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On Incompressible Two-Phase Flows with Surface Tension and Phase Transitions

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Notations

$\Omega \subset \mathbb{R}^n$ is an open bounded domain with smooth boundary $\partial\Omega$,
$\Omega_i(t)$ is the subdomain occupied by fluid $i = 1, 2$ at time t,
$\Gamma(t)$ is the interface separating the two phases.
We assume no boundary contact, i.e. $\Gamma(t) \cap \partial\Omega = \emptyset$, and no external forces.

Denote by

$u = u(t,x)$ velocity field, $\pi = \pi(t,x)$ pressure field
$S(t,x)$ stress tensor
$E(t,x) := \frac{1}{2}(\nabla u(t,x) + \nabla u(t,x)^{\mathsf{T}})$ rate of strain tensor
$\theta = \theta(t,x)$ (absolute) temperature field, $q = q(t,x)$ heat flux
$e = e(t,x)$ internal energy, $\psi = \psi(t,x)$ (Hemholtz) free energy
$\rho_i > 0$ densities, $\mu_i > 0$ viscosities in the phases, $\sigma > 0$ surface tension
$\nu_\Gamma(t,x)$ the normal at $x \in \Gamma(t)$ directed into $\Omega_2(t)$
$u_\Gamma = u_\Gamma(t,x)$ velocity of $\Gamma(t)$, $V_\Gamma = u_\Gamma \cdot \nu_\Gamma$ normal velocity of Γ
$H_\Gamma(t,x) = -\mathrm{div}_\Gamma \nu_\Gamma(t,x)$ curvature of $\Gamma(t)$
$[\![\phi]\!] = \lim_{h\to 0+}[\phi(t, x + h\nu_\Gamma(t,x)) - \phi(t, x - h\nu_\Gamma(t,x))]$,
the jump of the quantity ϕ accross $\Gamma(t)$.

I The Model

Balance of Mass

$$\partial_t \rho + \mathrm{div}\,(\rho u) = 0, \qquad t > 0,\ x \notin \Gamma(t),$$
$$[\![\rho(u - u_\Gamma)]\!] \cdot \nu_\Gamma = 0, \qquad t > 0,\ x \in \Gamma(t).$$

Define the interfacial mass flux (phase flux for short) by means of

$$j := \rho(u - u_\Gamma) \cdot \nu_\Gamma, \quad \text{which means} \quad [\![\frac{1}{\rho}]\!] j = [\![u \cdot \nu_\Gamma]\!].$$

Here we consider the completely incompressible case, i.e. we assume that the densities are constant in the phases. Then conservation of mass reduces to

$$\mathrm{div}\, u = 0, \quad t > 0,\ x \notin \Gamma(t).$$

No phase transition means $j \equiv 0$; then $[\![u \cdot \nu_\Gamma]\!] = 0$ and $V_\Gamma = u \cdot \nu_\Gamma$, i.e. the interface is advected with the flow.
Here we are interested in the case $j \not\equiv 0$!

Balance of Momentum

$$\begin{aligned} \rho(\partial_t u + u\cdot\nabla u) - \mathrm{div}\, T &= 0, \qquad t>0,\ x\notin\Gamma(t),\\ [\![u]\!]j - [\![T\nu_\Gamma]\!] &= \mathrm{div}_\Gamma T_\Gamma \qquad t>0,\ x\in\Gamma(t). \end{aligned}$$

Balance of momentum is complemented by the constitutive laws

$$T = 2\mu E - \pi I, \quad E = \frac{1}{2}(\nabla u + \nabla u^{\mathsf{T}}), \quad T_\Gamma = \sigma P_\Gamma,$$

where $P_\Gamma = I - \nu_\Gamma \otimes \nu_\Gamma$ and $\sigma > 0$ constant. Then $\mathrm{div}_\Gamma T_\Gamma = \sigma H_\Gamma \nu_\Gamma$.

This yields the two-phase Navier-Stokes problem

$$\begin{aligned} \rho(\partial_t u + u\cdot\nabla u) - \mu\Delta u + \nabla\pi &= 0, \qquad t>0,\ x\notin\Gamma(t),\\ [\![u]\!]j - [\![T\nu_\Gamma]\!] &= \sigma H_\Gamma \nu_\Gamma, \qquad t>0,\ x\in\Gamma(t). \end{aligned}$$

for the velocity u and the pressure π.
We further assume no tangential slip at the interface, i.e.

$$[\![P_\Gamma u]\!] = 0, \quad \text{equivalently} \quad [\![u]\!] = [\![\frac{1}{\rho}]\!] j\nu_\Gamma.$$

Note that u is continuous across the interface in the case of equal densities, but otherwise discontinuos!

