Non-autonomous Ornstein-Uhlenbeck type operators in exterior domains

Joint work with A. Rhandi (Salerno)

Tobias Hansel
IRTG 1529, TU Darmstadt

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Motivation:

The Navier-Stokes flow in the exterior of a rotating obstacle

Setup:

- $K \subset \mathbb{R}^3$ a compact obstacle
- the exterior of the obstacle $\Omega := \mathbb{R}^3 \setminus K$ filled with a viscous incompressible fluid
- K is rotating with prescribed time-dependent angular velocity:

$$m \in C([0,\infty); \mathbb{R}^{3\times 3}) \quad \text{with} \quad m(t)^{\mathsf{T}} = -m(t) \quad \text{for all} \quad t \geq 0$$

- exterior of the rotated obstacle at time t: $\Omega(t) := Q(t)\Omega$, where $Q(t)$ solves

$$\begin{cases} \partial_t Q(t) &= m(t)Q(t), \quad t > 0, \\ Q(0) &= \mathrm{Id}. \end{cases}$$

- fluid flow subject to an outflow condition: fluid has a prescribed velocity $v_\infty \in \mathbb{R}^3$ far away from the obstacle

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The equations for the fluid: Navier-Stokes Equations

$$\begin{cases} v_t - \Delta v + v \cdot \nabla v + \nabla q &= 0 & \text{in } \mathbb{R}_+ \times \Omega(t), \\ \operatorname{div} v &= 0 & \text{in } \mathbb{R}_+ \times \Omega(t), \\ v(t,y) &= m(t)y & \text{in } \mathbb{R}_+ \times \partial\Omega(t), \\ \lim_{|y|\to\infty} v(t,y) &= v_\infty \neq 0 & \text{for } t \in \mathbb{R}_+, \\ v(0,y) &= v_0(y) & \text{in } \Omega. \end{cases}$$

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Transformation: $x = Q(t)^{\mathsf{T}} y, \quad u(t,x) = Q(t)^{\mathsf{T}}(v(t,y) - v_\infty), \quad p(t,x) = q(t,y)$

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$$\begin{cases} \left.\begin{array}{r} u_t - \Delta u - (M(t)x) \cdot \nabla u + M(t)u \\ + \left(Q(t)^{\mathsf{T}} v_\infty\right) \cdot \nabla u + u \cdot \nabla u + \nabla p \end{array}\right\} &= 0 & \text{in } \mathbb{R}_+ \times \Omega, \\ \operatorname{div} u &= 0 & \text{in } \mathbb{R}_+ \times \Omega, \\ u(t,x) &= M(t)x - Q(t)^{\mathsf{T}} v_\infty & \text{in } \mathbb{R}_+ \times \partial\Omega, \\ \lim_{|x|\to\infty} u(t,x) &= 0 & \text{for } t \in \mathbb{R}_+, \\ u(0,x) &= v_0(x) & \text{in } \Omega, \end{cases}$$

where $M(t) := Q(t)^{\mathsf{T}} m(t) Q(t)$.

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Linearization leads to operators of the form

$$A(t)u(x) := \mathbb{P}\left(\Delta u + (M(t)x - Q(t)^{\mathsf{T}} v_\infty) \cdot \nabla u - M(t)u\right), \qquad t > 0,\ x \in \Omega.$$

Some known results:

- Autonomous case $M(t) \equiv M$ and $v_\infty = 0$
  - Hishida '99 (L^2-theory)
  - Geissert, Heck, Hieber '06 (L^p-theory)
  - Hishida, Shibata '09
- Autonomous case $M(t) \equiv M$ and $Mv_\infty = 0$
  - Shibata '08
- Non-autonomous case
  - Hishida '01 (L^2-theory and $M(t) = (0, 0, \omega_0(t)) \times \quad$)

Ornstein-Uhlenbeck operators

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Consider Ornstein-Uhlenbeck type operators (for functions $\varphi : \mathbb{R}^d \to \mathbb{R}$)

$$\mathcal{L}_\Omega(t)\varphi(x) = \Delta\varphi(x) + \langle M(t)x + c(t), \nabla\varphi(x)\rangle, \quad x \in \Omega, \quad t > 0,$$

The parabolic problem with Dirichlet boundary conditions

$$\begin{cases} u_t(t,x) - \mathcal{L}_\Omega(t)u(t,x) &= 0, \qquad t \in (s,\infty),\ x \in \Omega, \\ u(t,x) &= 0, \qquad t \in (s,\infty),\ x \in \partial\Omega, \\ u(s,x) &= f(x), \quad x \in \Omega. \end{cases}$$

