Characterization of two-dimensional point-vortex system in terms of statistically-defined temperature

Yuichi YATSUYANAGI

Faculty of Education, Shizuoka University

March 11, 2010

International Workshop on Mathematical Fluid Dynamics at Waseda University

Collaborator:

Prof. Tadatsugu HATORI

Prof. Yasuhito KIWAMOTO

Dr. Toshikazu EBISUZAKI

Agenda

1. Prelude of this research

- A vortex experiment using a non-neutral (pure electron) plasma
- Time-govering equation of guiding-center plasma = two-dimensional Euler equation

2. Negative absolute temperature state in two-dimensioanl point-vortex system confined in a finite area

- First introduced by Onsager 1949

3. Numerical results

- Massive numerical simulation using a special-purpose supercomputer

4. Analytical result

- Effective diffusion term in point-vortex system due to collisional processes of discrete point vortices

1. Prelude of this research

♦ Vortex experiments

using a non-neutral (pure electron) plasma

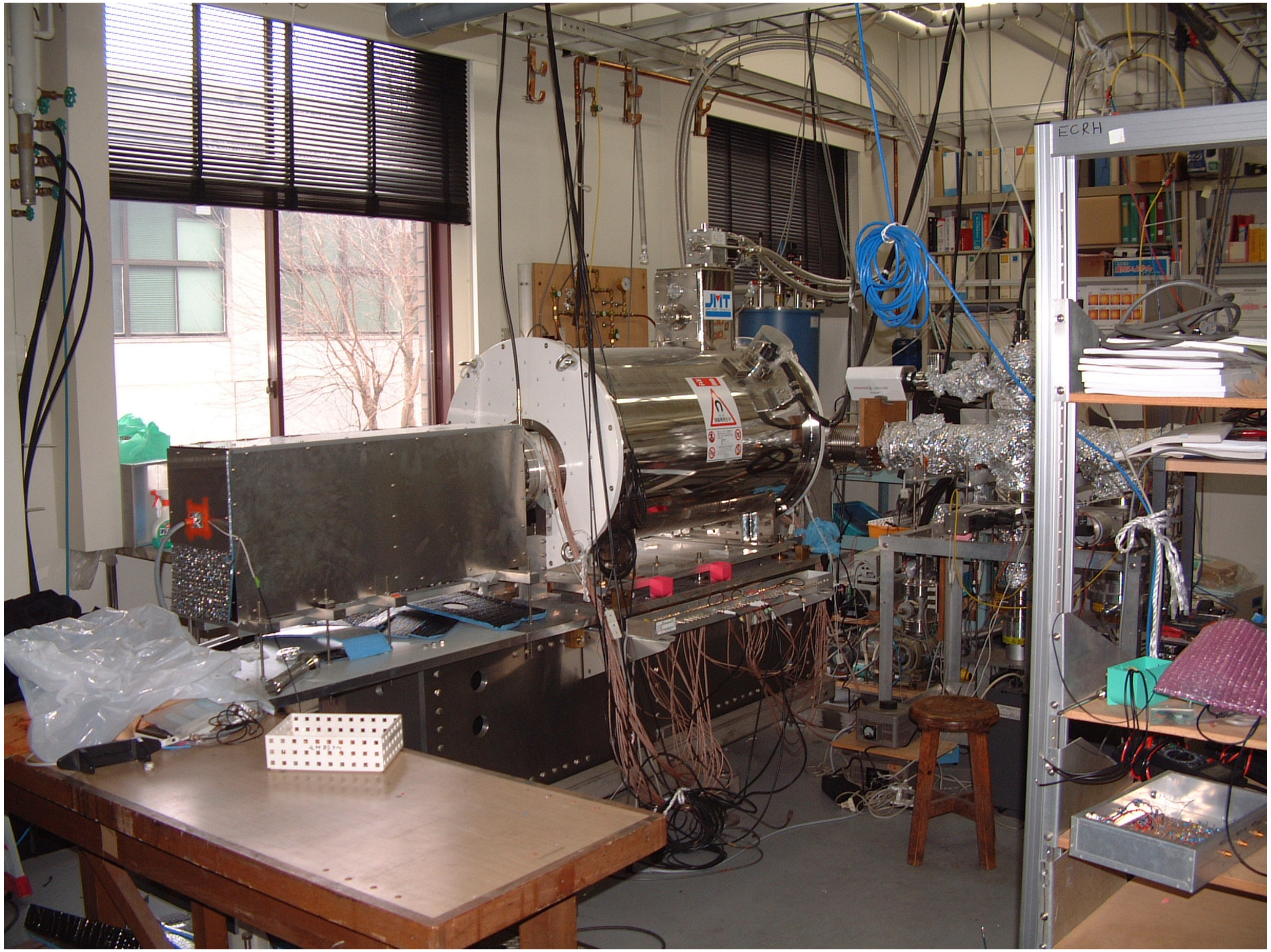


FIG. 1.1 A photo of non-neutral plasma trap in Kiwamoto Lab. at Kyoto Univ.