Balance of Energy

$$\rho(\partial_t e + u\cdot\nabla e) + \operatorname{div} q - T : \nabla u = 0, \qquad t>0,\ x\notin\Gamma(t),$$
$$([\![e]\!] + [\![\frac{1}{2}|u-u_\Gamma|_2^2]\!])j - [\![T\nu_\Gamma(u-u_\Gamma)]\!] + [\![q\cdot\nu]\!] = 0, \qquad t>0,\ x\in\Gamma(t).$$

Here we employ the constitutive laws

$$e(\theta) = \psi(\theta) + \theta\eta(\theta), \qquad \eta(\theta) = -\psi'(\theta),$$
$$\kappa(\theta) = e'(\theta) = -\theta\psi''(\theta), \qquad q = -d\nabla\theta,$$

considering the free energy $\psi(\theta)$ as given, and in Fourier's law $d>0$ is a constant, for simplicity. We assume $\kappa(\theta)>0$ below.

Observe that due to the constitutive law $[\![P_\Gamma u]\!] = 0$ we have

$$[\![|u-u_\Gamma|_2^2]\!] = [\![\frac{1}{\rho^2}]\!]j^2, \quad [\![T\nu_\Gamma(u-u_\Gamma)]\!] = [\![\frac{T\nu_\Gamma\cdot\nu_\Gamma}{\rho}]\!]j,$$

hence the boundary jump condition can be rewritten as

$$\{[\![\psi(\theta)]\!] + [\![\frac{1}{\rho^2}]\!]j^2 - [\![\frac{T\nu_\Gamma\cdot\nu_\Gamma}{\rho}]\!]\}j + \{[\![\theta\eta(\theta)]\!]j + [\![q\cdot\nu_\Gamma]\!]\} = 0.$$

We further assume

$$[\![\psi(\theta)]\!] + [\![\frac{1}{2\rho^2}]\!] j^2 - [\![\frac{T\nu_\Gamma \cdot \nu_\Gamma}{\rho}]\!] = 0, \quad [\![\theta]\!] = 0;$$

the first one is a generalized Gibbs-Thomson law. This gives the following problem for the temperature θ.

$$\begin{aligned} \rho\kappa(\theta)(\partial_t\theta + u\cdot\nabla\theta) - d\Delta\theta = 2\mu|E|_2^2, &\quad t>0,\ x\notin\Gamma(t), \\ [\![\theta\eta(\theta)]\!] j - [\![d\partial_{\nu_\Gamma}\theta]\!] = 0, &\quad t>0,\ x\in\Gamma(t), \\ [\![\theta]\!] = 0, &\quad t>0,\ x\in\Gamma(t). \end{aligned}$$

The second equation is a generalized Stefan law.

Note that in the case of equal densities the linearization of the generalized laws of Stefan and Gibbs-Thomson at $u_* = 0, j_* = 0, \theta_* = \theta_m$, where θ_m denotes the melting temperature defined by $[\![\psi(\theta_m)]\!] = 0$ yields the corresponding classical laws

$$[\![d\partial_{\nu_\Gamma}\theta]\!] = -l_m j, \quad \theta = -(\sigma\theta_m/l_m)H_\Gamma,$$

where $l_m = -\theta_m[\![\eta(\theta_m)]\!]$ means the latent heat at melting temperature. If $u = 0$ we have $j = -\rho V_\Gamma$.

Total Entropy Production

The total entropy is given by

$$\Psi(u,\theta,\Gamma) = \int_\Omega \rho\eta dx.$$

By the Reynolds transport theorem, it satisfies

$$\begin{aligned}
\frac{d}{dt}\Psi(\theta,\Gamma) &= \int_\Omega \rho\partial_t\eta(\theta)dx - \int_\Gamma [\![\rho\eta(\theta)]\!] V_\Gamma d\Gamma \\
&= \int_\Omega \frac{\eta'(\theta)}{e'(\theta)}\{2\mu|E|_2^2 - \mathrm{div}\, q - \rho u\cdot\nabla e\}dx - \int_\Gamma [\![\rho\eta(\theta)]\!] u_\Gamma\cdot\nu d\Gamma \\
&= \int_\Omega \{\frac{2\mu}{\theta}|E|_2^2 + \frac{d}{\theta^2}|\nabla\theta|_2^2\}dx + \int_\Gamma \{\theta[\![\eta]\!]j + [\![q\cdot\nu]\!]\}/\theta d\Gamma \geq 0.
\end{aligned}$$

Thus there is no entropy production on the interface if

$$\theta[\![\eta(\theta)]\!]j + [\![q\cdot\nu]\!] = 0.$$

This is the generalized Stefan law.

