On the Navier-Stokes equation with Robin boundary condition in a perturbed half space

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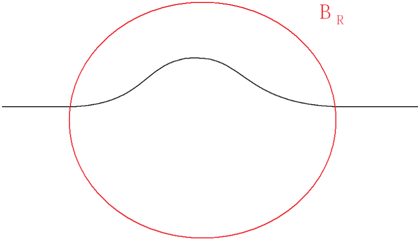
$$\mathbb{R}^n_+ = \{x \in \mathbb{R}^n \mid x_n > 0\}.$$

$$B_R = \{x \in \mathbb{R}^n \mid |x| < R\}.$$

A perturbed half space Ω

There exists $R > 0$ such that $\Omega \setminus B_R = \mathbb{R}^n_+ \setminus B_R$.



Navier-Stokes equation

$$\begin{cases} u_t - \Delta u + u \cdot \nabla u + \nabla p = 0 & \text{in } \Omega \times (0,\infty), \\ \nabla \cdot u = 0 & \text{in } \Omega \times (0,\infty), \\ u \cdot \nu = 0,\ B_{\alpha,\beta}(u,p) = 0 & \text{on } \partial\Omega \times (0,\infty), \\ u(x,0) = a(x) & \text{in } \Omega. \end{cases}$$

$u = (u_1, \cdots, u_n)$: Velosity, p : Pressure,
ν : A unit outer normal vector of $\partial\Omega$,
a : A initial value,
Robin boundary condition,
Ω : A perturbed half space with smooth boundary $\partial\Omega$,
Space dimension $n \geq 3$.

Robin boundary condition

$$u\cdot\nu = 0,\ B_{\alpha,\beta}(u,p) = \alpha u + \beta\{T(u,p)\nu - (T(u,p)\nu,\nu)\nu\} = 0\ (\alpha+\beta = 1).$$

$T(u,p) = D(u) - pI$: stress tensor of the Stokes flow

$D(u)_{jk} = \frac{\partial u_j}{\partial x_k} + \frac{\partial u_k}{\partial x_j}$: strain tensor.

$$B_{\alpha,\beta}(u,p) = B_{\alpha,\beta}(u).$$

Especially

$\alpha = 0$: Navier's slip condition, $\beta = 0$: non-slip condition.

In this talk

$\alpha > 0,\ \beta > 0.$

The main result

The global existence of a strong solution of Navier-Stokes equation with small data

Let $n \geq 3$.
There is a constant $\epsilon = \epsilon(\Omega, n) > 0$ such that if $a \in J^n(\Omega)$ satisfies

$$\|a\|_{L^n(\Omega)} \leq \epsilon,$$

Navier-Stokes equation admits a unique strong solution $u(t)$ on $(0, \infty)$.
Moreover as $t \longrightarrow \infty$,

- $\|u(t)\|_{L^p(\Omega)} = o(t^{-\frac{1}{2}+\frac{n}{2p}})$ for $n \leq p \leq \infty$,
- $\|\nabla u(t)\|_{L^n(\Omega)} = o(t^{-\frac{1}{2}})$.

With non-slip condition (Kubo-Shibata '04).

The global existence of solution to Navier-Stokes equation with small data.

I would like to extend these results to the Robin boundary condition.

To get main theorem

I shall analyze the following linear equation.

Stokes equation

$$\begin{cases} u_t - \Delta u + \nabla p = 0 & \text{in } \Omega \times (0,\infty), \\ \nabla \cdot u = 0 & \text{in } \Omega \times (0,\infty), \\ u \cdot \nu = 0,\ B_{\alpha,\beta}(u,p) = 0 & \text{on } \partial\Omega \times (0,\infty), \\ u(x,0) = a(x) & \text{in } \Omega. \end{cases}$$

$$J^p(\Omega) = \{u \in L^p(\Omega)^n \mid \nabla \cdot u = 0 \text{ in } \Omega,\ u \cdot \nu = 0 \text{ on } \partial\Omega\}$$

P:a continuous projection from $L^p(\Omega)^n$ onto $J^p(\Omega)$.

$$Au = -P\Delta u \quad \text{for } u \in D(A)$$

$$D(A) = J^p \cap \{u \in W^{2,p}(\Omega) \mid B_{\alpha,\beta}(u,p) = 0 \text{ on } \partial\Omega\}$$

According to Kato's argument
to get the main theorem,
it suffices to show the following two results about Stokes equation :

1. A generates an anlytic semigroup $\{T(t)\}_{t\geq 0}$.
2. $L^n - L^q$ decay estimates of the Stokes semigroup $\{T(t)\}_{t\geq 0}$.

