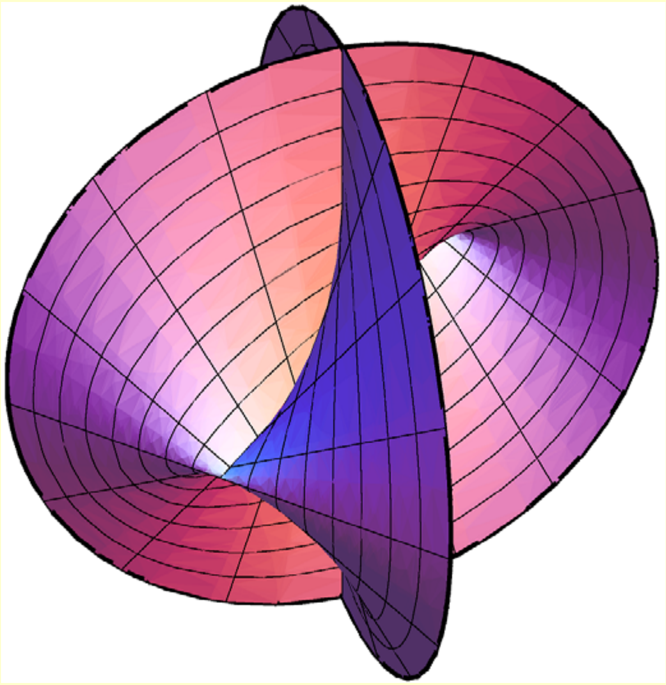
Thank you!



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