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On the stationary flow of viscous incompressible fluids past an obstacle

(Joint work with Hyunseok Kim and Hideo Kozono)

Overview

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Situation:

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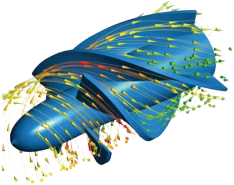
Viscous, incompressible fluid surrounding a moving obstacle.

1. Rotation with constant speed and axis
2. Axis of rotation parallel to translation
3. Translation with constant speed or fluid velocity at ∞ is constant



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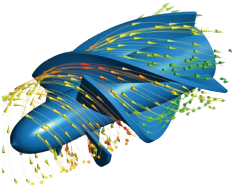

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The governing equations are given by

The Navier-Stokes equations

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$$u_t - \Delta u + u \cdot \nabla u + \nabla \pi = \operatorname{div} F \qquad \text{in } \Omega(t),\ t > 0$$

$$\begin{aligned} \operatorname{div} u &= 0 && \text{in } \Omega(t),\ t > 0 \\ u(x,t) &= Mx && \text{on } \partial\Omega(t),\ t > 0 \\ u(x,t) &\to U && \text{for } |x| \to \infty \end{aligned}$$

u: velocity, π: pressure, "$U = u(\infty)$"

The Navier-Stokes equations

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Transformation $x \rightsquigarrow e^{Mt}x$ and $u \rightsquigarrow e^{Mt}(u - U)$ leads to the system

$$
\begin{aligned}
u_t - \Delta u + u \cdot \nabla u + \nabla \pi &= \operatorname{div} F && \text{in } \Omega \quad, t > 0 \\
+ U \cdot \nabla u - (Mx) \cdot \nabla u + Mu & && \\
\operatorname{div} u &= 0 && \text{in } \Omega \quad, t > 0 \\
u(x,t) &= Mx - U && \text{on } \partial\Omega \quad, t > 0 \\
u(x,t) &\to 0 && \text{for } |x| \to \infty
\end{aligned}
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u: velocity, π: pressure, "$U = u(\infty)$",
Ω exterior domain, e^{Mt}: rotation matrix, i.e. $Mx = \omega \times x$

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$$\begin{aligned} u_t - \Delta u + u \cdot \nabla u + \nabla \pi &= \operatorname{div} F & \text{in } \Omega \quad , t > 0 \\ + U \cdot \nabla u - (Mx) \cdot \nabla u + Mu & & \\ \operatorname{div} u &= 0 & \text{in } \Omega \quad , t > 0 \\ u(x,t) &= Mx - U & \text{on } \partial\Omega \quad , t > 0 \\ u(x,t) &\to 0 & \text{for } |x| \to \infty \end{aligned}$$

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Set

$$b(x) = \frac{1}{2} \operatorname{rot} \left[\eta(x) \left(U \times x - |x|^2 \omega \right) \right]$$

Homogeneous boundary data

Then $(u - b, p)$ satisfies

$$\begin{aligned} u_t - \Delta u + \nabla \pi + (U - Mx) \cdot \nabla u + Mu &= \operatorname{div}(F - Q_b(u)) && \text{in } \Omega, t > 0 \\ \operatorname{div} u &= 0 && \text{in } \Omega, t > 0 \\ u(x,t) &= 0 && \text{on } \partial\Omega, t > 0 \\ u(x,t) &\to 0 && \text{for } |x| \to \infty \end{aligned}$$

with

$$Q_b(u) = (u + b) \otimes (u + b) - \nabla b + (U - Mx) \otimes b + b \otimes (Mx)$$

Homogeneous boundary data

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Then $(u - b, p)$ satisfies
We consider stationary solutions here:

$$\begin{aligned} u_t - \Delta u + \nabla \pi + (U - Mx) \cdot \nabla u + Mu &= \operatorname{div}(F - Q_b(u)) && \text{in } \Omega \\ \operatorname{div} u &= 0 && \text{in } \Omega \\ u(x \quad) &= 0 && \text{on } \partial\Omega \\ u(x \quad) &\to 0 && \text{for } |x| \to \infty \end{aligned}$$

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Known results

- Galdi: Weak and strong solutions, $U = 0$,
  pointwise estimates under decay assumption on the force.
- Silvestre: Strong solutions, $U \neq 0$
- Galdi/Silvestre: Decay estimates of strong solutions, $U \neq 0$
- Farwig/Hishida/Müller: L^p-theory for a model problem with $U = 0$ in $\mathbb{R}^n$
- Hishida: L^p-setting for weak solutions, $U = 0$, $\Omega = \mathbb{R}^3$ or exterior
- Farwig: Strong solutions in L^p, $U \neq 0$
- Farwig/Hishida: Weak solutions in Lorentz spaces, $U = 0$
- Hishida/Shibata: Stability of stationary solutions in weak L^3
- Necasova/Kracmar/Penel: Weak solutions, L^p-setting, $U \neq 0$

Overview

Lorentz spaces

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The critical space $L^3(\Omega)$, is too restrictive.
One possible way is to switch to the class $L_{3,\infty}(\Omega)$.