$L^n - L^q$ estimates in a perturbed half space

- $\|u\|_{L^q(\Omega)} \leq C_q t^{-\frac{n}{2}(\frac{1}{n}-\frac{1}{q})}\|f\|_{L^n(\Omega)}, \ \ t > 0, \ f \in J^n(\Omega), \ \ q \geq n$
- $\|\nabla u\|_{L^n(\Omega)} \leq C t^{-\frac{1}{2}}\|f\|_{L^n(\Omega)}, \ \ t > 0, \ f \in J^n(\Omega)$

To get main theorem

1. A generates an anlytic semigroup $\{T(t)\}_{t\geq 0}$.
2. $L^n - L^q$ decay estimates of the Stokes semigroup $\{T(t)\}_{t\geq 0}$.

To show the two theorems about Stokes equation,
I consider the resolvent problem:

Resolvent problem of Stokes equation in a perturbed half space

$$\begin{cases} \lambda u - \Delta u + \nabla p = f & \text{in } \Omega, \\ \nabla \cdot u = 0 & \text{in } \Omega, \\ u \cdot \nu = 0, \ B_{\alpha,\beta}(u) = 0 & \text{on } \partial\Omega. \end{cases}$$

Resolvent estimates in a perturbed half space with large λ (Shimada-Shibata '05)

For all $\epsilon > 0$ there exists λ_0 and C_ϵ such that satisfies the following:

$$|\lambda| \, \|u\|_{L^q(\Omega)} + |\lambda|^{\frac{1}{2}} \|\nabla u\|_{L^q(\Omega)} + \|\nabla^2 u\|_{L^q(\Omega)} + \|\nabla p\|_{L^q(\Omega)} \leq C\|f\|_{L^q(\Omega)}$$

for $\lambda \in \Sigma_\epsilon$, $|\lambda| > \lambda_0$

where $\Sigma_\epsilon = \{\lambda \in \mathbb{C} \setminus \{0\} \mid |\arg \lambda| \leq \pi - \epsilon\}$.

$\Longrightarrow$

- O The generation of an anlytic semigroup $\{T(t)\}_{t\geq 0}$.
- × $L^n - L^q$ estimates of the Stokes semigroup $\{T(t)\}_{t\geq 0}$

To show the $L^n - L^q$ decay estimates,
I have to analyze the resolvent problem with small λ.

I shall show the following theorem.

The estimates that λ is small.

Let $n \geq 3$.

There exists a positive constant C such that u satisfies the following:

- $\|u\|_{L^q(\Omega)} \leq C|\lambda|^{-\frac{1}{2}-\frac{n}{2q}}\|f\|_{L^n(\Omega)}, \qquad q \geq n$
- $\|\nabla u\|_{L^n(\Omega)} \leq C|\lambda|^{-\frac{1}{2}}\|f\|_{L^n(\Omega)},$

for $\lambda \in \sum_\epsilon = \{\lambda \in \mathbb{C} \setminus \{0\} \mid |\arg \lambda| \leq \pi - \epsilon\}, \ |\lambda| < \lambda_0.$

To show these estimates,

I consider Generalized resolvent problem in a perturbed half space.

The key step of my argument is to consider the following generalized resolvent problem:

Generalized resolvent problem of Stokes equation in a perturbed half space

$$\begin{cases} \lambda u - \Delta u + \nabla p = f & \text{in } \Omega, \\ \nabla \cdot u = g & \text{in } \Omega, \\ u \cdot \nu = 0,\ B_{\alpha,\beta}(u) = h & \text{on } \partial\Omega. \end{cases}$$

Let us define the solution operator $U(\lambda)$ and $\Pi(\lambda)$ by the formula:

$$U(\lambda)F = u, \Pi(\lambda)F = p,$$

where I set $F = {}^t(f, g, h)$.

The solution operators in a perturbed half space

$$U(\lambda) : L^p_{R+3}(\Omega)^n \times W^{1,p}_{R+3,0}(\Omega) \times W^{1,p}_{R+3}(\Omega)^n \longrightarrow W^{2,p}_{loc}(\Omega),$$
$${}^t(f, g, h) \longmapsto u,$$
$$\Pi(\lambda) : L^p_{R+3}(\Omega)^n \times W^{1,p}_{R+3,0}(\Omega) \times W^{1,p}_{R+3}(\Omega)^n \longrightarrow W^{1,p}_{loc}(\Omega),$$
$${}^t(f, g, h) \longmapsto p.$$

Function spaces

$$L^p_{R+3}(\Omega) = \{f \in L^p(\Omega) \mid \mathrm{supp}\, f \subset B_{R+3}\},$$
$$W^{1,p}_{R+3,0}(\Omega) = \left\{f \in W^{1,p}(\Omega) \mid \mathrm{supp}\, f \subset B_{R+3}, \int_\Omega f dx = 0\right\},$$
$$W^{1,p}_{R+3}(\Omega) = \left\{f \in W^{1,p}(\Omega) \mid \mathrm{supp}\, f \subset B_{R+3}\right\}.$$