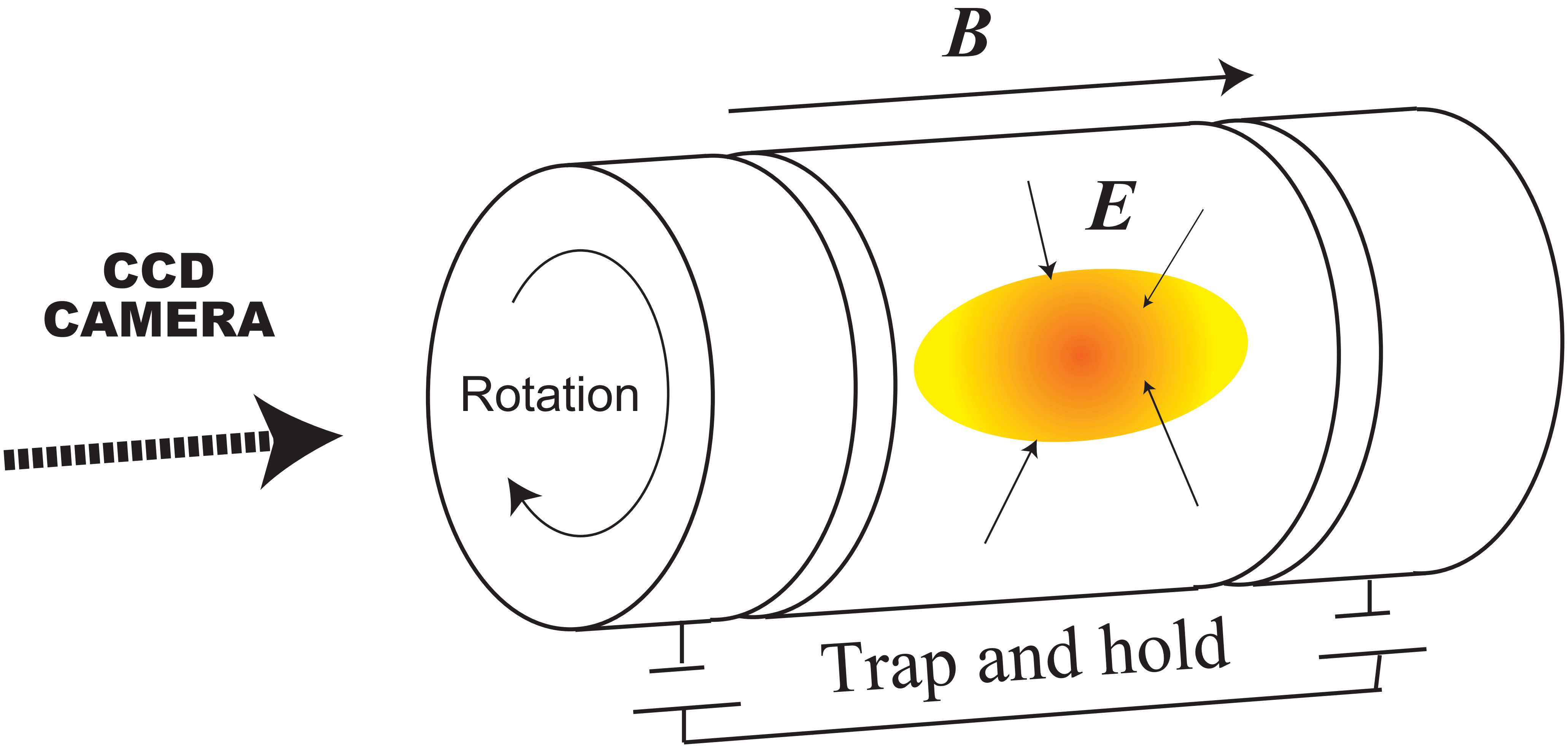


FIG 1.2 Schematics of the trap.

♦ Analogy of the 2D non-neutral plasma equation to the Euler equation

The equation of motion of an electron in the trap:

$$m\frac{d\boldsymbol{v}}{dt} = -e(\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B})$$

$$\boldsymbol{B} = B_0\hat{\boldsymbol{z}}$$

The time-averaged Eq. (1) over a gyro-motion around $\boldsymbol{B}$:

$$\boldsymbol{v} = \frac{\boldsymbol{E}\times\boldsymbol{B}}{|\boldsymbol{B}|^2} = \frac{1}{B_0}\hat{\boldsymbol{z}}\times\nabla\phi \quad \Leftrightarrow \quad \boldsymbol{u} = \hat{\boldsymbol{z}}\times\nabla\psi$$

where ϕ is the electrostatic potential:

$$\boldsymbol{E} = -\nabla\phi$$

ψ is the stream function for the 2D flow

$$\frac{\phi}{B_0} = \psi$$

The vorticity is proportional to the number density of electron n:

$$\omega_z \hat{\boldsymbol{z}} = \nabla \times \boldsymbol{v} = \frac{\hat{\boldsymbol{z}}}{B_0} \nabla^2 \phi = \frac{en}{\varepsilon_0 B_0} \hat{\boldsymbol{z}} \quad \Leftrightarrow \quad \omega_z \hat{\boldsymbol{z}} = \hat{\boldsymbol{z}} \nabla^2 \psi$$

