On the local in time solvability of the Navier-Stokes equations with phase transition

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joint work with

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Formulation of Problems

We consider the two-phase free boundary problem of incompressible capillary fluids with phase transition. We set

$$\begin{aligned}
&\Gamma(t) = \{x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \mid x_n = h(t, x')\},\\
&\Omega_i(t) = \{x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \mid (-1)^i(x_n - h(t, x')) > 0\}, \ \ i = 1, 2,\\
&\Omega(t) = \Omega_1(t) \cup \Omega_2(t).
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&\Gamma_0 = \{x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \mid x_n = h_0(x')\},\\
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initial velocity u_0 and initial absolute temperature θ_0 in Ω_0.

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initial velocity u_0 and initial absolute temperature θ_0 in Ω_0.

The unknowns are velocity $u(t, x)$, pressure $\pi(t, x)$, absolute temperature $\theta(t, x)$ in $\Omega(t)$, and free boundary $\Gamma(t)$.

Notations

- $u(t,x) = {}^t(u_1,\ldots,u_n)$: velocity , $\pi(t,x)$: pressure,
- $T(u,\pi) = \mu D(u) - \pi I$, $E(u) = \frac{1}{2}(\nabla u + {}^t(\nabla u))$,
- $\theta(t,x)$: absolute temperature,
- $\eta(t,x)$: entropy, $\psi(t,x)$: free energy, $\kappa(t,x)$: heat capacity,
- ν_Γ: unit normal directed into $\Omega_2(t)$,
- $j(t,x')$: phase flux, $H_\Gamma = -\mathrm{div}_\Gamma \nu_\Gamma$: mean curvature,
- V_Γ: normal velocity, $\sigma > 0$: surface tension,
- μ: viscosity, d: heat conductivity, ρ: density

$$\mu = \begin{cases} \mu_1 & \text{in } \Omega_1(t) \\ \mu_2 & \text{in } \Omega_2(t) \end{cases} \quad d = \begin{cases} d_1 & \text{in } \Omega_1(t) \\ d_2 & \text{in } \Omega_2(t) \end{cases} \quad \rho = \begin{cases} \rho_1 & \text{in } \Omega_1(t) \\ \rho_2 & \text{in } \Omega_2(t) \end{cases}$$

$$[\![v]\!] = \left(v|_{\Omega_2(t)} - v|_{\Omega_1(t)} \right)\Big|_{\Gamma(t)}.$$

Navier-Stokes equations with phase transition

$$
\begin{aligned}
\rho(\partial_t u + (u \cdot \nabla)u) - \mu \Delta u + \nabla \pi &= 0 && \text{in } \Omega(t),\\
\operatorname{div} u &= 0 && \text{in } \Omega(t),\\
[\![u]\!] &= [\![\rho^{-1}]\!] j \nu_\Gamma && \text{on } \Gamma(t),\\
[\![\rho^{-1}]\!] j^2 \nu_\Gamma - [\![T\nu_\Gamma]\!] &= \sigma H_\Gamma \nu_\Gamma && \text{on } \Gamma(t),\\
u(0) &= u_0 && \text{in } \Omega_0,
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[\![\rho^{-1}]\!] j^2 \nu_\Gamma - [\![T\nu_\Gamma]\!] = \sigma H_\Gamma \nu_\Gamma \quad & \text{on } \Gamma(t),\\
u(0) = u_0 \quad & \text{in } \Omega_0,\\
\rho\kappa(\theta)(\partial_t\theta + u\cdot\nabla\theta) - d\Delta\theta - 2\mu|E|_2^2 = 0 \quad & \text{in } \Omega(t),\\
[\![\theta]\!] = 0 \quad & \text{on } \Gamma(t),\\
\theta[\![\eta(\theta)]\!] j - [\![d\partial_{\nu_\Gamma}\theta]\!] = 0 \quad & \text{on } \Gamma(t),\\
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\rho\kappa(\theta)(\partial_t\theta + u\cdot\nabla\theta) - d\Delta\theta - 2\mu|E|_2^2 &= 0 && \text{in } \Omega(t),\\
[\![\theta]\!] &= 0 && \text{on } \Gamma(t),\\
\theta[\![\eta(\theta)]\!] j - [\![d\partial_{\nu_\Gamma}\theta]\!] &= 0 && \text{on } \Gamma(t),\\
\theta(0) &= \theta_0 && \text{in } \Omega_0,\\
[\![\psi(\theta)]\!] + [\![2^{-1}\rho^{-2}]\!] j^2 - [\![(T\nu_\Gamma\cdot\nu_\Gamma)/\rho]\!] &= 0 && \text{on } \Gamma(t),\\
V_\Gamma - \mathcal{R}u\cdot\nu_\Gamma + j/\rho &= 0 && \text{on } \Gamma(t),\\
h(0) &= h_0 && \text{on } \mathbb{R}^{n-1}.
\end{aligned}
\tag{1}
$$

