Global solvability of the rotating Navier-Stokes-Boussinesq equation with stratification effect with decaying initial data

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Geophysical fluid dynamics

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- Geophysical fluids: large-scale fluids as the earth's atmosphere, oceans and climate.

Geophysical fluid dynamics

- Fluids: gases and liquids as water and air.
- Geophysical fluids: large-scale fluids as the earth's atmosphere, oceans and climate.
- Features of geophysical fluids: Rotation and Stratification.
- Rotation: the earth's rotation (Coriolis force).
- Stratification: Density and Temperature.

Purpose and Goal

Purpose

To study the influence of rotational and stratification effects.

Goal

To construct a global-in-time unique smooth solution of a equation governing geophysical fluids (The equation is a rotating Navier-Stokes Boussinesq equation with stratification effects).

Coriolis force

- Rotating Navier-Stokes equations:

$$(RNS)\begin{cases}\partial_t u - \nu\Delta u + (u,\nabla)u + \nabla p = -\Omega e_3 \times u,\\ \nabla\cdot u = 0,\end{cases}$$

where $\Delta = \partial_1^2 + \partial_2^2 + \partial_3^2$, $\nabla = {}^t(\partial_1, \partial_2, \partial_3)$, $\times$: outer product,
$u = {}^t(u^1, u^2, u^3)$: the velocity,
p : the pressure,
$\nu > 0$: the viscosity coefficient ,
$e_3 = {}^t(0, 0, 1)$: the axis of earth,
Ω: the rotation rate(rotation parameter),
$\nabla\cdot u = 0$: incompressibility condition.

Stratification effects

- Navier-Stokes Boussinesq-type equations:

$$\begin{cases} \partial_t u - \nu \Delta u + (u, \nabla) u + \nabla p = g e_3 \theta, \\ \partial_t \theta - \kappa \Delta \theta + (u, \nabla)\theta = -N^2 u^3, \end{cases}$$

where θ: the temperature (distribution),
$\kappa > 0$: the heat diffusion rate,
$g > 0$: gravity,
N: Brunt-Väisälä frequency(stratification parameter).

Our system

- A rotating Navier-Stokes-Boussinesq equation with stratification effects:

$$
(S)\begin{cases}
\partial_t u - \nu\Delta u + (u,\nabla)u + \Omega d \times u + \nabla p & = g e_3 \theta, \\
\partial_t \theta - \kappa\Delta\theta + (u,\nabla)\theta = -N^2 u^3, (t > 0, & x \in \mathbb{R}^3), \\
\nabla \cdot u = 0, & x \in \mathbb{R}^3, \\
u|_{t=0} = {}^t(u_0^1, u_0^2, u_0^3), & x \in \mathbb{R}^3, \\
\theta|_{t=0} = \theta_0, & x \in \mathbb{R}^3.
\end{cases}
$$

- References: B.Cushman-Roisin, “Introduction to geophysical fluid dynamics”, Prentice Hall, 1994.
  J.Pedolosky, “Geophysical Fluid Dynamics”, volume 2nd edition. Springer verlag,1987.

Preliminary results

- Global solvability of (RNS);

  Giga-Inui-Mahalov-Saal2008, Hieber-Shibata2009:
  The case when initial data is sufficiently small.

  Chemin-Desjardins-Gallagher-Grenier2006:
  The case when initial data is not restricted, but $\Omega >> 1$.

Formulation

- $v = {}^t(v^1, v^2, v^3, v^4) := {}^t(u^1, u^2, u^3, \frac{\sqrt{g}}{N}\theta)$

$$(S)\begin{cases} \dfrac{dv}{dt} + \widetilde{P}(A+B)v = -\widetilde{P}(v,\nabla)v, \\ v|_{t=0} = \widetilde{P}v_0, \end{cases}$$

$$v_0 = {}^t(v_0^1, v_0^2, v_0^3, v_0^4) := {}^t(u_0^1, u_0^2, u_0^3, \tfrac{\sqrt{g}}{N}\theta_0),$$

$$A = \begin{pmatrix} -\nu\Delta & 0 & 0 & 0 \\ 0 & -\nu\Delta & 0 & 0 \\ 0 & 0 & -\nu\Delta & 0 \\ 0 & 0 & 0 & -\kappa\Delta \end{pmatrix}, B = \begin{pmatrix} 0 & -\Omega & 0 & 0 \\ \Omega & 0 & 0 & 0 \\ 0 & 0 & 0 & -N\sqrt{g} \\ 0 & 0 & N\sqrt{g} & 0 \end{pmatrix},$$

$\widetilde{P} = P \oplus 1$ with P: a Helmholtz projection,
i.e. $\widetilde{P} : [L^p]^4 \ni v \mapsto \widetilde{P}v \in [L^p]^4$ with $\nabla \cdot (\widetilde{P}v) = 0$.

