Stationary problem for a cross-diffusion system of a prey-predator type with a protection zone

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Prey-Predator model (P)

$$
\begin{cases}
u_t = \Delta[(1+k\alpha(x)v)u] + u(\lambda - u - b(x)v) & \text{in } \Omega \times (0,\infty), \\
v_t = \Delta v + v(\mu + cu - v) & \text{in } \Omega \backslash \bar{\Omega}_0 \times (0,\infty), \\
\partial_n u = \partial_n v = 0 & \text{on } \partial\Omega \times (0,\infty), \\
\partial_n v = 0 & \text{on } \partial\Omega_0 \times (0,\infty).
\end{cases}
$$

Ω, Ω_0 : bounded domains in $\mathbb{R}^N (\bar{\Omega}_0 \subset \Omega,\ N \leq 3)$.

$\partial\Omega, \partial\Omega_0$: smooth boundaries of Ω, Ω_0.

$u(x,t)$: population density of a prey species.

$v(x,t)$: population density of a predator species.

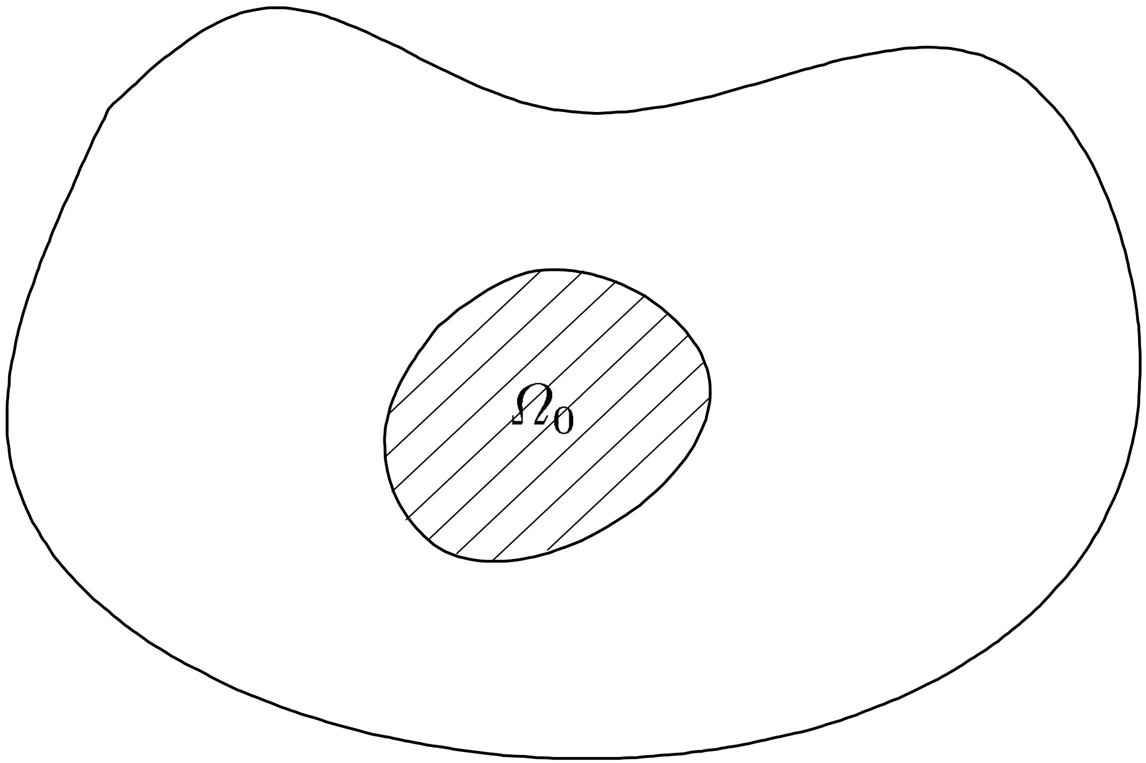


Fig.1. Protection zone Ω_0.

The prey species u can enter and leave Ω_0 freely.

The predator species v can not enter Ω_0.