Definition

$\theta \in (0,1)$, $1 < r < s < \infty$, $q \in [1,\infty]$, $\frac{1}{p} = \frac{(1-\theta)}{r} + \frac{\theta}{s}$

- Lorentz spaces: $L_{p,q}(\Omega) := (L^r(\Omega), L^s(\Omega))_{\theta,q}$
- Homogeneous Sobolev-Lorentz spaces: $\dot{H}^1_{p,q}(\Omega) := (\dot{H}^1_r(\Omega), \dot{H}^1_s(\Omega))_{\theta,q}$
  Note: $C_0^\infty(\Omega)$ is not dense in $\dot{H}^1_{p,\infty}(\Omega)$, set $\widehat{H}^1_{p,q}(\Omega) := \overline{C_0^\infty(\Omega)}$
- negative spaces: $\dot{H}^{-1}_{p,q}(\Omega) := (\widehat{H}^1_{p',q'}(\Omega))^*$

Weak solutions

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Remark

- Sobolev imbedding and real interpolation gives $\dot{H}^1_{3/2,\infty}(\Omega) \subset L_{3,\infty}(\Omega)$
- For $[u] \in \dot{H}^1_{n,1}(\Omega)$ there exists a representative $u \in BC(\Omega)$ with $u(x) \to 0$ as $|x| \to \infty$

Definition

We say that $(u, \pi) \in \dot{H}^1_{p,q}(\Omega) \times L_{p,q}(\Omega)$, $\operatorname{div} u = 0$, is a weak solution if $(\omega \times x - U) \cdot \nabla u - \omega \times u \in \dot{H}^{-1}_{p,q}(\Omega)$ and

$$\langle \nabla u, \nabla \varphi \rangle - \langle (\omega \times x - U) \cdot \nabla u - \omega \times u, \varphi \rangle - \langle Q_b(u), \nabla \varphi \rangle - \langle \pi, \operatorname{div} \varphi \rangle = \langle F, \nabla \varphi \rangle$$

for all $\varphi \in C_0^\infty(\Omega)$

Theorem (H., Kim, Kozono)

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$V_{p,q} := \dot{H}^1_{3/2,\infty}(\Omega) \cap \dot{H}^1_{p,q}(\Omega), \qquad \Pi_{p,q} := L_{\frac{3}{2},\infty}(\Omega) \cap L_{p,q}(\Omega)$

Let $(p, q) = (\frac{3}{2}, \infty)$ or $(p, q) \in (\frac{3}{2}, 3) \times [1, \infty]$.

1. $F \in \Pi_{p,q}$ with $|U| + |\omega| + \|F\|_{3/2,\infty}$ small enough

   $\Rightarrow$ Existence of $(u, \pi) \in V_{p,q} \times \Pi_{p,q}$ with

$$\|u\|_{3,\infty} + \|\nabla u\|_{3/2,\infty} + \|\pi\|_{3/2,\infty} \leq C \left(|U| + |\omega| + \|F\|_{3/2,\infty}\right)$$

$$\|u\|_{p^*,q} + \|\nabla u\|_{p,q} + \|\pi\|_{p,q} \leq C \left(|U| + |\omega| + \|F\|_{p,q}\right)$$

   (here: $p^* = 3p/(3 - p)$)

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   (here: $p^* = 3p/(3-p)$)

2. If $(p,q) \in (\frac{3}{2}, 3) \times [1, \infty]$ the solution is unique
3. If $(p,q) = (\frac{3}{2}, \infty)$ there exists η such that

$$\|u\|_{3,\infty} \leq \eta \quad \Rightarrow \quad \text{Uniqueness}$$

Overview

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The linear problem

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Set $Lu = -\Delta u + (U - \omega \times x) \cdot \nabla u + \omega \times u$, and consider

$$\begin{aligned} Lu + \nabla \pi &= f \qquad \text{in } \Omega \\ \operatorname{div} u &= g \qquad \text{in } \Omega \\ u &= 0 \qquad \text{on } \partial\Omega \end{aligned} \tag{1}$$

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Proposition

(i) $p = \frac{3}{2},\ q = \infty$ *or* *(ii)* $\frac{3}{2} < p < 3,\ 1 \leq q \leq \infty$ *or* *(iii)* $p = 3,\ q = 1$.