Main result

Theorem

Let $\nu, \kappa, \Omega, N, g > 0$ such that $\nu = \kappa$. Assume that $u_0 \in \dot{H}^{\frac{1}{2}}_\sigma$ and $\theta_0 \in \dot{H}^{\frac{1}{2}}$ with $\partial_2 u_0^1 - \partial_1 u_0^2 = 0$ in $\mathbb{R}^3$. Then there exists a positive number $M_0 = M_0(\nu, \|u_0\|_{\dot{H}^{\frac{1}{2}}}, \|\theta_0\|_{\dot{H}^{\frac{1}{2}}})$ such that there exists a global-in-time unique solution (u, θ) to the system (S) with initial data (u_0, θ_0),

$$\begin{cases} (u, \theta) \in C([0, \infty); \dot{H}^{\frac{1}{2}}_\sigma \times \dot{H}^{\frac{1}{2}}), \\ (\nabla u, \nabla \theta) \in L^2(\mathbb{R}_+ : \dot{H}^{\frac{1}{2}}), \end{cases}$$

for $N\sqrt{g} > M_0$ with $\Omega < N\sqrt{g}$.

$$\dot{H}^s(\mathbb{R}^3) = \{u \in \mathcal{S}' : \|u\|_{\dot{H}^s} \equiv \|(-\Delta)^{\frac{s}{2}} u\|_{L^2} < \infty\},$$

$$\dot{H}^s_\sigma(\mathbb{R}^3) = \{u \in \dot{H}^s : \xi_1 \hat{u}^1 + \xi_2 \hat{u}^2 + \xi_3 \hat{u}^3 = 0\}.$$

Outline the proof

- Linearized system:

$$(LS)\begin{cases}\dfrac{dw}{dt}+\widetilde{P}(A+B)w=0,\\ w|_{t=0}=w_0=\widetilde{P}\mathcal{P}_{r,R}v_0.\end{cases}$$

- Perturbation system: Set $\delta := v - w$,

$$(PS)\begin{cases}\dfrac{d\delta}{dt}+\widetilde{P}(A+B)\delta=-\widetilde{P}(w,\nabla)w-\widetilde{P}(\delta,\nabla)w\\ \qquad\qquad\qquad\qquad -\widetilde{P}(w,\nabla)\delta-\widetilde{P}(\delta,\nabla)\delta,\\ \delta|_{t=0}=\delta_0=\widetilde{P}(I-\mathcal{P}_{r,R})v_0,\end{cases}$$

I is an 4×4 identity matrix.

Fourier cut-off operators $\mathcal{P}_{r,R}$

Definition

Let $w \in \dot{H}^s (s \in \mathbb{R})$. The operators $\mathcal{P}_{r,R}$ are called Fourier cut-off operators if
(1) supp $\widehat{\mathcal{P}_{r,R} w} \subset D_{r,R}$,
(2)$\mathcal{P}_{r,R} w \to w$ in $\dot{H}^s$ as $r \to 0$ and $R \to \infty$,
with $D_{r,R} := \{\xi \in \mathbb{R}^3; |\xi| \le R, |\xi_3| \ge r, |\xi_h| \ge r\}$.
Here $\xi = (\xi_1, \xi_2, \xi_3)$ and $\xi_h = (\xi_1, \xi_2)$.

Remark

By the definition of $\mathcal{P}_{r,R}$, we see that
(3)$\|\mathcal{P}_{r,R} w\|_{\dot{H}^{s'}} \le C_{r,R,s,s'} \|w\|_{\dot{H}^s}$ for any $s, s' \in \mathbb{R}$,
where $C_{r,R,s,s'}$ does not depend on w.

A priori estimate

- Multiplying (PS) by the solution δ to (PS), we obtain

$$\frac{1}{2}\|\delta(t)\|_{\dot{H}^{\frac{1}{2}}}^2 + \nu\int_0^t \|\nabla\delta(s)\|_{\dot{H}^{\frac{1}{2}}}^2 ds = \frac{1}{2}\|\delta_0\|_{\dot{H}^{\frac{1}{2}}}^2 - \int_0^t ((w,\nabla)w + (\delta,\nabla)w + (w,\nabla)\delta + (\delta,\nabla)\delta, \delta)_{\dot{H}^{\frac{1}{2}}}(s)ds.$$

To use absorbing argument, we need estimates of the solution w of (LS).
(1)Energy inequality of (LS).
(2)Strichartz-type estimates of (LS).