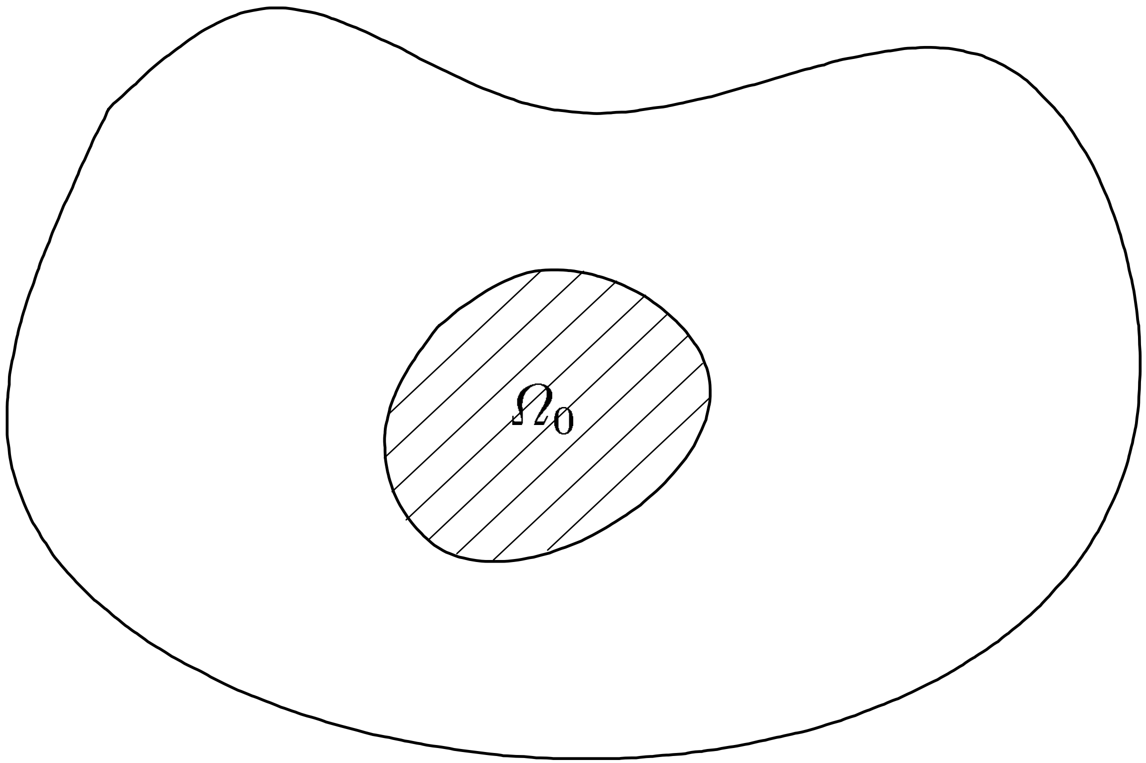


Fig.1. Protection zone Ω_0.

In the case of linear diffusion, population models with a protection zone have been studied by Du-Shi(2006) etc.

Prey-Predator model (P)

$$\begin{cases} u_t = \Delta[(1+k\alpha(x)v)u] + u(\lambda - u - b(x)v) & \text{in } \Omega \times (0,\infty), \\ v_t = \Delta v + v(\mu + cu - v) & \text{in } \Omega \setminus \bar{\Omega}_0 \times (0,\infty), \\ \partial_n u = \partial_n v = 0 & \text{on } \partial\Omega \times (0,\infty), \\ \partial_n v = 0 & \text{on } \partial\Omega_0 \times (0,\infty). \end{cases}$$

α, b : $\alpha = b = 0$ in $\bar{\Omega}_0$, $\alpha, b > 0$ in $\bar{\Omega} \setminus \bar{\Omega}_0$.

k, λ, c, μ : constants satisfying $k \geq 0$, $\lambda > 0$, $c > 0$, $\mu \in \mathbb{R}$.

Prey-Predator model (P)

$$\begin{cases} u_t = \Delta[(1+k\alpha(x)v)u] + u(\lambda - u - b(x)v) & \text{in } \Omega \times (0,\infty), \\ v_t = \Delta v + v(\mu + cu - v) & \text{in } \Omega \backslash \bar{\Omega}_0 \times (0,\infty), \\ \partial_n u = \partial_n v = 0 & \text{on } \partial\Omega \times (0,\infty), \\ \partial_n v = 0 & \text{on } \partial\Omega_0 \times (0,\infty). \end{cases}$$

Cross-diffusion $k\Delta[\alpha(x)vu]$

The prey species u tends to leave high-density areas of the predator species v.

Stationary problem (SP)

$$\begin{cases} \Delta[(1+k\alpha(x)v)u] + u(\lambda - u - b(x)v) = 0 & \text{in } \Omega, \\ \Delta v + v(\mu + cu - v) = 0 & \text{in } \Omega \backslash \bar{\Omega}_0, \\ \partial_n u = \partial_n v = 0 & \text{on } \partial\Omega, \\ \partial_n v = 0 & \text{on } \partial\Omega_0. \end{cases}$$

Biological problem

Do the two species can coexist?