References

- Kusaka and Tani, SIAM J. Math. Anal. 33 (1999)
  Navier-Stokes & Classical Stefan, local sol., Hölder spaces
- Kusaka and Tani, Math. Models Methods Appl. Sci. 12 (2002)
  Navier-Stokes & Classical two-phase Stefan, local sol., Hölder spaces
- Gibou, Chen, Nguyen, and Banerjee, J. Comput. Phys. 222 (2007)
  Numerical simulation

Linearization at Equilibria

The equilibrium state we consider here is

$$u_* = 0, \quad \theta_* = \mathrm{const.}, \quad \pi_* = \mathrm{const.} \quad [\![\pi_*]\!] = 0, \quad j = 0,$$
$$\Gamma_* = \{x = (x', x_n) \in \mathbb{R}^n \mid x_n = 0\} =: \mathbb{R}^n_0,$$
$$[\![\psi(\theta_*)]\!] + [\![\pi_*/\rho]\!] = 0 \text{ (} [\![\rho]\!] = 0 \text{ case it is } [\![\psi(\theta_*)]\!] = 0).$$

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We set

$$\mathbb{R}^n_\pm = \{x = (x', x_n) \in \mathbb{R}^n \mid \pm x_n > 0\},$$
$$\dot{\mathbb{R}}^n = \mathbb{R}^n_+ \cup \mathbb{R}^n_- = \mathbb{R}^n \setminus \mathbb{R}^n_0,$$
$$\nu_* = {}^t(0, \ldots, 0, 1), \quad P_* = I - \nu_* \otimes \nu_*,$$

and use the relation

$$\ell(\theta) = -\theta[\![\eta(\theta)]\!] = \theta[\![\psi'(\theta)]\!].$$

Linearization at Equilibria

In the bulk $\dot{\mathbb{R}}^n$

$$\begin{aligned} \rho\partial_t u - \mu\Delta u + \nabla\pi &= f_u, \\ \operatorname{div} u &= f_d, \\ \rho\kappa_*\partial_t\theta - d\Delta\theta &= f_\theta, \qquad \kappa_* = \kappa(\theta_*). \end{aligned}$$

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On the interface $\mathbb{R}^n_0$

$$\begin{aligned} -[\![\mu(\partial_n u' + \nabla u_n)]\!] &= g_\tau, \quad -2[\![\mu\partial_n u_n]\!] + [\![\pi]\!] - \sigma\Delta' h = g_n, \\ [\![u]\!] - [\![\rho^{-1}]\!] j\nu_* &= g_p, \quad [\![\theta]\!] = 0, \\ -\ell_* j - [\![d\partial_n\theta]\!] &= g_\theta, \quad (\ell_*/\theta_*)\theta - [\![2\rho^{-1}\mu\partial_n u_n]\!] + [\![\rho^{-1}\pi]\!] = g_j, \\ \partial_t h - \mathcal{R} u_n + j/\rho &= g_h, \qquad \ell_* = \ell(\theta_*). \end{aligned}$$

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Supplemented by initial conditions

$$u(0) = u_0, \quad \theta(0) = \theta_0 \quad \text{in } \dot{\mathbb{R}}^n, \qquad h(0) = h_0 \quad \text{on } \dot{\mathbb{R}}^n_0.$$

L_p-Maximal Regularity

$$\begin{aligned} \rho\partial_t u - \mu\Delta u + \nabla\pi &= f_u \\ \mathrm{div}\, u &= f_d \\ \rho\kappa_*\partial_t\theta - d\Delta\theta &= f_\theta, \qquad \kappa_* = \kappa(\theta_*) \end{aligned}$$

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$J = (0, T)$. Let

$$\begin{aligned} f_u &\in L_p(J; L_p(\mathbb{R}^n))^n, && =: \mathbb{F}_1, \\ f_d &\in H^1_p(J; \hat{H}^{-1}_p(\mathbb{R}^n)) \cap L_p(J; H^1_p(\dot{\mathbb{R}}^n)) && =: \mathbb{F}_2, \\ f_\theta &\in L_p(J; L_p(\mathbb{R}^n)) && =: \mathbb{F}_3, \end{aligned}$$

L_p-Maximal Regularity

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$$f_u \in L_p(J; L_p(\mathbb{R}^n))^n, \qquad =: \mathbb{F}_1,$$
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$$f_\theta \in L_p(J; L_p(\mathbb{R}^n)) \qquad =: \mathbb{F}_3,$$

then

$$u \in H_p^1(J; L_p(\mathbb{R}^n))^n \cap L_p(J; H_p^2(\dot{\mathbb{R}}^n))^n \qquad =: \mathbb{E}_1,$$
$$\pi \in L_p(J; \hat{H}_p^1(\dot{\mathbb{R}}^n)) \qquad =: \mathbb{E}_2,$$
$$\theta \in H_p^1(J; L_p(\mathbb{R}^n)) \cap L_p(J; H_p^2(\dot{\mathbb{R}}^n)) \qquad =: \mathbb{E}_3.$$