Conservation of Total Energy

We have for the total energy $\Phi(u,\theta,\Gamma) = \int_\Omega \{\frac{\rho}{2}|u|_2^2 + \rho e\}dx + \sigma \,\mathrm{mes}\,\Gamma$ by the Reynolds transport theorems

$$\partial_t\Phi = \int_\Omega \{u\cdot\rho\partial_t u + \rho\partial_t e\}dx - \int_\Gamma \{[\![\frac{\rho}{2}|u|_2^2 + \rho e]\!]\}V_\Gamma d\Gamma$$

$$= -\int_\Omega \{(\rho(u\cdot\nabla)\cdot u - \mathrm{div}\,T\cdot u) + (\rho(u\cdot\nabla)e + \mathrm{div}\,q - T:\nabla u)\}dx$$

$$-\int_\Gamma \{[\![\frac{\rho}{2}|u|_2^2 + \rho e]\!]\}u_\Gamma\cdot\nu_\Gamma d\Gamma$$

$$= \int_\Gamma \{[\![\frac{1}{2}|u|_2^2 + \psi(\theta)]\!]j - [\![Tu\cdot\nu_\Gamma]\!]\} + \{[\![\theta\eta(\theta)]\!]j + [\![q\cdot\nu_\Gamma]\!]\}d\Gamma$$

$$= \int_\Gamma \{[\![\psi(\theta) + \frac{j^2}{2\rho^2} - T\nu_\Gamma\cdot\nu_\Gamma/\rho]\!]\}j + \{[\![\theta\eta(\theta)]\!]j + [\![q\cdot\nu_\Gamma]\!]\}d\Gamma = 0$$

Thus there is conservation of energy across the interface if the generalized Stefan law holds, and

$$[\![\psi(\theta) + \frac{j^2}{2\rho^2} - \frac{1}{\rho}T\nu_\Gamma\cdot\nu_\Gamma]\!] = 0,$$

i.e. the generalized Gibbs-Thomson law is valid.

The Complete Model

In the bulk $\Omega \setminus \Gamma(t)$

$$\begin{aligned} \rho(\partial_t u + u \cdot \nabla u) - \mu\Delta u + \nabla\pi &= 0, \\ \operatorname{div} u &= 0, \\ \rho\kappa(\theta)(\partial_t\theta + u \cdot \nabla\theta) - d\Delta\theta - 2\mu|E|_2^2 &= 0 \qquad . \end{aligned}$$

On the interface $\Gamma(t)$

$$\begin{aligned} [\![\frac{1}{\rho}]\!]j^2\nu - [\![T\nu_\Gamma]\!] &= \sigma H_\Gamma \nu_\Gamma, \qquad [\![u]\!] = [\![\frac{1}{\rho}]\!]j\nu_\Gamma, \\ \theta[\![\eta(\theta)]\!]j - [\![d\partial_\nu\theta]\!] &= 0, \qquad [\![\theta]\!] = 0, \\ [\![\psi(\theta)]\!] + [\![\frac{1}{2\rho^2}]\!]j^2 - [\![\frac{1}{\rho}T\nu_\Gamma \cdot \nu_\Gamma]\!] &= 0, \qquad V_\Gamma = u \cdot \nu_\Gamma - \frac{1}{\rho}j. \end{aligned}$$

On the outer boundary $\partial\Omega$

$$u = 0, \quad \partial_\nu\theta = 0.$$

Initial conditions

$$\Gamma(0) = \Gamma_0, \quad u(0) = u_0, \quad \theta(0) = \theta_0.$$

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II Equilibria

The negative entropy is a strict Ljapunov functional:
if $\partial_t \Psi = 0$ on a time-interval $(t_0, t_1) \neq \emptyset$, then

$$E = 0, \quad \nabla\theta = 0 \quad \text{in } (t_0, t_1),$$

hence θ is constant. If latent heat $l(\theta) := -\theta[\![\eta(\theta)]\!] \neq 0$ then $j = 0$, hence $[\![u]\!] = 0$. Korn's inequality then implies $u = 0$, hence $\nabla\pi = 0$. This in turn yields H_Γ constant, i.e. Γ is a sphere if connected.