$\Downarrow$

Mathematical problem

Study the structure of the set of positive solutions of (SP).

Stationary problem (SP)

$$\begin{cases} \Delta[(1+k\alpha(x)v)u] + u(\lambda - u - b(x)v) = 0 & \text{in } \Omega, \\ \Delta v + v(\mu + cu - v) = 0 & \text{in } \Omega \backslash \bar{\Omega}_0, \\ \partial_n u = \partial_n v = 0 & \text{on } \partial\Omega, \\ \partial_n v = 0 & \text{on } \partial\Omega_0. \end{cases}$$

The purpose of this talk

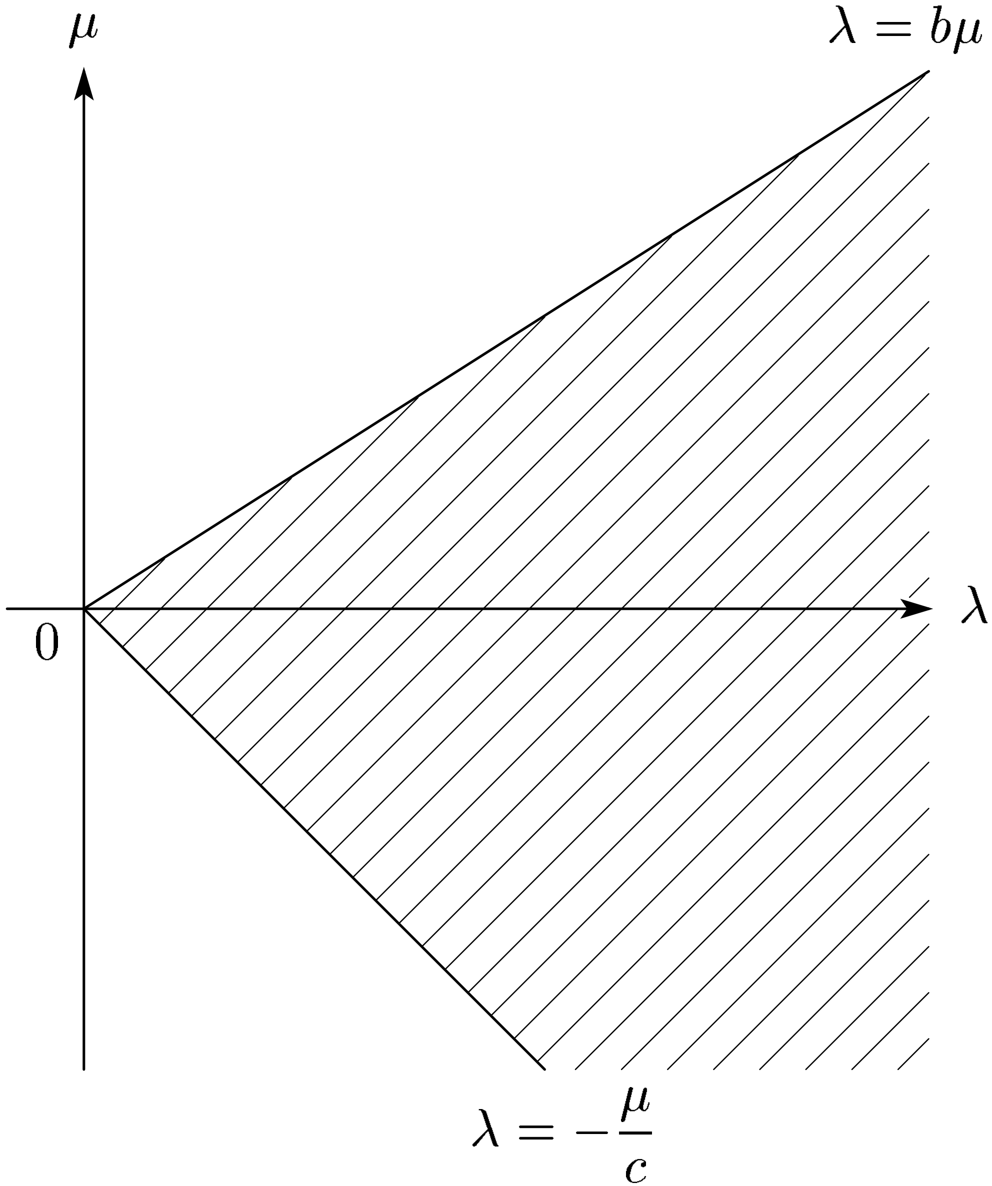
To study the effect of cross-diffusion on the set of positive solutions of (SP).