Interesting case: $\Omega \subset \mathbb{R}^d$ is an exterior domain, i.e. $\Omega := \mathbb{R}^d \setminus K$ and $\partial\Omega$ smooth

Assumptions:

- $M \in C^\alpha_{loc}(\mathbb{R}_+, \mathbb{R}^{d\times d})$, $c \in C^\alpha_{loc}(\mathbb{R}_+, \mathbb{R}^d)$ for some $\alpha \in (0,1)$

Non-autonomous abstract Cauchy problem

Set

$$\mathcal{D}(L_\Omega(t)) := \{u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) : \langle M(t)x, \nabla u(x)\rangle \in L^p(\Omega)\},$$
$$L_\Omega(t)u := \mathcal{L}_\Omega(t)u.$$

Non-autonomous abstract Cauchy Problem:

$$(CP) \qquad u'(t) = L_\Omega(t)u(t), \quad 0 \le s < t, \qquad u(s) = f \in L^p(\Omega),$$

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$$(CP) \qquad u'(t) = L_\Omega(t)u(t), \quad 0 \le s < t, \qquad u(s) = f \in L^p(\Omega),$$

Definition. $T_\Omega(\cdot,\cdot) \in \mathscr{L}(L^p(\Omega))$ is an evolution system if

- $T_\Omega(s,s) = \mathrm{Id}$, and $T_\Omega(t,s) = T_\Omega(t,r)T_\Omega(r,s)$,
- $(t,s) \mapsto T_\Omega(t,s)$ is strongly continuous, for $0 \le s \le r \le t$.

(CP) well-posed if there exists an evolution system $T_\Omega(\cdot,\cdot)$ and dense subspaces Y_s of $L^p(\Omega)$ such that $T_\Omega(t,s)Y_s \subset Y_t \subset \mathcal{D}(L_\Omega(t))$ for $0 \le s \le t$ and the function $u(t) := T_\Omega(t,s)f$ is a solution of (CP) for $f \in Y_s$. We say $L_\Omega(\cdot)$ generates $T_\Omega(\cdot,\cdot)$.

Non-autonomous abstract Cauchy problem

Well known: For fixed t, $(L_\Omega(t), \mathcal{D}(L_\Omega(t))$ generates C_0-semigroup (not analytic !)

Classical generation results for non-autonomous problems:

- Parabolic evolution systems
  $\rightsquigarrow$ generators of analytic semigroups
- Hyperbolic evolution systems
  $\rightsquigarrow$ stable family of generators

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Strategy:

- Study the whole space $\mathbb{R}^d$
  $\rightsquigarrow$ use representation formula (Da Prato/Lunardi '07, Geissert/Lunardi '08)
- Study bounded domains D
  $\rightsquigarrow$ apply standard results for parabolic problems
- Use some cut-off technique for exterior domain

Whole Space Case

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Lemma

Let $U(\cdot,\cdot)$ be the evolution system in $\mathbb{R}^d$ that satisfies $\partial_t U(t,s) = -M(t)U(t,s)$.
For $t > s$ set

$$T_{\mathbb{R}^d}(t,s)f(x) = \frac{1}{(4\pi)^{\frac{d}{2}}(\det Q_{t,s})^{\frac{1}{2}}} \int_{\mathbb{R}^d} f(U(s,t)x + g(t,s) - y) e^{-\frac{1}{4}\langle Q_{t,s}^{-1}y,y\rangle} \mathrm{d}y,$$

$$\text{where} \quad g(t,s) = \int_s^t U(s,r)c(r)\mathrm{d}r \quad \text{and} \quad Q_{t,s} = \int_s^t U(s,r)U^*(s,r)\mathrm{d}r.$$

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- $L_{\mathbb{R}^d}(\cdot)$ generates $T_{\mathbb{R}^d}(\cdot,\cdot)$.
- Let $T > 0$, $1 < p \leq q < \infty$ and $\beta \in \mathbb{N}_0^d$. There exists $C := C(T) > 0$ s.t.
  - $\|T_{\mathbb{R}^d}(t,s)f\|_q \leq C(t-s)^{-\frac{d}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}\|f\|_p, \qquad 0 \leq s < t < T,$
  - $\|\partial_x^\beta T_{\mathbb{R}^d}(t,s)f\|_p \leq C(t-s)^{-\frac{|\beta|}{2}}\|f\|_p, \qquad 0 \leq s < t < T.$

Bounded Domains

Lemma

Let $D \subset \mathbb{R}^d$ be a bounded domain with $C^{1,1}$-boundary.