Therefore the (non-degenerate) equilibria are

$$u = 0, \quad \theta = const, \quad \nabla\pi = 0,$$
$$V_\Gamma = 0, \quad j = 0, \quad \Gamma = S_R(x_0) \subset \Omega,$$
$$-\frac{\sigma(n-1)}{R} = \sigma H_\Gamma = [\![\pi]\!], \quad [\![\psi(\theta)]\!] + [\![\pi/\rho]\!] = 0,$$
$$\rho^1 e^1(\theta)\frac{\omega_n}{n}R^n + \rho^2 e^2(\theta)(\mathrm{mes}\,\Omega - \frac{\omega_n}{n}R^n) + \sigma\omega_n R^{n-1} = \Phi_0 := \Phi(0),$$

with $\omega_n = \mathrm{mes}\, S_1$. Denote the set of non-degenerate equilibria by $\mathcal{E}$.

Linearization at Equilibria

The linearized problem at a non-degenerate equilibrium $(0, \theta_*, \Gamma_*)$ reads:
In the bulk $\Omega \setminus \Gamma_*$

$$\begin{aligned} \rho\partial_t u - \mu\Delta u + \nabla\pi &= f_u, \\ \operatorname{div} u &= g_d, \\ \rho\kappa_*\partial_t\theta - d\Delta\theta &= f_\theta, \end{aligned}$$

on the interface Γ_*

$$\begin{aligned} -[\![T\nu_*]\!] + \sigma\mathcal{A}_* h\nu_* &= g_u, \qquad [\![u]\!] - [\![\frac{1}{\rho}]\!] j\nu_* = g_p, \\ -l_* j - [\![d\partial_\nu\theta]\!] &= g_\theta, \qquad [\![\theta]\!] = 0, \\ (l_*/\theta_*)\theta - [\![\frac{T\nu_*\cdot\nu_*}{\rho}]\!] &= g_j, \qquad \partial_t h - u\cdot\nu_* + \frac{1}{\rho} j = f_h, \end{aligned}$$

supplemented by boundary conditions on $\partial\Omega$ and initial conditions.
Here $\kappa_* = \kappa(\theta_*)$, $l_* = l(\theta_*)$, $\nu_* = \nu_{\Gamma_*}$, $\mathcal{A}_* = -H'(\Gamma_*) = -(\frac{n-1}{R_*^2} + \Delta_*)$.

Maximal L_p-Regularity

The strategy to solve the linear problem is as follows.
Suppose j and h are given.
(i) Solve the heat problem to determine θ as

$$\theta = \bar{\theta} + l_* N_H j;$$

here $\bar{\theta}$ is determined by the data alone, and N_H means the Neumann-to-Dirichlet operator for the heat problem.
(ii) Solve the Stokes problem with $h = 0$ and extract

$$u \cdot \nu_* - j/\rho = \bar{g}_1 - R_1 j, \quad -[\![T\nu_* \cdot \nu_*/\rho]\!] = \bar{g}_2 + [\![1/\rho]\!]^2 G j,$$

where $\bar{g}_k$ are given by the data, and R_1 and G are linear pseudo-differential operators to be studied.
(iii) Solve the Stokes problem with trivial data and $j = 0$ and extract

$$u \cdot \nu_* - j/\rho = -\sigma N_S \mathcal{A}_* h, \quad -[\![T\nu_* \cdot \nu_*/\rho]\!] = \sigma R_2 \mathcal{A}_* h.$$

Here N_S means the Neumann-to-Dirichlet operator for the Stokes problem and R_2 is a linear pseudo-differential operator.

(iv) Insert into the next-to-last equation to obtain an equation for j:

$$(l_*^2/\theta_*)N_H j + [\![1/\rho]\!]^2 G j = -\sigma R_2 \mathcal{A}_* h + \bar{g}_j.$$

Here $\bar{g}_j$ is determined by the data. Solve this equation for j.
(v) Insert into the last equation to obtain the final equation for the height function h:

$$\partial_t h + \sigma N_S \mathcal{A}_* h + \sigma R_1((l_*^2/\theta_*)N_H + [\![1/\rho]\!]^2 G)^{-1} R_2 \mathcal{A}_* h = \bar{g}_h,$$

where $\bar{g}_h$ is given by the data. Solve this equation for h.

Note that the operators N_H, N_S are of order $(-1/2, -1)$, i.e. $-1/2$ in time, 1 in space, R_1, R_2 are of order $(0,0)$, and the *phase-flux operator* G is of order $(1/2, 1)$.

Therefore the evolution equation for h has order $(1,1)$ if $[\![\rho]\!] \neq 0$; in this case it is velocity dominated.

It has order $(1,3)$ and in addition has mixed order $1/2$ in time and 1 in space, if $[\![\rho]\!] = 0$; in this case it is temperature dominated.