λ is small, F has a compact supprt

Let $n \geq 2$ and $1 < p < \infty$.

$$G_\Omega = \mathcal{L}(L^p_{R+3}(\Omega)^n \times W^{1,p}_{R+3,0}(\Omega) \times W^{1,p}_{R+3}(\Omega)^n, W^{2,p}_{loc}(\Omega) \times W^{1,p}_{loc}(\Omega))$$

Then solution operators $(U(\lambda), \Pi(\lambda)) \in G_\Omega$ for $\lambda \in U_{\frac{\alpha^2}{(1+\sqrt{2})^2\beta^2}}$, moreover they have the following expansion formula

$$(U(\lambda)F, \Pi(\lambda)F)$$

$$= \lambda^{\frac{n-2}{2}} H_1(\lambda)F + \lambda^{\frac{n-1}{2}} H_2(\lambda)F + (\lambda \log \lambda) H_3(\lambda)F + H_4(\lambda)F,$$

where H_j $(j = 1, 2, 3, 4)$ are G_Ω-valued holomorphic functions in $U_{\frac{\alpha^2}{(1+\sqrt{2})^2\beta^2}}$, $F = {}^t(f, g, h)$.

$$U_\lambda = \{\lambda \in \mathbb{C} \mid |\lambda| < r\}.$$

A model problem –a half space–

Moreover I need the results of a half space problem.

Generalized resolvent problem in a half space

$$\begin{cases} \lambda v_h - \Delta v_h + \nabla \theta_h = f & \text{in } \mathbb{R}^n_+, \\ \nabla \cdot v_h = g & \text{in } \mathbb{R}^n_+, \\ \alpha v_{hi} - \beta \partial_n v_{hi} = h_j \ (j = 1, \cdots n-1), \ v_{hn} = 0 & \text{on } \partial \mathbb{R}^n_+. \end{cases}$$

In a half space, the unit outer normal vector becomes $\nu = (0, \cdots, 0, -1)$. Using the Gagliardo-Nirenberg-Sobolev theorem, I can get the inequalities:

$L^p - L^\infty$ estimates in a half space

- $\|v_h\|_{L^\infty(\mathbb{R}^n_+)} \le C|\lambda|^{-\frac{1}{2}-\frac{n}{2p}} \|f\|_{L^p(\mathbb{R}^n_+)}$,
- $\|\nabla v_h\|_{L^\infty(\mathbb{R}^n_+)} \le C|\lambda|^{-\frac{1}{2}} \|f\|_{L^p(\mathbb{R}^n_+)}$,

for $\lambda \in \sum_\epsilon$, $|\lambda| < \lambda_0$, $p \neq n$.

Under the preparations I shall show the theorem:

The estimates when λ is small.

Let $n \geq 3$.
There exists a positive constant C such that u satisfies the following:

- $\|u\|_{L^q(\Omega)} \leq C|\lambda|^{-\frac{1}{2}-\frac{n}{2q}}\|f\|_{L^n(\Omega)}$,
- $\|\nabla u\|_{L^n(\Omega)} \leq C|\lambda|^{-\frac{1}{2}}\|f\|_{L^n(\Omega)}$,

for $\lambda \in \sum_\epsilon = \{\lambda \in \mathbb{C} \setminus \{0\} \mid |\arg\lambda| \leq \pi - \epsilon\}, \ \ |\lambda| < \lambda_0$.

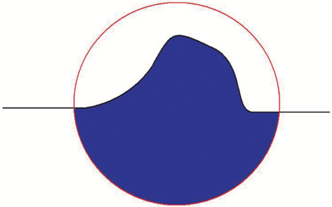


Figure: Ω_R

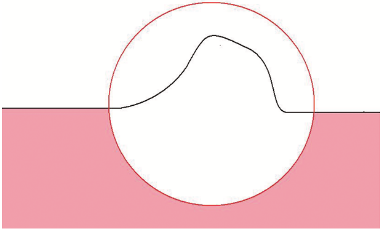


Figure: $\Omega \setminus \Omega_R$

To divide a domain into two parts,
I use cutoff function.

Cutoff function

$$\psi_R^\infty(x) \in C^\infty,\ \psi_R^\infty(x) = \begin{cases} 1 & \text{for } |x| \geq R+1 \\ 0 & \text{for } |x| \leq R \end{cases}$$

To get the estimates when λ is small.

First I shall consider a bounded part $\Omega_R = \Omega \cap B_R$.