The class of boundary data

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\begin{aligned}
-[\![\mu(\partial_n u' + \nabla u_n)]\!] &= g_\tau, \quad -2[\![\mu\partial_n u_n]\!] + [\![\pi]\!] - \sigma\Delta' h = g_n,\\
[\![u]\!] - [\![\rho^{-1}]\!] j\nu_* &= g_p \quad [\![\theta]\!] = 0,\\
-\ell_* j - [\![d\partial_n\theta]\!] &= g_\theta \quad (\ell_*/\theta_*)\theta - [\![2\rho^{-1}\mu\partial_n u_n]\!] + [\![\rho^{-1}\pi]\!] = g_j,\\
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\end{aligned}
$$

By trace theory the resulting data spaces are

$$
\begin{aligned}
g_\tau \in W_p^{\frac{1}{2}-\frac{1}{2p}}(J; L_p(\mathbb{R}^{n-1}))^{n-1} \cap L_p(J; W_p^{1-\frac{1}{p}}(\mathbb{R}^{n-1}))^{n-1} \quad & =: \mathbb{F}_5,\\
g_n, g_\theta \in W_p^{\frac{1}{2}-\frac{1}{2p}}(J; L_p(\mathbb{R}^{n-1})) \cap L_p(J; W_p^{1-\frac{1}{p}}(\mathbb{R}^{n-1})) \quad & =: \mathbb{F}_6 = \mathbb{F}_7.
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\end{aligned}
$$

We also know

$$
[\![\pi]\!] \in W_p^{\frac{1}{2}-\frac{1}{2p}}(J; L_p(\mathbb{R}^{n-1})) \cap L_p(J; W_p^{1-\frac{1}{p}}(\mathbb{R}^{n-1})) =: \mathbb{E}_4
$$

$[\![\rho]\!] \neq 0$ case

$$
\begin{aligned}
&[\![u]\!] - [\![\rho^{-1}]\!] j\nu_* = g_p, \quad [\![\theta]\!] = 0,\\
&-\ell_* j - [\![d\partial_n\theta]\!] = g_\theta, \quad (\ell_*/\theta_*)\theta - [\![2\rho^{-1}\mu\partial_n u_n]\!] + [\![\rho^{-1}\pi]\!] = g_j.
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g_j \in W_p^{\frac{1}{2}-\frac{1}{2p}}(J; L_p(\mathbb{R}^{n-1})) \cap L_p(J; W_p^{1-\frac{1}{p}}(\mathbb{R}^{n-1})) = \mathbb{F}_8.
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The regularity class of j is equal to the class $[\![u]\!]$, which implies

$$j, g_p \in W_p^{1-\frac{1}{2p}}(J; L_p(\mathbb{R}^{n-1})) \cap L_p(J; W_p^{2-\frac{1}{p}}(\mathbb{R}^{n-1})) =: \mathbb{E}_5 = \mathbb{F}_4.$$

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We express $\partial_t h - \mathcal{R}u_n + j/\rho = g_h$ symbolically: $S(t,\partial)h = g_h$,

$$g_h \in W_p^{1-\frac{1}{2p}}(J; L_p(\mathbb{R}^{n-1})) \cap L_p(J; W_p^{2-\frac{1}{p}}(\mathbb{R}^{n-1})) =: \mathbb{F}_9,$$
$$C_1(|\lambda| + |\xi'|) \geq s(\lambda, \xi') \geq C_2(|\lambda| + |\xi'|),$$
$$h \in W_p^{2-\frac{1}{2p}}(J; L_p(\mathbb{R}^{n-1})) \cap H_p^1(J; W_p^{2-\frac{1}{p}}(\mathbb{R}^{n-1})) \cap L_p(J; W_p^{3-\frac{1}{p}}(\mathbb{R}^{n-1})) =: \mathbb{E}_6.$$

$[\![\rho]\!] \neq 0$ case is dominated by velocity.

$[\![\rho]\!] = 0$ case

$$
\begin{aligned}
&[\![u]\!] = 0, \qquad -2[\![\mu\partial_n u_n]\!] + [\![\pi]\!] - \sigma\Delta' h = g_n, \\
&-\ell_* j - [\![d\partial_n\theta]\!] = g_\theta, \quad (\ell_*/\theta_*)\theta - \rho^{-1}[\![2\mu\partial_n u_n]\!] + \rho^{-1}[\![\pi]\!] = g_j.
\end{aligned}
$$

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The regularity class of j is equal to the class $[\![\partial_n\theta]\!]$, which implies

$$j \in W_p^{\frac{1}{2}-\frac{1}{2p}}(J; L_p(\mathbb{R}^{n-1})) \cap L_p(J; W_p^{1-\frac{1}{p}}(\mathbb{R}^{n-1})) =: \mathbb{E}_5.$$