The two-dimensional electron fluid is imcompressible:

$$\nabla \cdot \boldsymbol{v} = \nabla \cdot \left(\frac{1}{B_0} \hat{\boldsymbol{z}} \times \nabla \phi \right) = 0 \quad \Leftrightarrow \quad \nabla \cdot \boldsymbol{u} = \nabla \cdot [\hat{\boldsymbol{z}} \times \nabla \psi] = 0$$

Thus, 2D electron fluid is identical to the inviscid and incompressible Euler equation

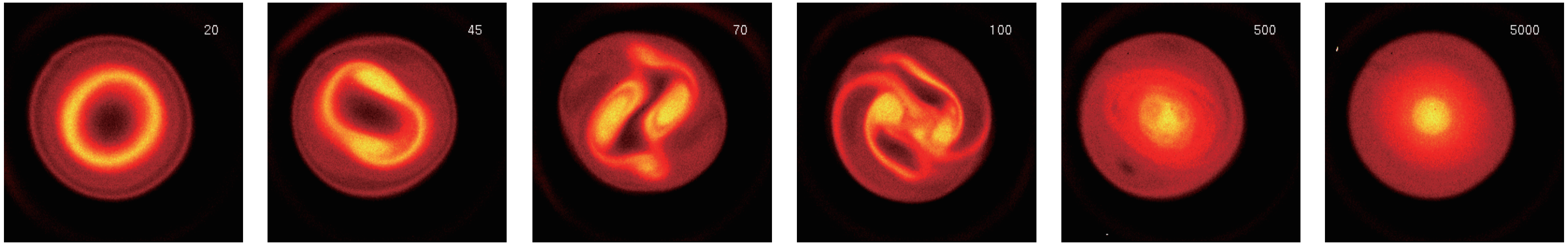
$$\frac{\partial \omega_z}{\partial t} + \boldsymbol{u} \cdot \nabla \omega_z = 0$$

The electron motion can be traced by the point vortex method !

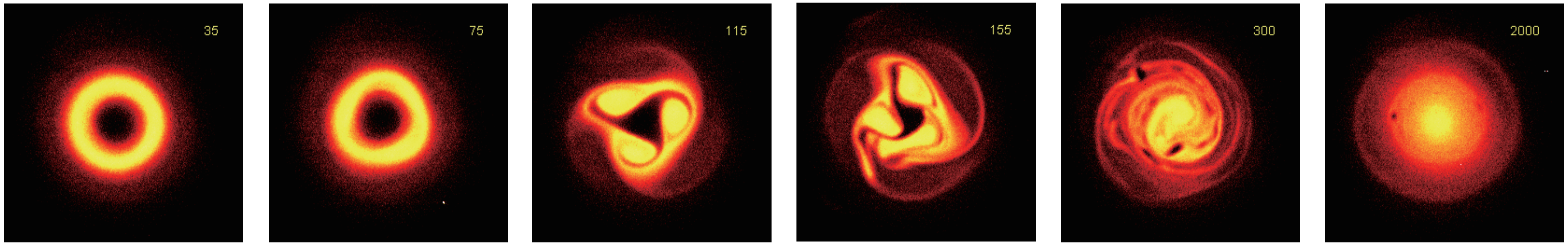
♦Some experimental results at Kiwamoto Lab.

Diocotron (Kelvin-Helmholtz) instability:

mode 2:



mode 3:



mode 4:

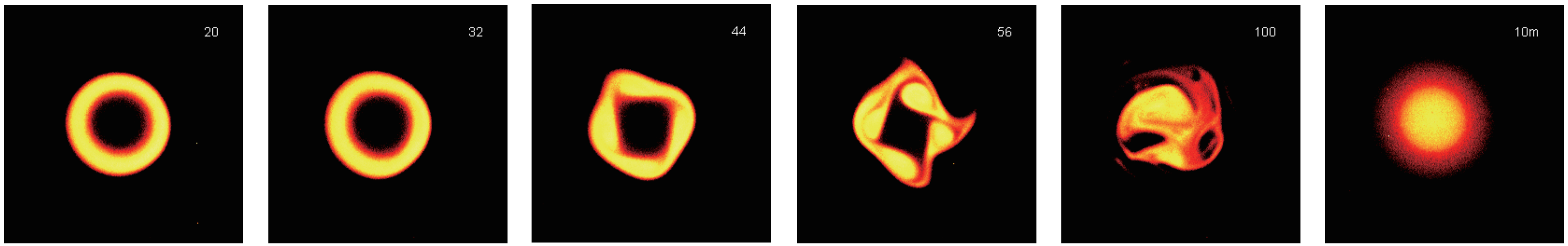
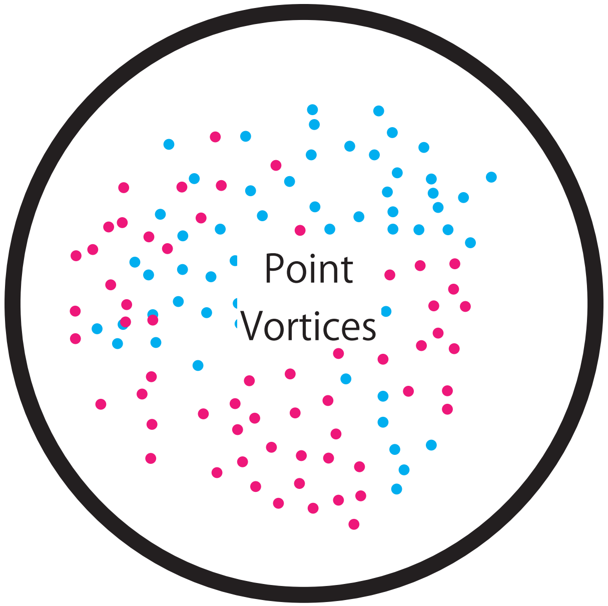