- $L_D(\cdot)$ generates an evolution system $T_D(\cdot,\cdot)$.
- Let $T > 0$, $1 < p < \infty$ and $p \leq q < \infty$. There exists $C := C(T) > 0$ s.t.
  - $\|T_D(t,s)f\|_q \leq C(t-s)^{-\frac{d}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}\|f\|_p, \qquad 0 \leq s < t < T,$
  - $\|\nabla T_D(t,s)f\|_p \leq C(t-s)^{-\frac{1}{2}}\|f\|_p, \qquad 0 \leq s < t < T,$
  - $\|\nabla^2 T_D(t,s)f\|_p \leq C(t-s)^{-1}\|f\|_p, \qquad 0 \leq s < t < T.$

Cut-off technique

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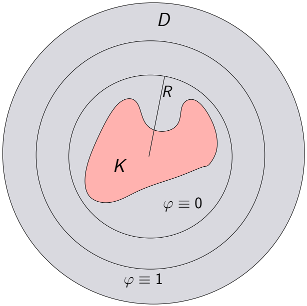
- Consider exterior domain $\Omega = \mathbb{R}^d \setminus K$.
- Let $R > 0$ s.t. $K \subset B(R)$.
- Set $D := \Omega \cap B(R+3)$.
- Consider a cut-off fct. $\varphi \in C^\infty(\Omega)$ s.t. $0 \leq \varphi \leq 1$ and

$$\varphi(x) := \begin{cases} 1, & |x| \geq R+2, \\ 0, & |x| \leq R+1, \end{cases}$$



- For $f \in L^p(\Omega)$ set $W(t,s)f = \varphi T_{\mathbb{R}^d}(t,s)f_0 + (1-\varphi)T_D(t,s)f_D$.

Cut-off technique

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Short calculation: $W(t,s)f$ solves inhomogeneous problem

$$\begin{cases} \partial_t W(t,s)f &= L_\Omega(t)W(t,s)f - F(t,s)f, \quad 0 \le s < t, \\ W(s,s)f &= f, \end{cases}$$

with error terms

$$F(t,s)f = 2\nabla\varphi\left(\nabla T_{\mathbb{R}^d}(t,s)f_0 + \nabla T_D(t,s)f_D\right) + \left(\Delta\varphi + (M(t)x + c(t))\nabla\varphi\right)\left(T_{\mathbb{R}^d}(t,s)f_0 + T_D(t,s)f_D\right).$$

Estimate for error terms:

$$\|F(t,s)f\|_p \le C(t-s)^{-\frac{1}{2}}\|f\|_p, \qquad 0 \le s < t < T,$$

for every $T > 0$.

Cut-off technique

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Consider the integral equation (Variation of constant formula)

$$T_\Omega(t,s)f = W(t,s)f + \int_s^t T_\Omega(t,r)F(r,s)f\,dr, \quad t \geq s \geq 0,\ f \in L^p(\Omega).$$

Solution to integral equation:

$$T_\Omega(t,s)f = \sum_{k=0}^{\infty} T_k(t,s)f$$

where $T_0(t,s)f = W(t,s)f$ and

$$T_{k+1}(t,s)f = \int_s^t T_k(t,r)F(r,s)f\,\mathrm{d}r.$$

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$$T_{k+1}(t,s)f = \int_s^t T_k(t,r)F(r,s)f\mathrm{d}r.$$

(Technical) calculation yields:

$$\partial_t T_\Omega(t,s)f = L_\Omega(t)T_\Omega(t,s)f \qquad \text{and} \qquad \partial_s T_\Omega(t,s)f = -T_\Omega(t,s)L_\Omega(s)f$$

for "nice" functions f.

Result:

Well-posedness and L^p-estimates in exterior domains

Theorem

Let $\Omega \subset \mathbb{R}^d$ be an exterior domain with $C^{1,1}$-boundary and $1 < p < \infty$.

- $L_\Omega(\cdot)$ generates $T_\Omega(\cdot,\cdot)$.
- Let $T > 0$, $1 < p < \infty$ and $p \leq q < \infty$. There exists $C := C(T) > 0$ s.t.
  - $\|T_\Omega(t,s)f\|_{L^q(\Omega)} \leq C(t-s)^{-\frac{d}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}\|f\|_{L^p(\Omega)}, \quad 0 \leq s < t < T,$
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