$f \in \dot{H}^{-1}_{p,q}(\Omega)$ *and* $g \in L_{p,q}(\Omega)$ *with* $(U - \omega \times x)g \in \dot{H}^{-1}_{p,q}(\Omega)$,
then there exists a unique weak solution $(u, \pi) \in \dot{H}^{1}_{p,q}(\Omega) \times L_{p,q}(\Omega)$ *of* (1).
Moreover, for $|U| + |\omega| \leq M < \infty$

$$\|u\|_{p^*,q} + \|\nabla u\|_{p,q} + \|\pi\|_{p,q} \leq C \left(\|f\|_{-1,p,q} + \|g\|_{p,q} + \|(U - \omega \times x)g\|_{-1,p,q} \right). \tag{2}$$

Proof of the proposition

- Existence is proved by localization using corresponding results in $\mathbb{R}^3$ and bounded domains.
- The uniqueness follows from

Lemma

Let (p_i, q_i), $i = 1, 2$ *satisfy (i), (ii) or (iii). Let* $s > 3/2$.
$f = \operatorname{div} F \in L_2(\Omega)$ *with* $(1 + |x|^2)\, F \in L_\infty(\Omega)$.
Then a weak solution $u \in \dot{H}^1_{p_1,q_1}(\Omega) + \dot{H}^1_{p_2,q_2}(\Omega)$, $\pi \in L_{p_1,q_1}(\Omega) + L_{p_2,q_2}(\Omega)$ *satisfies*

$$\boxed{u \in \dot{H}^1_2(\Omega),} \quad \boxed{(1 + |x|)\, u \in L_\infty(\Omega),} \quad \boxed{\nabla^2 u \in L_2(\Omega),} \quad \boxed{\pi \in H^1_2(\Omega) \cap L_s(\Omega)}$$

and

$$\int_\Omega |\nabla u|^2 \, dx = -\int_\Omega F \cdot \nabla u \, dx. \tag{3}$$

It thus follows that if $f \equiv 0$, *then* $(u, \pi) = (0, 0)$ *in* Ω.

The nonlinear problem: proof of the main result

Existence: For $u \in V_{p,q}$ define $v = T(u)$ where

$$\begin{aligned} Lv + \nabla\pi &= \operatorname{div}(F - Q_b(u)) \quad \text{in } \Omega \\ \operatorname{div} v &= 0 \quad \text{in } \Omega \\ v &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

The nonlinear problem: proof of the main result

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Estimates for the linear problem yields

$$\begin{aligned} \|\nabla T(u)\|_{3/2,\infty} &\leq 2C\left(|U| + |\omega| + \|F\|_{3/2,\infty}\right) \\ \|\nabla T(u)\|_{V_{p,q}} &\leq 2C\left(|U| + |\omega| + \|F\|_{p,q}\right) \\ \|T(u_1) - T(u_2)\|_{V_{p,q}} &\leq C\left(\|\nabla u_1\|_{3/2,\infty} + \|\nabla u_2\|_{3/2,\infty}\right)\|u_1 - u_2\|_{V_{p,q}} \end{aligned}$$

provided $\|\nabla u\|_{3/2,\infty} \leq 2C\left(|U| + |\omega| + \|F\|_{3/2,\infty}\right)$
and $\|\nabla u\|_{p,q} \leq 2C\left(|U| + |\omega| + \|F\|_{p,q}\right)$.
Then use contraction mapping principle

Proof of the main result

Uniqueness:
Let (u_1, π_1), (u_2, π_2) be two solutions. Then for $(u, \pi) := (u_1 - u_2, \pi_1 - \pi_2)$

$$Lu + \nabla\pi = \operatorname{div} G$$

where $G = Q_b(u_2) - Q_b(u_1) = u \otimes (u_1 + b) + (u_2 + b) \otimes v$.

A bootstrap argument with $V_{p,q} \subset V_{p,\infty} \subset L_{3,\infty}(\Omega) \cap L_{p^*,\infty}(\Omega)$ let us find

$$(u, \pi) \in \dot{H}_2^1(\Omega) \times L_2(\Omega)$$

Estimating G finally yields

$$||\nabla u||_2^2 \leq C_0 \left(|U| + |\omega| + ||F||_{3/2,\infty}\right) ||\nabla u||_2^2$$

$\rightsquigarrow ||\nabla u||_2^2 = 0.$ (if $|U| + |\omega| + \|F\|_{3/2,\infty}$ is small)