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$$g_h \in W_p^{\frac{1}{2}-\frac{1}{2p}}(J; L_p(\mathbb{R}^{n-1})) \cap L_p(J; W_p^{1-\frac{1}{p}}(\mathbb{R}^{n-1})) =: \mathbb{F}_9,$$
$$C_1(|\lambda| + |\lambda|^{\frac{1}{2}}|\xi'|^2 + |\xi'|^3) \geq s(\lambda,\xi') \geq C_2(|\lambda| + |\lambda|^{\frac{1}{2}}|\xi'|^2 + |\xi'|^3),$$
$$h \in W_p^{\frac{3}{2}-\frac{1}{2p}}(J; L_p(\mathbb{R}^{n-1})) \cap W_p^{1-\frac{1}{2p}}(J; H_p^2(\mathbb{R}^{n-1})) \cap L_p(J; W_p^{4-\frac{1}{p}}(\mathbb{R}^{n-1})) =: \mathbb{E}_6,$$
$$g_j \in W_p^{1-\frac{1}{2p}}(J; L_p(\mathbb{R}^{n-1})) \cap L_p(J; W_p^{2-\frac{1}{p}}(\mathbb{R}^{n-1})) =: \mathbb{F}_8.$$

$[\![\rho]\!] = 0$ case is dominated by temperature.

Compatibility Conditions (C)

$$\begin{aligned}
&\mathrm{div}\, u_0 = f_d(0), \\
&-[\![\mu(\partial_n u_0' + \nabla' u_0)]\!] = g_\tau(0) && \text{if} \quad p > 3, \\
&[\![\theta_0]\!] = 0 && \text{if} \quad p > 3/2.
\end{aligned}$$

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\end{aligned}$$

Additionally for $[\![\rho]\!] \neq 0$ case

$$\begin{aligned}
&[\![P_* u_0]\!] = P_* g_p(0) && \text{if} \quad p > 3/2, \\
&-\ell_* [\![\rho^{-1}]\!]^{-1}([\![u_0 \cdot \nu_*]\!] - g_p(0) \cdot \nu_*) - [\![d \partial_n \theta_0]\!] = g_\theta(0) && \text{if} \quad p > 3,
\end{aligned}$$

Compatibility Conditions (C)

$$
\begin{aligned}
&\mathrm{div}\, u_0 = f_d(0), \\
&-[\![\mu(\partial_n {u_0}' + \nabla' u_0)]\!] = g_\tau(0) && \text{if} \quad p > 3, \\
&[\![\theta_0]\!] = 0 && \text{if} \quad p > 3/2.
\end{aligned}
$$

Additionally for $[\![\rho]\!] \neq 0$ case

$$
\begin{aligned}
&[\![P_* u_0]\!] = P_* g_p(0) && \text{if} \quad p > 3/2, \\
&-\ell_* [\![\rho^{-1}]\!]^{-1}([\![u_0 \cdot \nu_*]\!] - g_p(0) \cdot \nu_*) - [\![d \partial_n \theta_0]\!] = g_\theta(0) && \text{if} \quad p > 3,
\end{aligned}
$$

for $[\![\rho]\!] = 0$ case

$$
\begin{aligned}
&[\![u_0]\!] = g_p(0) && \text{if} \quad p > 3/2, \\
&(\ell_*/\theta_*)\theta_0 + (\sigma/\rho)\Delta' h_0 = g_j(0) - g_n(0)/\rho && \text{if} \quad p > 3/2, \\
&\mathcal{R} u_n(0) + 1/(\ell_* \rho)([\![d \partial_n \theta_0]\!] + g_\theta(0)) + g_h(0) \in W_p^{2-6/p} && \text{if} \quad p > 3.
\end{aligned}
$$

The class of initial data

By trace theory, the spaces of initial data are

$$(u,\theta) \in H^1_p(J; L_p(\mathbb{R}^n))^{n+1} \cap L_p(J; H^2_p(\dot{\mathbb{R}}^n))^{n+1},$$
$$(u_0,\theta_0) \in (L_p(\dot{\mathbb{R}}^n), W^2_p(\dot{\mathbb{R}}^n))^{n+1}_{1-1/p,p} = W^{2-2/p}_p(\dot{\mathbb{R}}^n)^{n+1} =: \mathbb{F}_{10}.$$

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$[\![\rho]\!] \neq 0$ case

$$h \in W^{2-\frac{1}{2p}}_p(J; L_p(\mathbb{R}^{n-1})) \cap H^1_p(J; W^{2-\frac{1}{p}}_p(\mathbb{R}^{n-1})) \cap L_p(J; W^{3-\frac{1}{p}}_p(\mathbb{R}^{n-1})),$$
$$h_0 \in (W^{2-\frac{1}{p}}_p(\mathbb{R}^{n-1}), W^{3-\frac{1}{p}}_p(\mathbb{R}^{n-1}))_{1-1/p,p} = W^{3-2/p}_p(\mathbb{R}^{n-1}) =: \mathbb{F}_{11}.$$