The Underlying Semigroups

Consider first the velocity dominated case. Set

$$X_0 = L_{p,\sigma}(\Omega) \times L_p(\Omega) \times W_p^{2-1/p}(\Gamma_*),$$

and define the operator A by

$$A(u,\theta,h) = (-(\mu/\rho)\Delta u + \nabla\pi/\rho, -(d/\rho\kappa_*)\Delta\theta, -u\cdot\nu_* + [\![1/\rho]\!]^{-1}[\![u\cdot\nu_*]\!]/\rho).$$

To define the domain $D(A)$ of A, we set

$$X_1 = \{(u,\theta,h) \in H_p^2(\Omega\setminus\Gamma_*) \times H_p^2(\Omega\setminus\Gamma_*) \times W_p^{3-1/p}(\Gamma_*) : \\ \operatorname{div} u = 0 \ \text{ in } \ \Omega\setminus\Gamma_*, \ [\![P_* u]\!] = [\![\theta]\!] = 0 \ \text{ on } \ \Gamma_*\},$$

and

$$D(A) = \{(u,\theta,h) \in X_1 : \ [\![\mu P_* E\nu_*]\!] = [\![d\partial_{\nu_*}\theta]\!] + l_*[\![1/\rho]\!]^{-1}[\![u\cdot\nu_*]\!] = 0 \text{ on } \Gamma_*\}.$$

Here π is determined as the solution of the weak elliptic problem

$$(\nabla\pi/\rho|\nabla\phi)_2 = ((\mu/\rho)\Delta u|\nabla\phi)_2, \quad \phi \in \dot{H}^1_{p'}(\Omega),$$
$$[\![\pi/\rho]\!] = [\![2\mu E\nu_*\cdot\nu_*/\rho]\!] - (l_*/\theta_*)\theta, \quad [\![\pi]\!] = -\sigma\mathcal{A}_* h + 2[\![\mu(E\nu_*|\nu_*)]\!].$$

Next we consider the temperature dominated case. Set

$$X_0 = L_{p,\sigma}(\Omega) \times L_p(\Omega) \times W_p^{2-2/p}(\Gamma_*),$$

and define the operator A by

$$A(u,\theta,h) = (\,-(\mu/\rho)\Delta u + \nabla\pi/\rho, -d\Delta\theta/\rho\kappa_*, -u\cdot\nu_* - (l_*\rho)^{-1}[\![d\partial_{\nu_*}\theta]\!]).$$

To define the domain $D(A)$ of A, we set

$$\begin{aligned} X_1 = \{(u,\theta,h) \in H_p^2(\Omega\setminus\Gamma_*) \times H_p^2(\Omega\setminus\Gamma_*) \times W_p^{4-1/p}(\Gamma_*) : \\ \operatorname{div} u = 0 \ \text{ in } \ \Omega\setminus\Gamma_*,\ [\![u]\!] = [\![\theta]\!] = 0 \ \text{ on } \ \Gamma_*\}, \end{aligned}$$

and

$$D(A) = \{(u,\theta,h) \in X_1 : [\![\mu P_* E\nu_*]\!] = (l_*/\theta_*)\theta - (\sigma/\rho)\mathcal{A}_* h = 0 \ \text{ on } \ \Gamma_*\}.$$

Here π is determined as the solution of the weak transmission problem

$$\begin{aligned} (\nabla\pi|\nabla\phi/\rho)_2 = ((\mu/\rho)\Delta u|\nabla\phi)_2, \quad \phi \in \dot{H}^1_{p'}(\Omega), \\ [\![\pi]\!] = -\sigma\mathcal{A}_* h + 2[\![\mu(E\nu_*|\nu_*)]\!]. \end{aligned}$$

Then the linearized problem can be rewritten as an evolution equation in X_0 as

$$\dot{z} + Az = f, \quad t > 0, \quad z(0) = z_0, \tag{1}$$

where $z = (u, \theta, h)$, $f = (f_u, f_\theta, f_h)$, $z_0 = (u_0, \theta_0, h_0)$, provided $g_d = g_u = g_p = g_\theta = g_j = 0$.

The linearized problem has maximal L_p-regularity, hence (1) has this property as well. Therefore, $-A$ generates an analytic C_0-semigroup in X_0.

Since the embedding $X_1 \hookrightarrow X_0$ is compact, the semigroup e^{-At} as well as the resolvent $(\lambda + A)^{-1}$ of $-A$ are compact, too. Therefore the spectrum $\sigma(A)$ of A consists only of countably many eigenvalues of finite algebraic multiplicity.