FIG 1.3 Two-dimensional electron distribution perpendicular to the magnetic field

2. Negative absolute temperature state in two-dimensioanl point-vortex system confined in a finite area

♦ Target point vortex system

N positive point vortices and N negative point vortices are confined in a circular wall with radius R



Circular Wall with radius R

Vorticity field is descritized as:

$$\omega_z(\boldsymbol{r}, t) = \sum_i \Omega_i \delta\left(\boldsymbol{r} - \boldsymbol{r}_i(t)\right)$$

Circulation of each point vortex $\Omega_i = \Omega_0$ or $-\Omega_0$ (Ω_0=constant)

Position vector $\boldsymbol{r}_i$

Equation of motion of the point vortex:

$$\Omega_i \frac{dx_i}{dt} = \frac{\partial}{\partial y_i} H, \quad \Omega_i \frac{dy_i}{dt} = -\frac{\partial}{\partial x_i} H$$

$$H = -\frac{1}{4\pi}\sum_{i=1}^{N}\sum_{j\neq i}^{N}\Omega_i\Omega_j \ln\left|\boldsymbol{r}_i - \boldsymbol{r}_j\right| - \frac{1}{4\pi}\sum_{i=1}^{N}\sum_{j}^{N}\Omega_i\Omega_j\left(\ln\left|\boldsymbol{r}_i - \bar{\boldsymbol{r}}_j\right| - \ln\frac{R}{\left|\boldsymbol{r}_j\right|}\right)$$

Explicit Biot-Savart integral form:

$$\frac{d\boldsymbol{r}_i}{dt} = -\frac{1}{2\pi}\sum_{j\neq i}^{2N}\Omega_j \frac{\left(\boldsymbol{r}_i - \boldsymbol{r}_j\right)\times\hat{\boldsymbol{z}}}{\left|\boldsymbol{r}_i - \boldsymbol{r}_j\right|^2} + \frac{1}{2\pi}\sum_{j}^{2N}\Omega_j \frac{\left(\boldsymbol{r}_i - \bar{\boldsymbol{r}}_j\right)\times\hat{\boldsymbol{z}}}{\left|\boldsymbol{r}_i - \bar{\boldsymbol{r}}_j\right|^2}$$

Wall effect is introduced by the image vortex at

$$\bar{\boldsymbol{r}}_j = \frac{R^2}{\left|\boldsymbol{r}_j\right|^2}\boldsymbol{r}_j$$

♦ Negative temperature state in the 2D point-vortex system

The statistical definition of (inverse) temperature:

$$\beta = \frac{dS}{dE} = \frac{d \log W(E)}{dE}$$

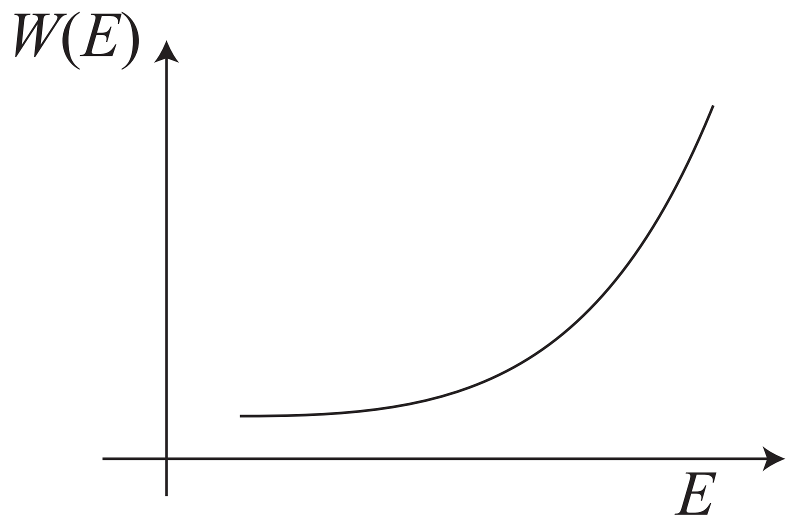


FIG 2.1 "Normal" relation between a density of state and $\mathrm{d}S / \mathrm{d}E \geqq 0$