Stationary problem (SP)

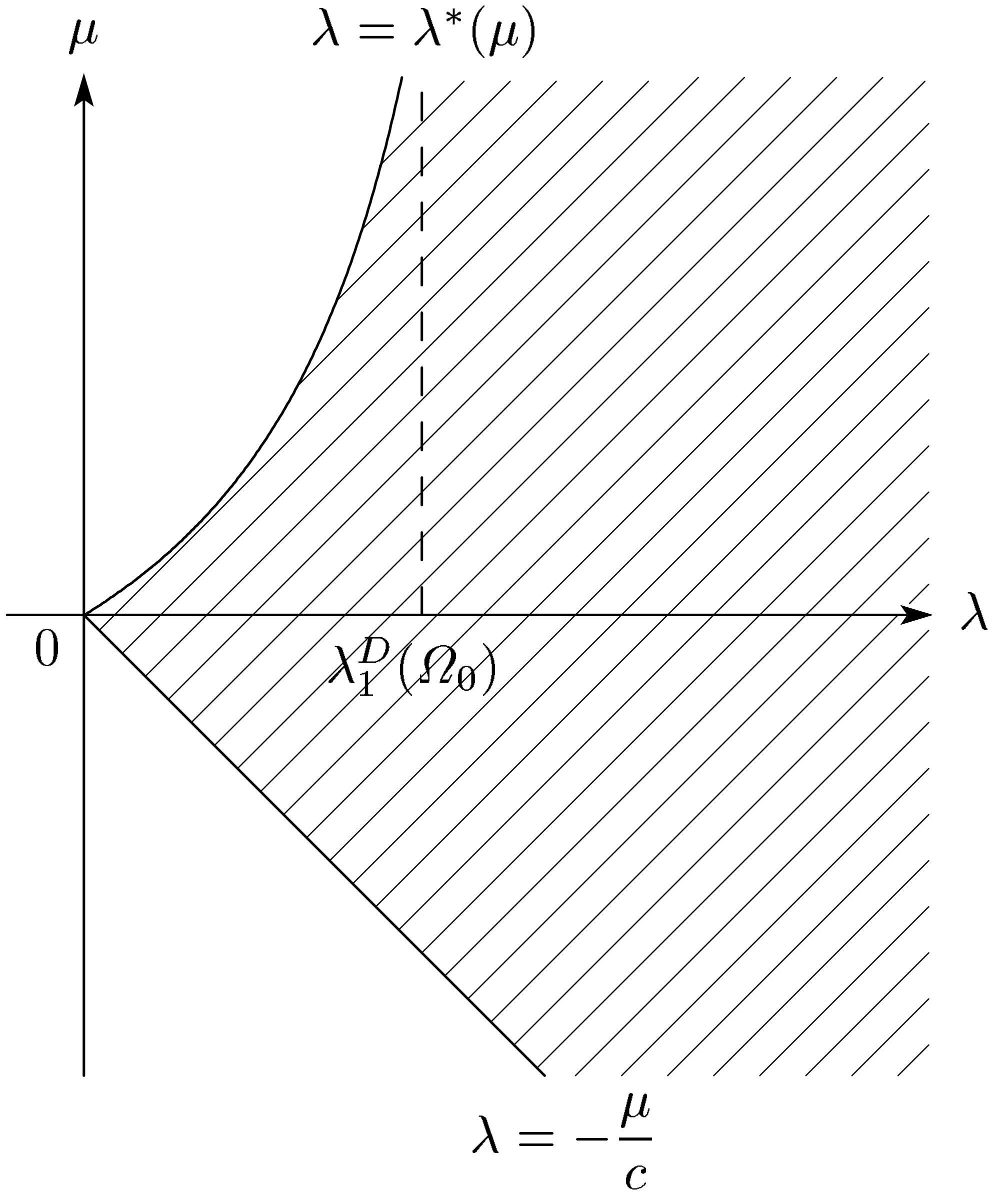
$$\begin{cases} \Delta[(1+k\alpha(x)v)u] + u(\lambda - u - b(x)v) = 0 & \text{in } \Omega, \\ \Delta v + v(\mu + cu - v) = 0 & \text{in } \Omega \backslash \bar{\Omega}_0, \\ \partial_n u = \partial_n v = 0 & \text{on } \partial\Omega, \\ \partial_n v = 0 & \text{on } \partial\Omega_0. \end{cases}$$

λ : intrinsic growth rate of the prey species.

μ : intrinsic growth rate of the predator species.