$u = \psi_R^\infty v_h + w$, $p = \psi_R^\infty \theta_h + \pi$

$$\begin{cases} (\lambda - \Delta)w + \nabla\pi = K_1 & \text{in } \Omega, \\ \nabla \cdot w = K_2 & \text{in } \Omega, \\ w \cdot \nu = 0, \ B_{\alpha,\beta}(w) = K_3 & \text{on } \partial\Omega. \end{cases}$$

K_j : The linear functions of $\nabla\psi_R^\infty \cdot V_h, (\Delta\psi_R^\infty)v_h, \nabla\psi_R^\infty\theta_h, (1 - \psi_R^\infty)f$.

Since supp $K_1, K_2, K_3 \subset B_R$, I can use resolvent expansion:

$$\begin{aligned} ||w||_{L^\infty(\Omega_R)} &\leq C|\lambda|^{\frac{n-2}{2}}||(K_1, K_2, K_3)||_{L^\infty(\Omega_R)} \\ &\leq C|\lambda|^{\frac{n-2}{2}}\left(|\lambda|^{-1+\frac{n}{2p}}||f||_{L^p(\Omega)}\right) \\ &\leq C|\lambda|^{-2+\frac{n}{2}+\frac{n}{2p}}||f||_{L^p(\Omega)} \end{aligned}$$

To get the estimates when λ is small.

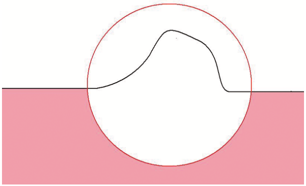


Figure: $\Omega \setminus \Omega_R$

Next I shall consider (w, π) in $\Omega \setminus \Omega_R$.

$\psi^\infty_{R-2} w = z,\ \psi^\infty_{R-2} \pi = \theta.$

$$\begin{cases} (\lambda - \Delta) z + \nabla \theta = H_1 & \text{in } \mathbb{R}^n_+, \\ \nabla \cdot z = H_2 & \text{in } \mathbb{R}^n_+, \\ \alpha z_i - \beta \partial_n z_i = H_{3j}\ (j = 1, \cdots n-1),\ z_n = 0 & \text{on } \partial \mathbb{R}^n_+. \end{cases}$$

H_j : The linear functions of $\nabla \psi^\infty_{R-2} \cdot w, (\Delta \psi^\infty_{R-2}) w, \nabla \psi^\infty_{R-2} \pi$.

$\psi^{\infty}_{R-2} w = z,\ \psi^{\infty}_{R-2}\pi = \theta.$

$$\|z\|_{L^p(\mathbb{R}^n_+)} \le C\|\nabla^2 z\|_{L^q(\mathbb{R}^n_+)}^{\frac{n}{2}(\frac{1}{q}-\frac{1}{p})}\|z\|_{L^q(\mathbb{R}^n_+)}^{1-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})}$$

$$\le \|(H_1,H_2,H_3)\|_{L^q(\mathbb{R}^n_+)}^{\frac{n}{2}(\frac{1}{q}-\frac{1}{p})}$$

$$\times \Big\{|\lambda|^{-1}\|H_1\|_{L^n(\mathbb{R}^n_+)} + \|H_2\|_{W^{-1,q}(\mathbb{R}^n_+)} + |\lambda|^{-\frac{1}{2}}\|(H_2,H_3)\|_{L^q(\mathbb{R}^n_+)}$$

$$+ |\lambda|^{-1}\|(\nabla H_2,\nabla H_3)\|_{L^q(\mathbb{R}^n_+)}\Big\}^{1-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})}$$

$$\le C|\lambda|^{1-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})}\|(H_1,H_2,H_3)\|_{L^q(\Omega_R)}$$

$$\le C|\lambda|^{-1+\frac{n}{2}(\frac{1}{q}-\frac{1}{p})}\left(|\lambda|^{-2+\frac{n}{2}+\frac{n}{2q}}\|f\|_{L^q(\Omega)}\right)$$

$$\le C|\lambda|^{-3+\frac{n}{2}+\frac{n}{q}-\frac{n}{2p}}\|f\|_{L^q(\Omega)}$$

$$\le C|\lambda|^{-1+\frac{n}{2}(\frac{1}{q}-\frac{1}{p})}\|f\|_{L^q(\Omega)} \qquad \text{for } q \le p.$$

The main result

The global existence of a strong solution of Navier-Stokes equation with small data

Let $n \geq 3$.
There is a constant $\epsilon = \epsilon(\Omega, n) > 0$ such that if $a \in J^n(\Omega)$ satisfies

$$||a||_{L^n(\Omega)} \leq \epsilon,$$

Navier-Stokes equation admits a unique strong solution $u(t)$ on $(0, \infty)$.
Moreover as $t \longrightarrow \infty$,

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- $||\nabla u(t)||_{L^n(\Omega)} = o(t^{-\frac{1}{2}})$.