Suppose a system whose total phase space volume is finite:

$$\int_{-\infty}^{\infty} W(E)dE < \infty$$

In such a system, $W(E)$ has at leaset a peak at an energy value E_0 and $\mathrm{d}S / \mathrm{d}E < 0$ at $E > E_0$.

$$\mathrm{d}S / \mathrm{d}E < 0 \quad \Leftrightarrow \quad \beta < 0$$

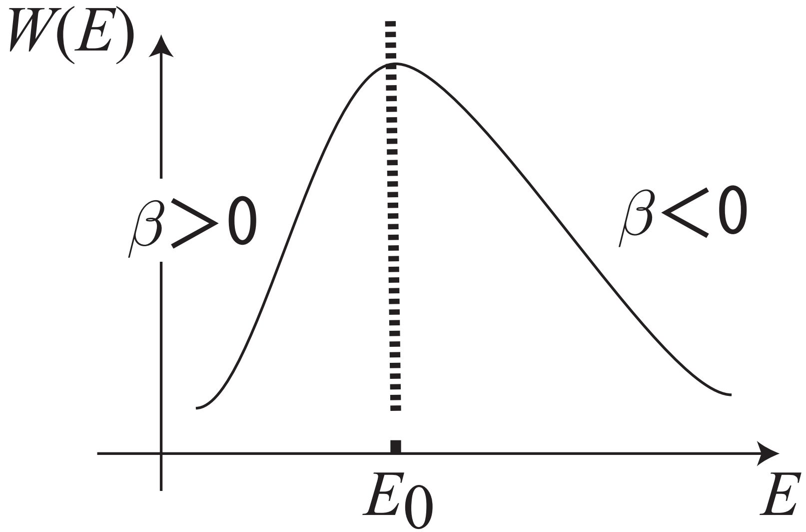


FIG 2.2 "Special" relation for a system that has a negative temperature state.

$\mathrm{d}S / \mathrm{d}E < 0$ at $E > E_0$

Onsager pointed out that:

The phase space equals the configuration space for the 2D point-vortex system by making an analogy to the usual Hamilton equation.

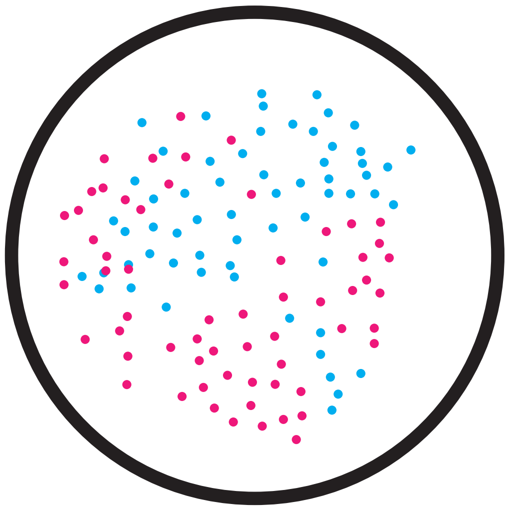
$$\frac{dq_i}{dt} = \frac{\partial}{\partial p_i} H, \quad \frac{dp_i}{dt} = -\frac{\partial}{\partial q_i} H \quad \Leftrightarrow \quad \Omega_i \frac{dx_i}{dt} = \frac{\partial}{\partial y_i} H, \quad \Omega_i \frac{dy_i}{dt} = -\frac{\partial}{\partial x_i} H$$

$$(q, p) \quad \Leftrightarrow \quad (x, y)$$

and

The total phase space volume is finite for the system confined in a finite space.

Phase space volume = $(\pi R^2)^{2N}$



=Configufation spae

=Phase spae !

→ 2D point vortex system has a negative temperature state.

3. Numerical results

- Massive numerical simulation using a special-purpose supercomputer, MDGRAPE-3

FIG. 3.1 Hardware accelerator: MDGRAPE-3

PCI-X board

Approx. 300 times faster than a normal PC

Density of state

Direct calculation of the density of state as functions of the system energy E and