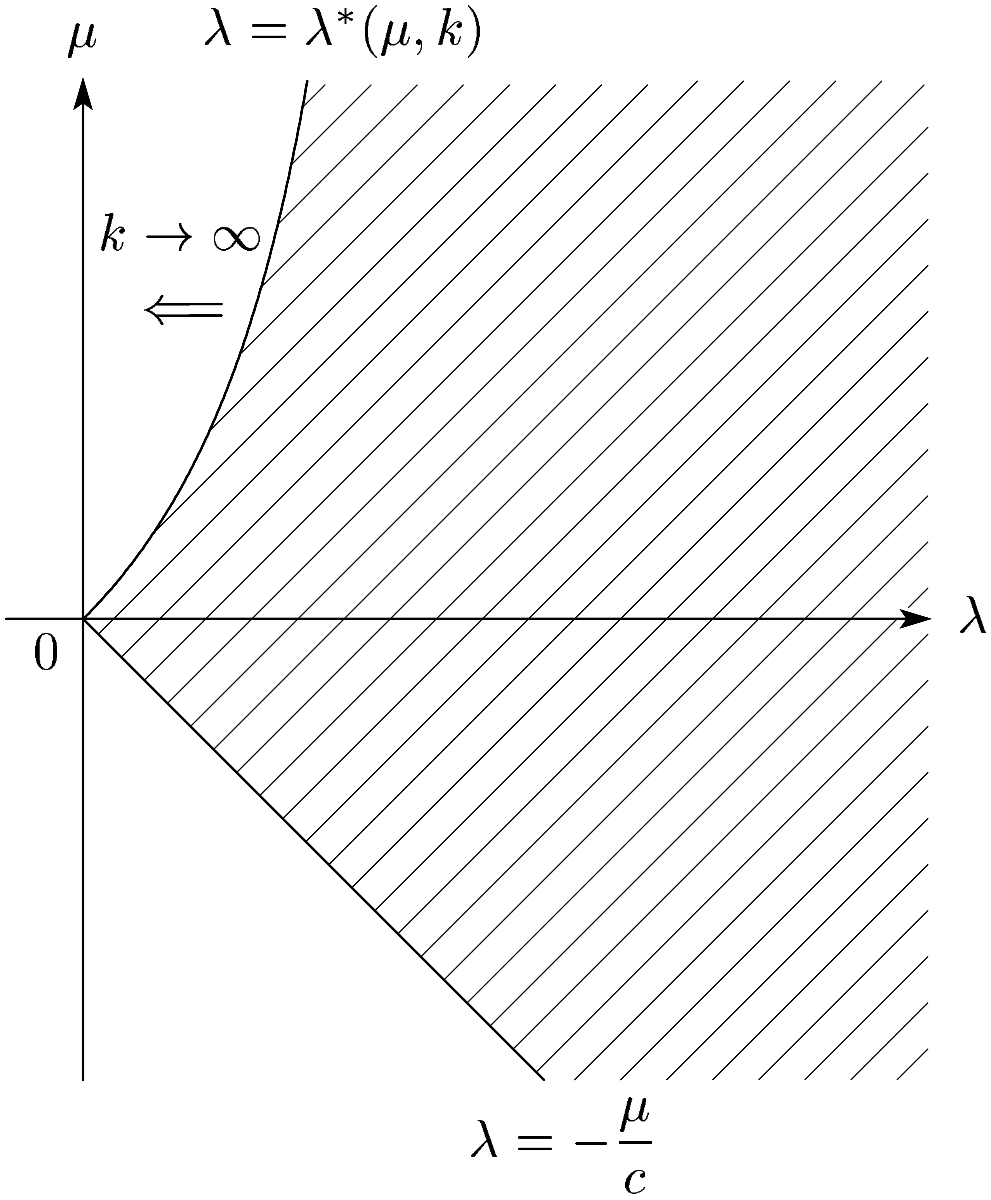


Coexistence region of (SP)
with no protection zone ($\Omega_0 = \emptyset$).



Coexistence region of (SP)

with a protection zone ($\Omega_0 \neq \emptyset$) and no cross-diffusion ($k = 0$).



Coexistence region of (SP)
with a protection zone ($\Omega_0 \neq \emptyset$) and cross-diffusion ($k > 0$).

Main results

$\lambda^*(\mu, k, \alpha, b, \Omega)$: positive number satisfying $\lambda_1^N\left(\frac{b(x)\mu-\lambda^*}{1+k\alpha(x)\mu}, \Omega\right) = 0$,

where $\lambda_1^N\left(\frac{b(x)\mu-\lambda^*}{1+k\alpha(x)\mu}, \Omega\right)$ is the least eigenvalue of $-\Delta+\frac{b(x)\mu-\lambda^*}{1+k\alpha(x)\mu}$ over Ω with Neumann boundary condition.