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$[\![\rho]\!] = 0$ case

$$h \in W^{\frac{3}{2}-\frac{1}{2p}}_p(J; L_p(\mathbb{R}^{n-1})) \cap W^{1-\frac{1}{2p}}_p(J; H^2_p(\mathbb{R}^{n-1})) \cap L_p(J; W^{4-\frac{1}{p}}_p(\mathbb{R}^{n-1})),$$
$$h_0 \in (W^{2-\frac{1}{p}}_p(\mathbb{R}^{n-1}), W^{4-\frac{1}{p}}_p(\mathbb{R}^{n-1}))_{1-1/p,p} = W^{4-3/p}_p(\mathbb{R}^{n-1}) =: \mathbb{F}_{11}.$$

L_p-Maximal Regularity Theorem

$$\mathbb{E}(J) = \times_{j=1,\dots,5}\mathbb{E}_j(J), \quad \mathbb{F}(J) = \times_{j=1,\dots,11}\mathbb{F}_j(J).$$

Theorem (L_p-Maximal Regularity)

Let $1 < p < \infty$, $p \neq 3/2, 3$. In the case $[\![\rho]\!] = 0$, we assume $\ell_* = \ell(\theta_*) \neq 0$. The linearized problem admits a unique solution $(u, \pi, [\![\pi]\!], \theta, h) \in \mathbb{E}(J)$ if and only if the data $(f_u, f_d, f_\theta, g_p, g_\tau, g_n, g_\theta, g_j, g_h, u_0, \theta_0, h_0)$ belongs to the regularity class $\mathbb{F}(J)$ and satisfies the compatibility conditions (C).

Nonlinear Problems

Main Theorem ($[\![\rho]\!] \neq 0$ case)

Let $p > n+2$ and

$$(u_0, \theta_0, h_0) \in W_p^{2-2/p}(\Omega_0)^n \times W_p^{2-2/p}(\Omega_0) \times W_p^{3-2/p}(\mathbb{R}^{n-1})$$

be given. Assume that the compatibility conditions

$$\begin{aligned}
&\mathrm{div}\, u_0 = 0 && \text{in } \Omega_0 \\
&[\![\mu P_{\Gamma_0} E(u_0)\nu_0]\!] = 0, \quad [\![P_{\Gamma_0} u_0]\!] = 0 && \text{on } \Gamma_0 \\
&[\![\theta_0]\!] = 0, \quad [\![d\partial_{\nu_0}\theta_0]\!] + \ell(\theta_0)[\![\rho^{-1}]\!]^{-1}[\![u_0 \cdot \nu_0]\!] = 0 && \text{on } \Gamma_0
\end{aligned}$$

are satisfied. Then for each time $T > 0$ there exists $\eta > 0$ such that for

$$\|u_0\|_{W_p^{2-2/p}(\Omega_0)} + \|\theta_0 - \theta_*\|_{W_p^{2-2/p}(\Omega_0)} + \|h_0\|_{W_p^{3-2/p}(\mathbb{R}^{n-1})} < \eta,$$

there exists a unique solution $(u, \pi, \theta, h) \in \mathbb{E}(J, \Omega(t))$ for $t \in J$.

Main Results

Main Theorem ($[\![\rho]\!] = 0$ case)

Let $p > n+2$ and

$$(u_0, \theta_0, h_0) \in W_p^{2-2/p}(\Omega_0)^n \times W_p^{2-2/p}(\Omega_0) \times W_p^{4-3/p}(\mathbb{R}^{n-1})$$

be given. Assume that $\ell(\theta_*) \neq 0$ and the compatibility conditions

$$\begin{aligned} &\mathrm{div}\, u_0 = 0 && \text{in } \Omega_0 \\ &[\![\mu P_{\Gamma_0} E(u_0)\nu_0]\!] = 0, \quad [\![u_0]\!] = 0 && \text{on } \Gamma_0 \\ &[\![\theta_0]\!] = 0, \quad [\![\psi(\theta_0)]\!] + (\sigma/\rho) H_{\Gamma_0} = 0, \quad [\![d\partial_{\nu_0}\theta_0]\!] \in W_p^{2-6/p} && \text{on } \Gamma_0 \end{aligned}$$

are satisfied. Then for each time $T > 0$ there exists $\eta > 0$ such that for

$$\|u_0\|_{W_p^{2-2/p}(\Omega_0)} + \|\theta_0 - \theta_*\|_{W_p^{2-2/p}(\Omega_0)} + \|h_0\|_{W_p^{4-3/p}(\mathbb{R}^{n-1})} < \eta,$$

there exists a unique solution $(u, \pi, \theta, h) \in \mathbb{E}(J, \Omega(t))$ for $t \in J$.