inertia $I = \sum_{i}^{2N} \Gamma_i \left| \boldsymbol{r}_i \right|^2$

random sampling of states based on microcanonical statistics

number of vortices = 6724

number of total states sampled = 10^8

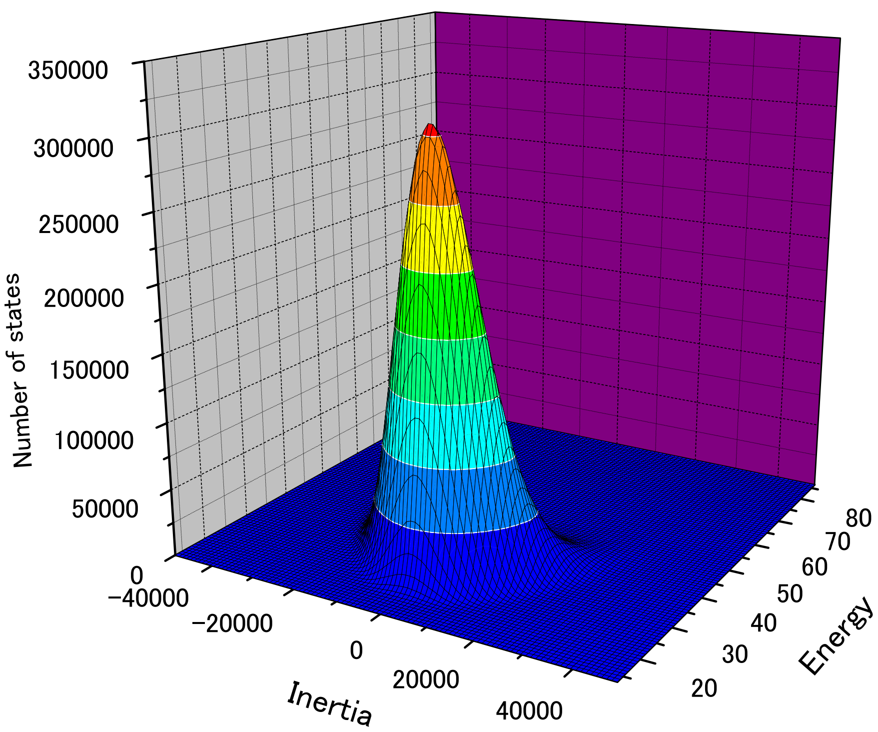


FIG. 3.2　Density of state

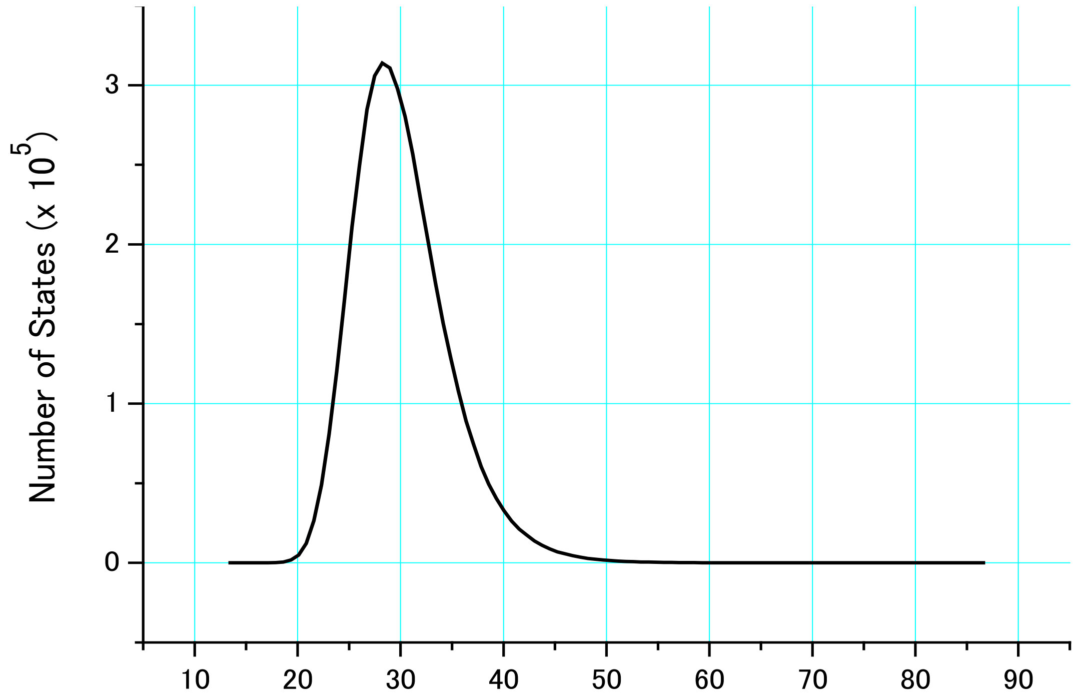


FIG. 3.3 Density of state (I=0)

Equilibrium distribution

Time-asymptotic equilibrium distributions are obtained by time-development simulations.

System energy is controlled by the initial distribution of the point vortices.