Theorem 1

(i) Case $\mu > 0$.

(SP) has a positive solution $\Leftrightarrow \lambda > \lambda^*$.

(ii) Case $\mu \leq 0$.

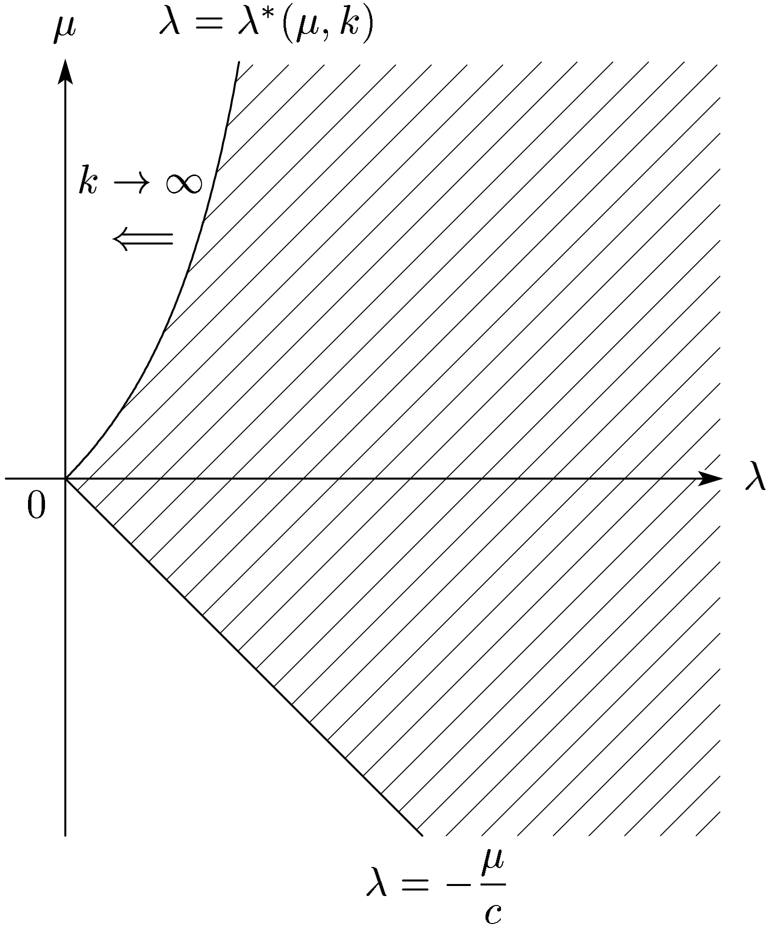
$\lambda > -\mu/c \Rightarrow$ (SP) has a positive solution.

Theorem 2

Suppose $\mu > 0$.

$^{\exists} k^* = k^*(\alpha, \lambda, \mu, b, \Omega, \Omega_0) \geq 0$ such that

$$k \geq k^* \Rightarrow \text{(SP) has a positive solution.}$$



Asymptotic behavior of positive solutions of (SP) as $k \to \infty$

$$
\begin{cases}
\Delta[(1 + k\alpha(x)v)u] + u(\lambda - u - b(x)v) = 0 & \text{in } \Omega, \\
\Delta v + v(\mu + cu - v) = 0 & \text{in } \Omega \backslash \bar{\Omega}_0, \\
\partial_n u = \partial_n v = 0 & \text{on } \partial\Omega, \\
\partial_n v = 0 & \text{on } \partial\Omega_0.
\end{cases}
$$

Theorem 3

Let (u_k, v_k) be any positive solution of (SP) for each k.

(i) If $\mu \geq 0$, then

$$\lim_{k\to\infty} u_k = \lambda \text{ in } C^1(\bar{\Omega}_0),$$
$$\lim_{k\to\infty} u_k = 0 \text{ uniformly on any compact subset of } \bar{\Omega}\setminus\bar{\Omega}_0,$$
$$\lim_{k\to\infty} v_k = \mu \text{ in } C^1(\bar{\Omega}\setminus\Omega_0).$$

(ii) If $\lambda > -\mu/c > 0$, then ${}^{\exists}\{k_i\}_{i=1}^{\infty}$ with $\lim_{i\to\infty} k_i = \infty$ s.t.