Outline of a Proof ($[\![\rho]\!] \neq 0$ case)

Step 1 (1) is transformed to a problem on a fixed domain $\dot{\mathbb{R}}^n$ by means of transformation

$$v(t,x) = u(t,x',h(t,x')+x_n)$$
$$\vartheta(t,x) = \theta(t,x',h(t,x')+x_n) - \theta_*$$
$$q(t,x) = \pi(t,x',h(t,x')+x_n) - \pi_*$$

Outline of a Proof ($[\![\rho]\!] \neq 0$ case)

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q(t,x) &= \pi(t,x',h(t,x')+x_n) - \pi_*
\end{aligned}
$$

(1) is deformed into

$$
\begin{aligned}
\rho\partial_t v - \mu\Delta v + \nabla q &= f_u(v,q,h) && \text{in} \quad \dot{\mathbb{R}}^n, \quad t>0\\
\operatorname{div} v &= f_d(v,h) && \text{in} \quad \dot{\mathbb{R}}^n, \quad t>0\\
[\![v]\!] - [\![\rho^{-1}]\!] j\nu_* &= g_p(h,j) && \text{on} \quad \mathbb{R}^{n-1}, \ t>0\\
-[\![\mu\,(\partial_n v' + \nabla' v_n)]\!] &= g_\tau(v,[\![q]\!],h,j) && \text{on} \quad \mathbb{R}^{n-1}, \ t>0\\
-2[\![\mu\partial_n v_n]\!] + [\![q]\!] - \sigma\Delta' h &= g_n(v,h) && \text{on} \quad \mathbb{R}^{n-1}, \ t>0\\
v(0) &= v_0 && \text{in} \quad \dot{\mathbb{R}}^n,
\end{aligned}
$$

$$\begin{aligned}
\rho\kappa_*\partial_t\vartheta - d\Delta\vartheta &= f_\theta(v,\vartheta,h) && \text{in} \quad \dot{\mathbb{R}}^n, \quad t>0 \\
[\![\vartheta]\!] &= 0 && \text{on} \quad \mathbb{R}^{n-1},\ t>0 \\
-\ell_* j - [\![d\partial_n\vartheta]\!] &= g_j(\vartheta,h,j) && \text{on} \quad \mathbb{R}^{n-1},\ t>0 \\
\vartheta(0) &= \vartheta_0 && \text{in} \quad \dot{\mathbb{R}}^n,
\end{aligned}$$

$$\begin{aligned}
(\ell_*/\theta_*)\vartheta - [\![2\rho^{-1}\mu\partial_n v_n]\!] + [\![\rho^{-1}q]\!] &= g_\theta(v,[\![q]\!],\vartheta,h,j) && \text{on} \quad \mathbb{R}^{n-1},\ t>0 \\
\partial_t h - \mathcal{R}v_n + j/\rho &= g_h(v,h,j) && \text{on} \quad \mathbb{R}^{n-1},\ t>0 \\
h(0) &= h_0 && \text{on} \quad \mathbb{R}^{n-1}. \qquad (2)
\end{aligned}$$

where $v_0(x) = u_0(x', h_0(x') + x_n)$, $\vartheta_0(x) = \theta_0(x', h_0(x') + x_n) - \theta_*$.

$$
\begin{aligned}
\rho\kappa_*\partial_t\vartheta - d\Delta\vartheta &= f_\theta(v,\vartheta,h) && \text{in} \quad \dot{\mathbb{R}}^n, \quad t>0\\
[\![\vartheta]\!] &= 0 && \text{on} \quad \mathbb{R}^{n-1},\ t>0\\
-\ell_* j - [\![d\partial_n\vartheta]\!] &= g_j(\vartheta,h,j) && \text{on} \quad \mathbb{R}^{n-1},\ t>0\\
\vartheta(0) &= \vartheta_0 && \text{in} \quad \dot{\mathbb{R}}^n,
\end{aligned}
$$

$$
\begin{aligned}
(\ell_*/\theta_*)\vartheta - [\![2\rho^{-1}\mu\partial_n v_n]\!] + [\![\rho^{-1}q]\!] &= g_\theta(v,[\![q]\!],\vartheta,h,j) && \text{on} \quad \mathbb{R}^{n-1},\ t>0\\
\partial_t h - \mathcal{R}v_n + j/\rho &= g_h(v,h,j) && \text{on} \quad \mathbb{R}^{n-1},\ t>0\\
h(0) &= h_0 && \text{on} \quad \mathbb{R}^{n-1}. \qquad (2)
\end{aligned}
$$

where $v_0(x) = u_0(x', h_0(x') + x_n)$, $\vartheta_0(x) = \theta_0(x', h_0(x') + x_n) - \theta_*$.

Since $[\![q]\!]$ and j can be expressed by v, the right members are given by functions of v, ϑ and h.