Energy estimate on $[T_0, \infty)$

Proposition (A)

Let $\varepsilon, \nu, r, R, \Omega, N, g$ be any positive numbers such that $r < R$. Assume that $v_0 \in \dot{H}^{\frac{1}{2}}_{\sigma} \times \dot{H}^{\frac{1}{2}}$ and w be the solution of (LS) with initial data $\mathcal{P}_{r,R} v_0$. Then there exist positive constants $r_0 = r_0(\varepsilon, \nu, \|v_0\|_{\dot{H}^{\frac{1}{2}}})$, $R_0 = R_0(\varepsilon, \nu, \|v_0\|_{\dot{H}^{\frac{1}{2}}}, r_0)$ and $T_0 = T_0(\varepsilon, \nu, \|v_0\|_{\dot{H}^{\frac{1}{2}}}, r_0, R_0)$ such that for any fixed $r < r_0$ and $R > R_0$

$$\frac{1}{2} \int_{T_0}^{\infty} \|\nabla w(\tau)\|^2_{\dot{H}^{\frac{1}{2}}} d\tau < \varepsilon.$$

Strichartz-type estimate on $[0, T_0]$

Proposition (B)

For any positive numbers $(\nu, T, r, R, \Omega, N, g)$ such that $r < R$ and $\Omega < N\sqrt{g}$. Let $v_0 \in \dot{H}^{\frac{1}{2}}_{\sigma} \times \dot{H}^{\frac{1}{2}}$ and w be the solution of (LS) with initial data $\mathcal{P}_{r,R} v_0$. Then there exist positive constants $C_1 = C_1(\nu, r, R)$ and $C_2 = C_2(\nu, r, R, g)$ such that then for all $p \in [1, \infty]$,

$$\|w\|_{L^p([0,T];L^\infty(\mathbb{R}^3))} \le M_1^{\frac{1}{p}} \|w_0\|_{\dot{H}^{\frac{1}{2}}}^{1-\frac{1}{p}},$$

with

$$M_1 := M_1(\nu, r, R, T, N, g, w_0)$$
$$:= C_1 T^{\frac{1}{4}} (\|\partial_2 w_0^1 - \partial_1 w_0^2\|_{L^2} + \|w_0^4\|_{\dot{H}^{\frac{1}{2}}}) + \frac{C_2}{N^{\frac{1}{4}}} \|w_0\|_{\dot{H}^{\frac{1}{2}}}.$$

What is Strichartz estimate?

- Strichartz estimate:
  a basic estimate on dispersive equations.

- The symbol of (LS):

$$\widehat{w} = [M_1 e^{-\nu|\xi|^2 t} \cos \alpha t + M_2 e^{-\nu|\xi|^2 t} \sin \alpha t + M_3 e^{-\nu|\xi|^2 t}]\widehat{w}_0,$$

$$\alpha = \frac{\sqrt{N^2 g \xi_1^2 + N^2 g \xi_2^2 + \Omega^2 \xi_3^2}}{\sqrt{\xi_1^2 + \xi_2^2 + \xi_3^2}}, \ M_j : 4 \times 4 \ \textit{coefficient matrix}.$$

Our Strichartz-type estimate

- The case when initial data with no condition: $(1 \leq p \leq \infty)$

$$\|w\|_{L^\infty([0,T];L^p(\mathbb{R}^3))} \leq C\Big(\frac{1}{N^\alpha} + T^\beta\Big)\|w_0\|_{\dot{H}^{\frac{1}{2}}},$$

$C = C(r, R), \alpha = \alpha(p), \beta = \beta(p) \geq 0.$

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- The case when initial data satisfies the condition: $\partial_2 w_0^1 - \partial_1 w_0^2 = 0 \ \ (1 \leq p \leq \infty)$

$$\|w\|_{L^\infty([0,T];L^p(\mathbb{R}^3))} \leq C\left(\frac{1 + T^\beta}{N^\alpha}\right)\|w_0\|_{\dot{H}^{\frac{1}{2}}},$$

$C = C(r,R), \alpha = \alpha(p), \beta = \beta(p) \geq 0.$

Conclusion

- For all $T' > 0$, we have

$$E(T') := \|\delta(T')\|^2_{\dot{H}^{\frac{1}{2}}} + \int_0^{T'} \|\nabla\delta(s)\|^2_{\dot{H}^{\frac{1}{2}}} ds$$

$$\leq C(T')(\|(I - \mathcal{P}_{r,R})v_0\|^2_{\dot{H}^{\frac{1}{2}}} + \|w\|_{L^\infty([0,T_0];L^2)} + \int_{T_0}^{\infty} \|\nabla w(s)\|^2_{\dot{H}^{\frac{1}{2}}} ds),$$

with $C(T') = C(1 + E(T'))$, $C = C(\|v_0\|_{\dot{H}^{\frac{1}{2}}}) > 0$.

- Let ε be any positive number fixed.
  By the definition of $\mathcal{P}_{r,R}$, we choose $r_0 << 1$ and $R_0 >> 1$.
  By Proposition(A), we can take $T_0 >> 1$.
  Finally, Proposition(B), we take $N >> 1$.
  Then we get the desired result:

$$\|\delta\|_{L^\infty([0,\infty);\dot{H}^{\frac{1}{2}})} + \|\nabla\delta\|_{L^2(\mathbb{R}_+:\dot{H}^{\frac{1}{2}})} < \varepsilon.$$