$$\lim_{i\to\infty}(u_{k_i}, v_{k_i}, k_i v_{k_i}) = (\bar{u}, 0, \bar{w}) \text{ in } C^1(\bar{\Omega}) \times C^1(\bar{\Omega}\setminus\Omega_0)^2,$$

where $\bar{u} \not\equiv$ const., $\bar{w} \not\equiv$ const., $(\bar{u}, \bar{w})$ is a positive solution of

$$\begin{cases} \Delta[(1+\alpha(x)\bar{w})\bar{u}]+\bar{u}(\lambda-\bar{u}) = 0 & \text{in } \Omega, \\ \Delta\bar{w} + \bar{w}(\mu + c\bar{u}) = 0 & \text{in } \Omega\setminus\bar{\Omega}_0, \\ \partial_n\bar{u} = \partial_n\bar{w} = 0 \text{ on } \partial\Omega, & \partial_n\bar{w} = 0 \text{ on } \partial\Omega_0. \end{cases}$$

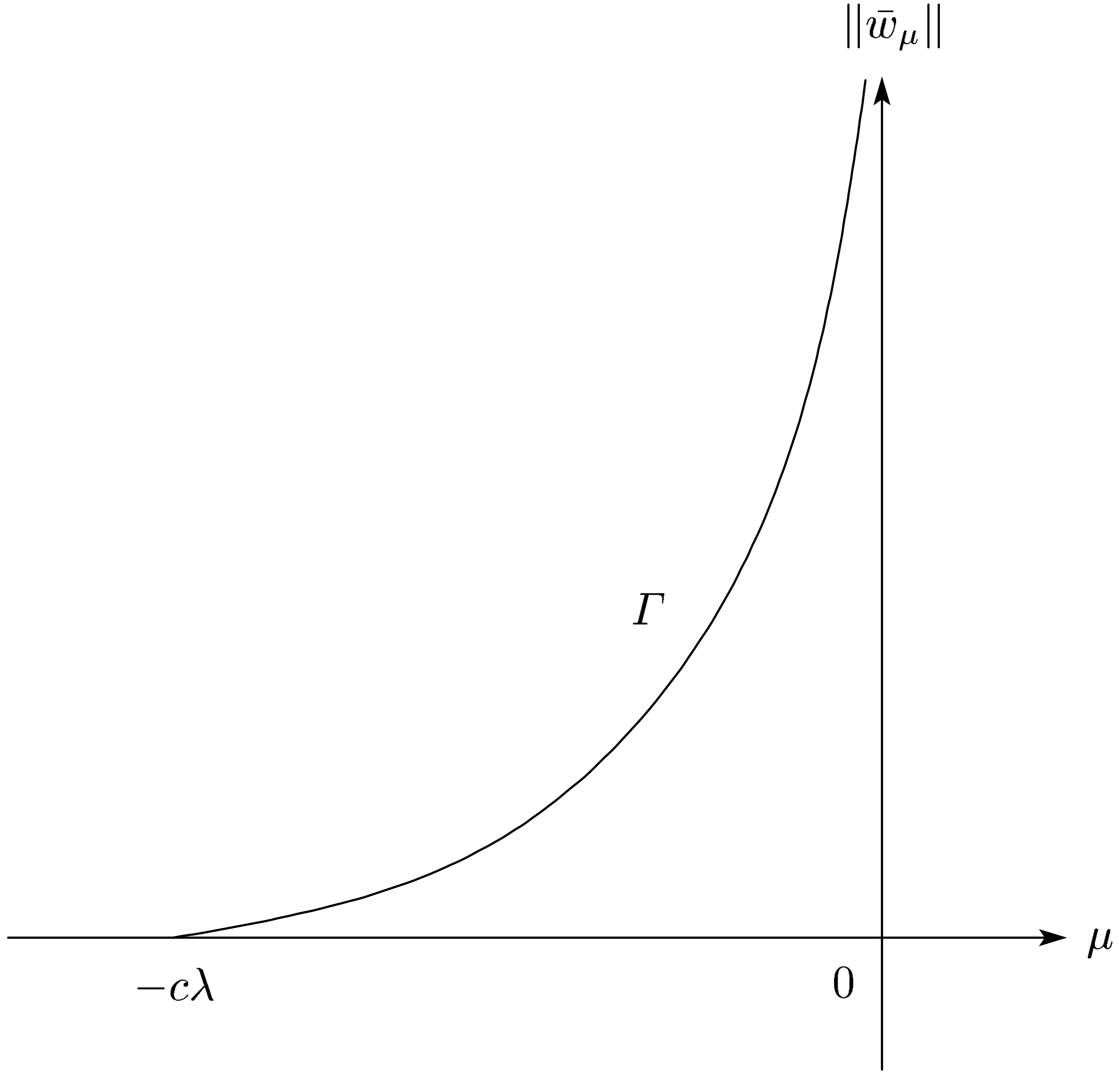
Theorem 4

The set of positive solutions of (LP) with bifurcation parameter μ contains an unbounded connected set Γ satisfying the following properties:

(i) Γ bifurcates from $\{(\mu, \bar{u}, \bar{w}) = (\mu, \lambda, 0) : \mu \in \mathbb{R}\}$ at $\mu = -c\lambda$.