Step 2 Reduction to time trace 0.

We construct $z^* = (v^*, q^*, [\![q^*]\!], \vartheta^*, h^*, j^*) \in \mathbb{E}(J)$ such that

$$v^*(0) = v_0, \quad \vartheta^*(0) = \vartheta_0, \quad h^*(0) = h_0.$$

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Set

$$\begin{aligned}
&f_d^* = e^{t\Delta} f_d(v_0, h_0) \in \mathbb{F}_2(J), \quad g_p^* = e^{t\Delta'} g_p(v_0, h_0) \in \mathbb{F}_4(J),\\
&g_\tau^* = e^{t\Delta'} g_\tau(v_0, h_0) \in \mathbb{F}_5(J), \quad g_n^* = e^{t\Delta'} g_n(v_0, h_0) \in \mathbb{F}_6(J),\\
&g_\theta^* = e^{t\Delta'} g_\theta(\vartheta_0, h_0) \in \mathbb{F}_7(J), \quad g_j^* = e^{t\Delta'} f_d(v_0, \vartheta_0, h_0) \in \mathbb{F}_8(J).\\
&g_h^* = e^{t\Delta'} g_h(v_0, h_0) \in \mathbb{F}_9(J).
\end{aligned}$$

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By L_p maximal regularity theorem, there exist a unique solution

$$Lz^* = (0, f_d^*, 0, g_p^*, g_\tau^*, g_n^*, g_\theta^*, g_j^*, g_h^*, u_0, \theta_0, h_0).$$

Step 3 We set

$$X = \{(v, \vartheta, h) \mid v \in W_p^{2-2/p}(\dot{\mathbb{R}}^n)^n,\ [\![P_* v]\!] = 0,$$
$$\vartheta \in W_p^{2-2/p}(\dot{\mathbb{R}}^n),\ [\![\vartheta]\!] = 0, \quad h \in W_p^{3-2/p}(\mathbb{R}^{n-1})\},$$

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$$\begin{aligned}\mathcal{PM}=\{(v,\vartheta,h)\in X\,|\mathrm{div}\,v=f_d(v,h),\ [\![v]\!]-[\![\rho^{-1}]\!]j(v)\nu_*=g_p(v,h)\\ -[\![\mu(\partial_n v'+\nabla' v_n)]\!]=g_\tau(v,h),\\ -2[\![\mu\partial_n v_n]\!]+[\![q(v)]\!]-\sigma\Delta' h=g_n(v,h)\\ -\ell_* j-[\![d\partial_n\vartheta]\!]=g_\theta(\vartheta,h),\\ (\ell_*/\theta_*)\vartheta-[\![2\rho^{-1}\mu\partial_n v_n]\!]+[\![\rho^{-1}q(v)]\!]=g_j(v,\vartheta,h)\},\end{aligned}$$

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$$\mathcal{PM} = \{(v, \vartheta, h) \in X \,|\mathrm{div}\, v = f_d(v, h),\ [\![v]\!] - [\![\rho^{-1}]\!] j(v)\nu_* = g_p(v, h)$$
$$- [\![\mu(\partial_n v' + \nabla' v_n)]\!] = g_\tau(v, h),$$
$$- 2[\![\mu\partial_n v_n]\!] + [\![q(v)]\!] - \sigma\Delta' h = g_n(v, h)$$
$$- \ell_* j - [\![d\partial_n \vartheta]\!] = g_\theta(\vartheta, h),$$
$$(\ell_*/\theta_*)\vartheta - [\![2\rho^{-1}\mu\partial_n v_n]\!] + [\![\rho^{-1} q(v)]\!] = g_j(v, \vartheta, h)\},$$

$$X^0 = \{(v, \vartheta, h) \in X \mid \mathrm{div}\, v = 0,\ [\![v]\!] - [\![\rho^{-1}]\!] j(v)\nu_* = 0$$
$$- [\![\mu(\partial_n v' + \nabla' v_n)]\!] = 0,\ -2[\![\mu\partial_n v_n]\!] + [\![q(v)]\!] - \sigma\Delta' h = 0$$
$$- \ell_* j - [\![d\partial_n \vartheta]\!] = 0,\ (\ell_*/\theta_*)\vartheta - [\![2\rho^{-1}\mu\partial_n v_n]\!] + [\![\rho^{-1} q(v)]\!] = 0\}.$$

For given $(v_0, \vartheta_0, h_0) \in \mathcal{PM}$, it is possible to parametrize over $(\tilde{v}_0, \tilde{\vartheta}_0, \tilde{h}_0) \in X^0$, namely there exists $\phi \in C^3$ such that

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As in the argument in Step 2, for given $(v_0, \vartheta_0, h_0) \in \mathcal{PM}$, we construct

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As in the argument in Step 2, for given $(v_0, \vartheta_0, h_0) \in \mathcal{PM}$, we construct

$$z^* = z^*(v_0, \vartheta_0, h_0) \in \mathbb{E}(J).$$

For the simplicity we set

$$w = (v_0, \vartheta_0, h_0), \quad \tilde{w} = (\tilde{v}_0, \tilde{\vartheta}_0, \tilde{h}_0)$$

Then $z^* = z^*(\phi(\tilde{w}))$.