Nonzero Eigenvalues

Suppose that $\lambda \neq 0$, $\mathrm{Re}\lambda \geq 0$ is an eigenvalue of the linear problem. Then taking the inner product of the equation for u with u we get:

$$
\begin{aligned}
0 &= \lambda\|\rho^{1/2}u\|_2^2 + \int_\Omega T : \nabla\bar{u}dx + \int_{\Gamma_*}[\![T\nu\bar{u}]\!]d\Gamma \\
&= \lambda\|\rho^{1/2}u\|_2^2 + 2\|\mu^{1/2}E\|_2^2 + \sigma\bar{\lambda}(\mathcal{A}_*h|h)_2 + ([\![\rho^{-1}T\nu\cdot\nu]\!]|j)_2.
\end{aligned}
$$

Similarly, for θ we obtain

$$
\begin{aligned}
0 &= \lambda\|(\rho\kappa_*)^{1/2}\theta\|_2^2 + \|d^{1/2}\nabla\theta\|_2^2 + ([\![d\partial_\nu\theta]\!]|\theta)_2 \\
&= \lambda\|(\rho\kappa_*)^{1/2}\theta\|_2^2 + \|d^{1/2}\nabla\theta\|_2^2 - \theta_*(j|[\![\rho^{-1}T\nu\cdot\nu]\!])_2,
\end{aligned}
$$

Adding the real parts yields

$$
\begin{aligned}
0 &= \mathrm{Re}\,\lambda\|\rho^{1/2}u\|_2^2 + 2\|\mu^{1/2}E\|_2^2 + \sigma\mathrm{Re}\,\lambda(\mathcal{A}_*h|h)_2 \\
&+\theta_*^{-1}(\mathrm{Re}\,\lambda\|(\rho\kappa_*)^{1/2}\vartheta\|_2^2 + \|d^{1/2}\nabla\theta\|_2^2).
\end{aligned}
$$

If $\mathrm{Im}\,\lambda \neq 0$, taking imaginary parts we get

$$
\sigma(\mathcal{A}_*h|h)_2 = \|\rho^{1/2}u\|_2^2 - \|(\rho\kappa_*)^{1/2}\theta\|_2^2/\theta_*.
$$

We deduce from this

$$0 = 2\mathrm{Re}\,\lambda\|\rho^{1/2}u\|_2^2 + 2\|\mu^{1/2}E\|_2^2 + \|d^{1/2}\nabla\theta\|_2^2/\theta_*,$$

which shows that such eigenvalues are real. Next decompose

$$\theta = \hat\theta + \theta_0, \quad j = \hat j + j_0, \quad h = \hat h + h_0,$$

with the weighted means

$$\hat\vartheta = (\kappa_*\vartheta|\rho)_2/(\kappa_*|\rho), \quad \hat h = (h|1)_2/\mathrm{mes}, \Gamma_* \quad \hat j = (j|1)_2/\mathrm{mes}\,\Gamma_*.$$

By the equations we have

$$\lambda\hat h = -\hat j/\rho, \quad \lambda(\kappa_*|\rho)_2\hat\theta = l_*\mathrm{mes}\,\Gamma_*\hat j,$$

in particular $\hat h = \hat j = \hat\theta = 0$ in case $[\![\rho]\!] \neq 0$. Otherwise

$$\lambda\|\rho^{1/2}u\|_2^2 + 2\|\mu^{1/2}E\|_2^2 + \sigma\lambda(\mathcal{A}_* h_0|h_0)_2 + (\lambda\|(\rho\kappa_*)^{1/2}\theta_0\|_2^2$$
$$+\|d^{1/2}\nabla\theta_0\|_2^2)/\theta_* + \lambda\rho\mathrm{mes}\,\Gamma_*\{\frac{l_*^2\mathrm{mes}\,\Gamma_*}{\theta_*(\kappa_*|\rho)_2} - \frac{(n-1)\sigma}{\rho^2 R_*^2}\}|\hat h|^2 = 0.$$

As a consequence we have

Theorem. Let the phases be connected. Then
(i) *In case* $[\![\rho]\!] \neq 0$, *there are no eigenvalues* $\mathrm{Re}\lambda \geq 0, \lambda \neq 0$.

(ii) *In case* $[\![\rho]\!] = 0$, *there are no eigenvalues* $\mathrm{Re}\lambda \geq 0, \lambda \neq 0$ *if*

$$(S) \qquad \frac{l_*^2 \,\mathrm{mes}\,\Gamma_*}{\theta_*(\kappa_*|\rho)} - \frac{(n-1)\sigma}{\rho^2 R_*^2} \geq 0.$$

This is the Stability Condition for equilibria in the temperature dominated case.