(ii) $(-c\lambda, 0) \subset \{\mu : (\mu, \bar{u}_\mu, \bar{w}_\mu) \in \Gamma\} \subset (\tilde{\mu}, 0)$
for some $\tilde{\mu} \in (-\infty, -c\lambda]$.

(iii) $\lim_{\mu \to 0} \bar{u}_\mu = \lambda$ in $C^1(\bar{\Omega}_0)$,
$\lim_{\mu \to 0} \bar{u}_\mu = 0$ uniformly on any compact subset of $\bar{\Omega} \backslash \bar{\Omega}_0$,
$\lim_{\mu \to 0} \bar{w}_\mu = \infty$ uniformly in $\bar{\Omega} \backslash \Omega_0$,
where $(\mu, \bar{u}_\mu, \bar{w}_\mu) \in \Gamma$.


$\|\bar{w}_\mu\|$
Γ
μ
$-c\lambda$
0

Theorem 1(i) ($\mu > 0$)

(SP) has a positive sol.$\Leftrightarrow \lambda > \lambda^*$, where $\lambda_1^N\left(\frac{b(x)\mu-\lambda^*}{1+k\alpha(x)\mu}\right) = 0$.

$$\text{(SP) has a positive solution} \Leftrightarrow \lambda_1^N\left(\frac{b(x)\mu-\lambda}{1+k\alpha(x)\mu}\right) < 0.$$

$$\begin{aligned}
\lambda_1^N\left(\frac{b(x)\mu-\lambda}{1+k\alpha(x)\mu}\right) &= \inf_{\phi\in H^1(\Omega),\,\|\phi\|_{L^2(\Omega)}=1} \int_\Omega \left(|\nabla\phi|^2 + \frac{b(x)\mu-\lambda}{1+k\alpha(x)\mu}\phi^2\right) dx \\
&\le \frac{1}{|\Omega|}\int_\Omega \frac{b(x)\mu-\lambda}{1+k\alpha(x)\mu}dx \qquad \left(\phi \equiv 1/\sqrt{|\Omega|}\right) \\
&\le \frac{1}{|\Omega|}\int_{\Omega\setminus\bar{\Omega}_0} \frac{b(x)\mu}{1+k\alpha(x)\mu}dx - \lambda\frac{|\Omega_0|}{|\Omega|}.
\end{aligned}$$

Proof of $\lim_{\mu\to\infty} \lambda^*(\mu) \le \lambda_1^D(\Omega_0)$

Let ϕ_1 satisfy

$$-\Delta\phi_1 = \lambda_1^D(\Omega_0)\phi_1 \ \text{ in } \Omega_0, \quad \phi_1 = 0 \ \text{ on } \partial\Omega_0, \quad \int_{\Omega_0} \phi_1^2 dx = 1$$

and define $\tilde{\phi}_1 \in H^1(\Omega)$ as follows:

$$\tilde{\phi}_1 \equiv \phi_1 \ \text{ in } \Omega_0, \quad \tilde{\phi}_1 \equiv 0 \ \text{ in } \Omega \backslash \Omega_0.$$

Then

$$\begin{aligned}
0 &= \lambda_1^N\left(\frac{b(x)\mu - \lambda^*(\mu)}{1 + k\alpha(x)\mu}\right) \\
&= \inf_{\phi\in H^1(\Omega),\, \|\phi\|_{L^2(\Omega)}=1} \int_{\Omega} \left(|\nabla\phi|^2 + \frac{b(x)\mu - \lambda^*(\mu)}{1 + k\alpha(x)\mu}\phi^2\right) dx \\
&\le \int_{\Omega_0} \left(|\nabla\phi_1|^2 - \lambda^*(\mu)\phi_1^2\right) dx \qquad (\phi = \tilde{\phi}_1) \\
&= \lambda_1^D(\Omega_0) - \lambda^*(\mu) \quad \text{for } {}^\forall\mu \ge 0.
\end{aligned}$$