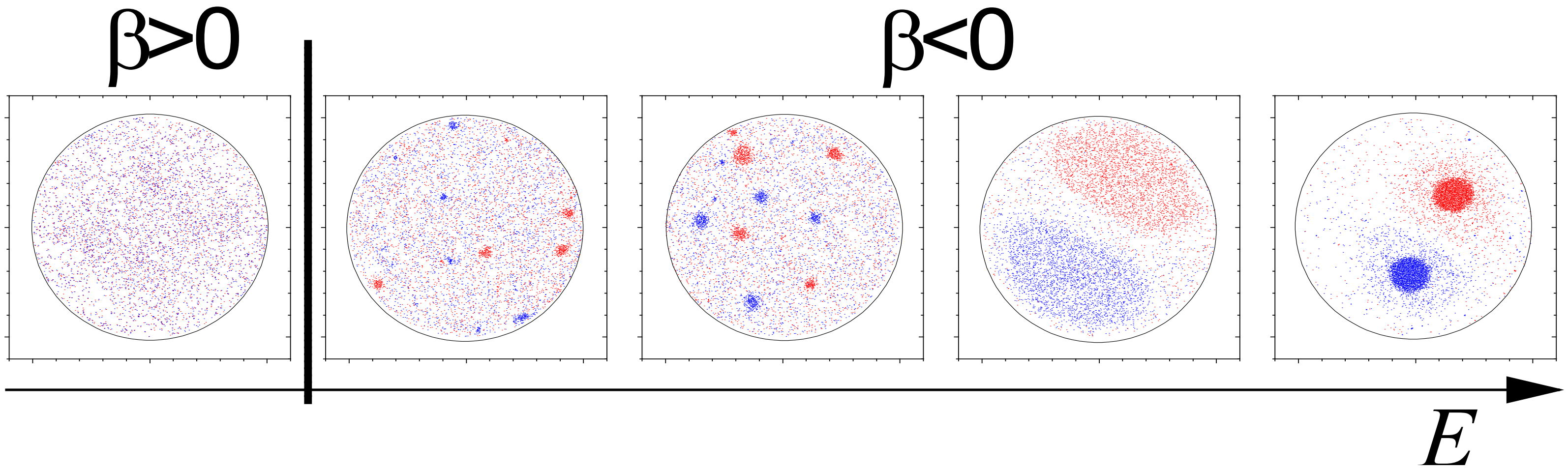
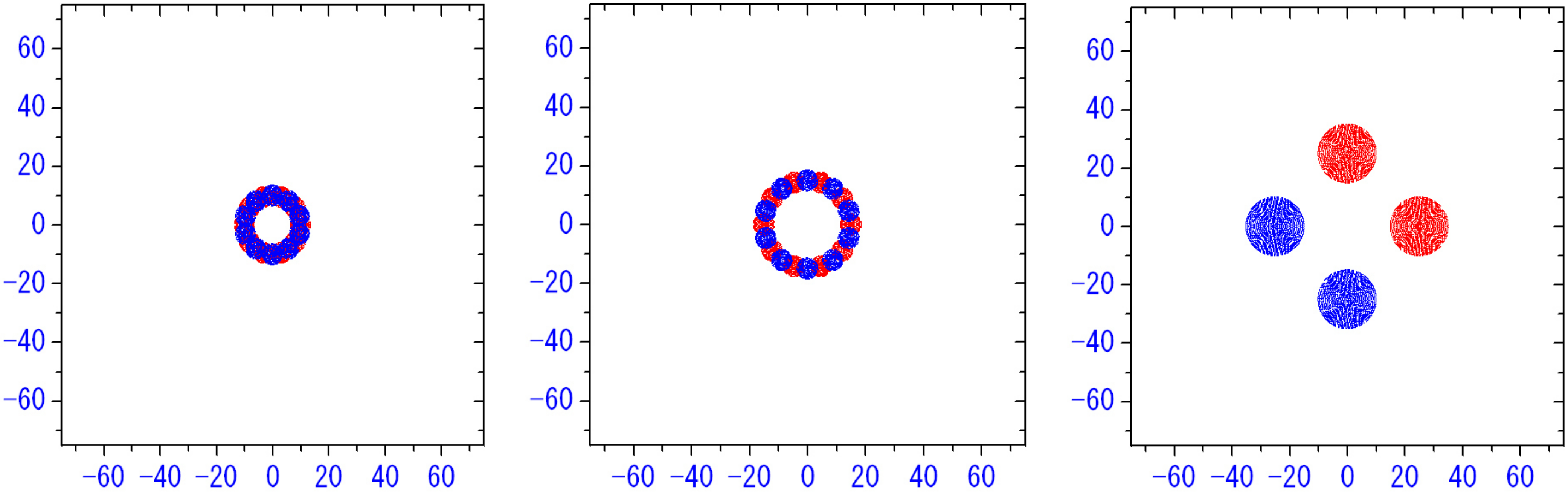


FIG. 3.4 Time-asymptotic equilibrium states

red: positive point vortex

blue: negative point vortex


60
40
20
0
−20
−40
−60
−60 −40 −20 0 20 40 60
60
40
20
0
−20
−40
−60
−60 −40 −20 0 20 40 60
60
40
20
0
−20
−40
−60
−60 −40 −20 0 20 40 60

4. Analytical result

♦ The vorticity equation has a microscopic solution.

The 2D inviscid vorticity equation has the EXACT point vortex solution:

$$\frac{\partial \omega_z}{\partial t} + \boldsymbol{u} \cdot \nabla \omega_z = 0$$

$$\omega_z(\boldsymbol{r}, t) = \sum_i \Omega_i \delta\left(\boldsymbol{r} - \boldsymbol{r}_i(t)\right)$$

$$\begin{aligned}
\frac{\partial}{\partial t}\omega_z(\boldsymbol{r}, t) &= \frac{\partial}{\partial t}\left(\sum_i \Omega_i \delta\left(\boldsymbol{r} - \boldsymbol{r}_i(t)\right)\right) \\
&= -\sum_i \Omega_i \left(\frac{\partial}{\partial t}\boldsymbol{r}_i(t)\right) \cdot \nabla \delta\left(\boldsymbol{r} - \boldsymbol{r}_i(t)\right) \\
&= -\sum_i \Omega_i \boldsymbol{u}(\boldsymbol{r}_i(t), t) \cdot \nabla \delta\left(\boldsymbol{r} - \boldsymbol{r}_i(t)\right) \\
&= -\boldsymbol{u}(\boldsymbol{r}, t) \cdot \nabla \sum_i \Omega_i \delta\left(\boldsymbol{r} - \boldsymbol{r}_i(t)\right) \\
&= -\boldsymbol{u}(\boldsymbol{r}, t) \cdot \nabla \omega_z(\boldsymbol{r}, t)
\end{aligned}$$

This is a **rare case** that a macroscopic equation has a microscopic, particle solution.