If the stability condition is violated there will be a positive eigenvalue which is algebraically simple.

Note that the condition (S) does neither involve the diffusion coefficients d^k nor the viscosities μ^k!

The Eigenvalue $\lambda = 0$

Here the equations yield

$$0 = 2\|\mu^{1/2}E\|^2 + \|d^{1/2}\nabla\theta\|^2,$$

hence θ is constant, with $l_* \neq 0$ also $j = 0$, hence $u = 0$ by Korn's inequality. This further shows that π is constant in the phases, hence we are left with the equations

$$[\![\pi]\!] = \sigma\mathcal{A}_*h, \quad (l_*/\theta_*)\theta + [\![\pi/\rho]\!] = 0.$$

The kernel of $\mathcal{A}_*$ consists of the (orhonormalized) spherical harmonics $\{Y_k\}_{k=1}^n$ of degree one.

In the temperature dominated case, there is one further degree of freedom, namely $[\![\pi]\!]$, hence eigenvalue zero has geometric multiplicity $n+1$. It is semi-simple, unless equality holds in condition (S).

In the velocity dominated case, there are two further degree of freedom, namely π^1, π^2, hence eigenvalue zero has geometric multiplicity $n+2$. However, taking preservation of volume into account, we have $[\![\pi]\!] = 0$ which reduces the multiplicity to $n+1$; $\lambda = 0$ is semi-simple.

If one also takes into account conservation of energy, then the dimension of $\mathcal{E}$ and of the kernel are both equal to n.

Thus in both cases, the dimension of the eigenspace for $\lambda = 0$ of the linearization equals the dimension of the manifold $\mathcal{E}$ of equilibria, and its tangent space is isomorphic to this eigenspace.

This shows that the equilibrium in question is normally stable in the velocity dominated case, and also in the temperature dominated case if we have strict inequality in (S). If (S) does not hold, it is normally hyperbolic. In the temperature dominated case, the equilibrium is not normally hyperbolic if and only if we have equality in (S) (or $l_* = 0$).

III The Induced Semiflow

We first introduce the semiflow induced by the solutions. Recall that the closed C^2-hypersurfaces contained in Ω form a C^2-manifold, which we denote by $\mathcal{MH}^2$. Charts are obtained via parametrization over a fixed hypersurface, and the tangent spaces consist of the normal vector fields. As an ambient space for the phase-manifold $\mathcal{PM}$ of the two-phase problem with surface tension and phase transitions we consider the product space $X_0 := L_{p,\sigma}(\Omega) \times L_p(\Omega) \times \mathcal{MH}^2$.

In the velocity dominated case we define $\mathcal{PM}$ as follows

$$\mathcal{PM} := \{(u, \theta, \Gamma) \in X_0 : (u, \theta) \in W_p^{2-2/p}(\Omega \setminus \Gamma)^{n+1}, \Gamma \in W_p^{3-2/p}, \text{the compatibilities (3) hold}\}. \quad (2)$$

The charts for this manifold are obtained by the charts induced by $\mathcal{MH}^2$, followed by a Hanzawa transformation.

Observe that the compatibility conditions

$$\begin{aligned} &\mathcal{P}_\Gamma [\![\mu E]\!]\nu_\Gamma = \mathcal{P}_\Gamma [\![u]\!] = [\![\theta]\!] = 0 \quad \text{ on } \Gamma, \\ &l(\theta)[\![1/\rho]\!]^{-1}[\![u \cdot \nu_\Gamma]\!] + [\![d\partial_{\nu_\Gamma}\theta]\!] = 0 \text{ on } \Gamma \\ &u = \partial_\nu \theta = 0 \text{ on } \partial\Omega, \end{aligned} \tag{3}$$

as well as regularity are preserved by the solutions.

Applying the local existence result and re-parameterizing repeatedly, we obtain a local semiflow on $\mathcal{PM}$.

Theorem *Let $p > n+2$ and $[\![\rho]\!] \neq 0$. Then the two-phase problem with phase transitions generates a local semiflow on the phase-manifold $\mathcal{PM}$. Each solution (u, θ, Γ) exists on a maximal time interval $[0, t_*)$.*

In the temperature dominated case we define $\mathcal{PM}$ as follows

$$\mathcal{PM} := \{(u,\theta,\Gamma) \in X_0 : (u,\theta) \in W_p^{2-2/p}(\Omega\setminus\Gamma)^{n+1},\ \Gamma \in W_p^{4-3/p},$$
$$[\![d\partial_{\nu_\Gamma}\theta]\!] \in W_p^{2-6/p}(\Gamma), \quad l(\theta) \neq 0, \text{ on } \Gamma \tag{4}$$
$$\text{and the compatibilities (5) hold }\}.$$

The charts for this manifold are obtained by the charts induced by $\mathcal{MH}^2$, followed by a Hanzawa transformation.