Step 4 (2) is equivalent to

$$L(z + z^*(\phi(\tilde{w})) = N(z + z^*(\phi(\tilde{w})), \quad z \in {}_0\mathbb{E}(J).$$

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If we set

$$G(z, \tilde{w}) := Lz + Lz^*(\phi(\tilde{w})) - N(z + z^*(\phi(\tilde{w})),$$

then it holds that $G(z, \tilde{w}) = 0$. Since

$$G : {}_0\mathbb{E}(J) \times X^0 \to {}_0\mathbb{F}(J),$$
$$G(0,0) = 0, \quad D_z G(0,0) = L,$$

L_p maximal regularity shows L is an isomorphism from ${}_0\mathbb{E}(J)$ to ${}_0\mathbb{F}(J)$.

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L_p maximal regularity shows L is an isomorphism from ${}_0\mathbb{E}(J)$ to ${}_0\mathbb{F}(J)$.
By implicit function theorem,

$$\begin{aligned} &{}^\exists B_r(0) \in X^0, {}^\exists B_\delta(0) \in {}_0\mathbb{E}(J), \\ &F : B_r(0) \to B_\delta(0) \quad \text{is } C^3, \\ &F(0) = 0, \quad G(F(\tilde{w}), \tilde{w}) = 0 \text{ for } {}^\forall w \in B_r(0). \end{aligned}$$

Therefore $F(\tilde{w}) + z^*(\phi(\tilde{w}))$ is the solution.

$p \neq q$ case

Theorem ($p \neq q$, $[\![\rho]\!] \neq 0$ case) (S)

Let $1 < p, q < \infty$. If the data satisfy the following regularity conditions:

$$\begin{aligned}
&f_u \in L_p(J, L_q(\mathbb{R}^n))^n, \quad f_\theta \in L_p(J, L_q(\mathbb{R}^n)),\\
&f_d \in W_p^1(J, \hat{W}_q^{-1}(\mathbb{R}^n)) \cap L_p(J, W_q^1(\dot{\mathbb{R}}^n)),\\
&(g_\tau, g_n) \in H_p^{\frac{1}{2}}(J, L_q(\mathbb{R}^n))^n \cap L_p(J, W_q^1(\dot{\mathbb{R}}^n))^n,\\
&g_j \in H_p^{\frac{1}{2}}(J, L_q(\mathbb{R}^n)) \cap L_p(J, W_q^1(\dot{\mathbb{R}}^n)),\\
&g_p, g_\theta, g_h \in W_p^1(J, L_q(\mathbb{R}^n)) \cap L_p(J, W_q^2(\dot{\mathbb{R}}^n)), \quad g_h \in L_p(J, W_q^2(\dot{\mathbb{R}}^n)),\\
&u_0 \in (L_q(\mathbb{R}^n), W_q^2(\dot{\mathbb{R}}^n))^n_{1-1/p,p} = B_{q,p}^{2(1-1/p)}(\dot{\mathbb{R}}^n)^n, \quad \theta_0 \in B_{q,p}^{2(1-1/p)}(\dot{\mathbb{R}}^n),\\
&h_0 \in (W_q^{2-1/q}(\mathbb{R}^{n-1}), W_q^{3-1/q}(\mathbb{R}^{n-1}))_{1-1/p,p} = B_{q,p}^{3-1/p-1/q}(\mathbb{R}^{n-1})
\end{aligned}$$

and compatibility conditions (C)

$p \neq q$ case

then the linearized problem admits a unique solution

$$
\begin{aligned}
&u \in W_p^1(J, L_q(\mathbb{R}^n))^n \cap L_p(J, W_q^2(\dot{\mathbb{R}}^n))^n,\\
&\pi \in L_p(J, \hat{W}_q^1(\dot{\mathbb{R}}^n)),\\
&[\![\pi]\!] \in H_p^{\frac{1}{2}}(J, L_q(\mathbb{R}^n)) \cap L_p(J, W_q^1(\dot{\mathbb{R}}^n)),\\
&\theta \in W_p^1(J, L_q(\mathbb{R}^n)) \cap L_p(J, W_q^2(\dot{\mathbb{R}}^n)),\\
&h \in W_p^1(J, W_q^{2-1/q}(\mathbb{R}^{n-1})) \cap L_p(J, W_q^{3-1/q}(\mathbb{R}^{n-1})).
\end{aligned}
$$

We use operator-valued Foulier multiplier theorem as the same method in Shibata-S ('08 J. reine angew. Math.).