There should be an effective viscous effect in the microscopic particle system.

♦ Effective viscous effect due to "collison" between the point vortices

To identify microscopic and macroscopic physical quantities, new notations are introduced:

Macroscopic vorticity field:

$$\omega_z(\boldsymbol{r},t) \equiv \langle \hat{\omega}_z(\boldsymbol{r},t) \rangle = \left\langle \sum_i \Omega_i \delta\left(\boldsymbol{r} - \boldsymbol{r}_i(t)\right) \right\rangle_{S\cdot E}$$

Microscopic vorticity field:

$$\hat{\omega}_z(\boldsymbol{r},t) = \sum_i \Omega_i \delta\left(\boldsymbol{r} - \boldsymbol{r}_i(t)\right) = \langle \hat{\omega}_z(\boldsymbol{r},t) \rangle_{S\cdot E} + \delta\omega_z(\boldsymbol{r},t)$$

A hat "^" means a microscopic variable.

The microscopic vorticity consists of a macroscopic part and a fluctuation part.

Operator $\langle \cdot \rangle_E$ is an ensemble average and operator $\langle \cdot \rangle_S$ is an space average:

$$\langle \hat{\omega}_z(\boldsymbol{r},t) \rangle_S = \frac{1}{|\Lambda|} \int_{\Lambda(\boldsymbol{r})} d\boldsymbol{r}' \hat{\omega}_z(\boldsymbol{r}',t)$$

♦ Evaluation of the diffusion term

The space-averaged vorticity equation:

$$\frac{\partial}{\partial t}\omega_z(\boldsymbol{r},t)+\nabla\cdot\left[\boldsymbol{u}(\boldsymbol{r},t)\omega_z(\boldsymbol{r},t)\right]=-\nabla\cdot\left\langle\delta\boldsymbol{u}(\boldsymbol{r},t)\delta\omega_z(\boldsymbol{r},t)\right\rangle_{S\cdot E}$$

The right hand side of this Eq. is the effective diffusion term.

To obtain an explicit formula of the diffusion term, use the linearlized vorticity equation:

$$\frac{\partial}{\partial t}\delta\omega_z(\boldsymbol{r},t)+\boldsymbol{u}(\boldsymbol{r},t)\cdot\nabla\delta\omega_z(\boldsymbol{r},t)=-\delta\boldsymbol{u}(\boldsymbol{r},t)\cdot\nabla\omega_z(\boldsymbol{r},t)$$

Assuming the macroscopic variables approximately constant in the microscopic scale, a formal solution is obtained:

$$\delta\omega_z(\boldsymbol{r},t)=-\int_{-\infty}^{t}d\tau\delta\boldsymbol{u}(\boldsymbol{r}-(t-\tau)\boldsymbol{u}(\boldsymbol{r},t),\tau)\cdot\nabla\omega_z(\boldsymbol{r},t)$$

Note that velocity correlation time is sufficiently short.

Using this result, explicit diffusion term is written as:

$$\frac{\partial}{\partial t}\omega_z(\boldsymbol{r},t)+\nabla\cdot\left[\boldsymbol{u}(\boldsymbol{r},t)\omega_z(\boldsymbol{r},t)\right]=-\nabla\cdot\left\langle\delta\boldsymbol{u}(\boldsymbol{r},t)\delta\omega_z(\boldsymbol{r},t)\right\rangle_{S\cdot E}$$

$$-\nabla\cdot\left\langle\delta\boldsymbol{u}(\boldsymbol{r},t)\delta\omega_z(\boldsymbol{r},t)\right\rangle_{S\cdot E}=-\nabla\cdot\left(\overleftrightarrow{\eta}\cdot\nabla\omega_z\right)$$

$$\overleftrightarrow{\eta}=\left\langle\int_{-\infty}^{t}d\tau\delta\boldsymbol{u}(\boldsymbol{r},t)\delta\boldsymbol{u}(\boldsymbol{r}-(t-\tau)\boldsymbol{u},\tau)\right\rangle_{S\cdot E}$$

The result is analogous to the well-known Kubo formula.

5. Conclusion

- The existence of the negative temperature state in two-dimensional point vortex system is confirmed through the massive numerical simulations.

- In the negative temperature state, point vortices of the same sign makes a clump, while in the positive temperature state, both sign vortices mix up with each other and forms a uniform distribution.

- Effective diffusion coefficient is derived by using the Klimontvich formalism.

- The diffusion term arises from the discreteness of the vorticity.

- Our result is an extension of the well-known Kubo formula under a presence of flow.