Observe that the compatibility conditions

$$\mathcal{P}_\Gamma[\![\mu E]\!]\nu_\Gamma = [\![u]\!] = [\![\theta]\!] = 0 \quad \text{ on } \Gamma, \tag{5}$$
$$[\![\psi(\theta)]\!] + (\sigma/\rho)H_\Gamma = 0 \text{ on } \Gamma,$$
$$u = \partial_\nu\theta = 0 \text{ on } \partial\Omega,$$

as well as regularity are preserved by the solutions. However, the well-posedness condition $l(\theta) \neq 0$ may fail to be preserved by the semiflow!

By the local existence result and re-parameterizing repeatedly, we obtain also in case $[\![\rho]\!] = 0$ a local semiflow on $\mathcal{PM}$.

Nonlinear Stability

The properties of the linearized problem near an equilibrium call for the Generalized Principle of Linear Stability; cf. P., Simonett, Zacher 2009. Employing this technique we obtain

Theorem. *Let* $(0, \theta_*, \Gamma_*)$ *be an equilibrium such that* $l_* \neq 0$ *in case* $[\![\rho]\!] = 0$.

(i) *If* $[\![\rho]\!] \neq 0$ *then the equilibrium* $(0, \theta_*, \Gamma_*)$ *is stable in* $\mathcal{PM}$ *for the nonlinear problem. Each solution starting near* $(0, \theta_*, \Gamma_*)$ *converges to another equilibrium.*

(ii) *If* $[\![\rho]\!] = 0$ *and the Stability Condition* (S) *holds with strict inequality, then the equilibrium* $(0, \theta_*, \Gamma_*)$ *is stable in* $\mathcal{PM}$ *for the nonlinear problem. Each solution starting near* $(0, \theta_*, \Gamma_*)$ *converges to another equilibrium.*

(iii) *If* $[\![\rho]\!] = 0$ *and the Stability Condition* (S) *does not hold, then the equilibrium* $(0, \theta_*, \Gamma_*)$ *is unstable in* $\mathcal{PM}$ *for the nonlinear problem.*

Asymptotic behaviour

There are a number of obstructions to global existence of the solutions:

- regularity: the norms of either $u(t)$, $\theta(t)$ or $\Gamma(t)$ become unbounded;
- geometry: the topology of the interface changes;
  or the interface touches the boundary of Ω;
  or the interface shrinks to a point (in case $[\![\rho]\!] = 0$);
- well-posedness: $l(\theta(t))$ develops a zero (in case $[\![\rho]\!] = 0$).

Note that in case $[\![\rho]\!] \neq 0$ the phase volumes are preserved by the semiflow!

We say that a solution (u, θ, Γ) satisfies a uniform ball condition, if there is a radius $r > 0$ such that for each $t \in [0, t_*)$ and at every point $p \in \Gamma(t)$ we have

$$\bar{B}_r(p \pm r\nu_{\Gamma(t)}(p)) \cap \Gamma(t) = \{p\}.$$

Combining the above results, we obtain the following theorem on the asymptotic behavior of solutions.

Theorem *Let $p > n+2$. Suppose that (u, θ, Γ) is a solution of the two-phase problem with phase transition on the maximal time interval $[0, t_*)$. Assume the following on $[0, t_*)$:*

(i) $\|u(t)\|_{W_p^{2-2/p}} + \|\theta(t)\|_{W_p^{2-2/p}} + \|\Gamma(t)\|_{W_p^{3-2/p}} \leq M < \infty$;

(ii) (u, θ, Γ) *satisfies a uniform ball condition,*

and in case $[\![\rho]\!] = 0$ in addition:

(iii) $|l(\theta)| \geq 1/M$, $\|\Gamma(t)\|_{W_p^{4-3/p}} + \|[\![d\partial_{\nu_\Gamma}]\!]\|_{W_p^{2-6/p}} \leq M$.

Then $t_* = \infty$, i.e. the solution exists globally and its limit set $\omega(u, \theta, \Gamma) \subset \mathcal{E}$ is nonempty. If this limit set contains a stable equilibrium, then the solution converges in $\mathcal{PM}$ to this equilibrium. The converse is also true.

For the proof we combine a compactness argument with the entropy and apply the stability result.