Characterization of two-dimensional point-vortex system in terms of statistically-defined temperature

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Agenda

1. Prelude of this research
   - A vortex experiment using a non-neutral (pure electron) plasma
   - Time-governing equation of guiding-center plasma = two-dimensional Euler equation

2. Negative absolute temperature state in two-dimensional point-vortex system confined in a finite area
   - First introduced by Onsager 1949

3. Numerical results
   - Massive numerical simulation using a special-purpose supercomputer

4. Analytical result
   - Effective diffusion term in point-vortex system due to collisional processes of discrete point vortices
1. Prelude of this research

♦ Vortex experiments

using a non-neutral (pure electron) plasma

FIG. 1.1 A photo of non-neutral plasma trap in Kiwamoto Lab. at Kyoto Univ.
FIG 1.2 Schematics of the trap.
**Analogy of the 2D non-neutral plasma equation to the Euler equation**

The equation of motion of an electron in the trap:

\[
m \frac{dv}{dt} = -e(E + v \times B)
\]

\[
B = B_0 \hat{z}
\]

The time-averaged Eq. (1) over a gyro-motion around \(B\):

\[
v = \frac{E \times B}{|B|^2} = \frac{1}{B_0} \hat{z} \times \nabla \phi \quad \iff \quad u = \hat{z} \times \nabla \psi
\]

where \(\phi\) is the electrostatic potential: \(E = -\nabla \phi\)

\[
\frac{\phi}{B_0} = \psi
\]

\(\psi\) is the stream function for the 2D flow.
The vorticity is proportional to the number density of electron $n$:

$$\omega_z \hat{z} = \nabla \times \mathbf{v} = \frac{\hat{z}}{B_0} \nabla^2 \phi = \frac{en}{\varepsilon_0 B_0} \hat{z}$$

$$\iff \omega_z \hat{z} = \hat{z} \nabla^2 \psi$$

The two-dimensional electron fluid is incompressible:

$$\nabla \cdot \mathbf{v} = \nabla \cdot \left( \frac{1}{B_0} \hat{z} \times \nabla \phi \right) = 0$$

$$\iff \nabla \cdot \mathbf{u} = \nabla \cdot [\hat{z} \times \nabla \psi] = 0$$

Thus, 2D electron fluid is identical to the inviscid and incompressible Euler equation:

$$\frac{\partial \omega_z}{\partial t} + \mathbf{u} \cdot \nabla \omega_z = 0$$

The electron motion can be traced by the point vortex method!
Some experimental results at Kiwamoto Lab.

Diocotron (Kelvin-Helmholtz) instability:

mode 2:

mode 3:

mode 4:

FIG 1.3 Two-dimensional electron distribution perpendicular to the magnetic field
2. Negative absolute temperature state in two-dimensional point-vortex system confined in a finite area

♦ Target point vortex system

$N$ positive point vortices and $N$ negative point vortices are confined in a circular wall with radius $R$. 
Vorticity field is descritized as:

$$\omega_z(r, t) = \sum_i \Omega_i \delta(r - r_i(t))$$

Circulation of each point vortex

$$\Omega_i = \Omega_0 \text{ or } -\Omega_0 \quad (\Omega_0=\text{constant})$$

Position vector

$$r_i$$
Equation of motion of the point vortex:

\[
\Omega_i \frac{dx_i}{dt} = \frac{\partial}{\partial y_i} H, \quad \Omega_i \frac{dy_i}{dt} = -\frac{\partial}{\partial x_i} H
\]

\[
H = -\frac{1}{4\pi} \sum_{i=1}^{N} \sum_{j \neq i}^{N} \Omega_i \Omega_j \ln |\mathbf{r}_i - \mathbf{r}_j| - \frac{1}{4\pi} \sum_{i=1}^{N} \sum_{j}^{N} \Omega_i \Omega_j \left( \ln |\mathbf{r}_i - \mathbf{r}_j| - \ln \frac{R}{|\mathbf{r}_j|} \right)
\]

Explicit Biot-Savart integral form:

\[
\frac{d\mathbf{r}_i}{dt} = -\frac{1}{2\pi} \sum_{j \neq i}^{2N} \Omega_j \frac{\left(\mathbf{r}_i - \mathbf{r}_j\right) \times \mathbf{\hat{z}}}{|\mathbf{r}_i - \mathbf{r}_j|^2} + \frac{1}{2\pi} \sum_{j}^{2N} \Omega_j \frac{\left(\mathbf{r}_i - \mathbf{\bar{r}}_j\right) \times \mathbf{\hat{z}}}{|\mathbf{r}_i - \mathbf{\bar{r}}_j|^2}
\]

Wall effect is introduced by the image vortex at

\[
\mathbf{\bar{r}}_j = \frac{R^2}{|\mathbf{r}_j|^2} \mathbf{r}_j
\]
Negative temperature state in the 2D point-vortex system

The statistical definition of (inverse) temperature:

\[ \beta = \frac{dS}{dE} = \frac{d \log W(E)}{dE} \]

FIG 2.1 "Normal" relation between a density of state and \( dS / dE \geq 0 \)
Suppose a system whose total phase space volume is finite:

\[ \int_{-\infty}^{\infty} W(E) dE < \infty \]

In such a system, \( W(E) \) has at least a peak at an energy value \( E_0 \) and \( \frac{dS}{dE} < 0 \) at \( E > E_0 \).

\[ \frac{dS}{dE} < 0 \iff \beta < 0 \]

**FIG 2.2** "Special" relation for a system that has a negative temperature state.

\( dS / dE < 0 \) at \( E > E_0 \)
Onsager pointed out that:

**The phase space equals the configuration space for the 2D point-vortex system by making an analogy to the usual Hamilton equation.**

\[
\frac{dq_i}{dt} = \frac{\partial}{\partial p_i} H, \quad \frac{dp_i}{dt} = -\frac{\partial}{\partial q_i} H
\]

\[
\Omega_i \frac{dx_i}{dt} = \frac{\partial}{\partial y_i} H, \quad \Omega_i \frac{dy_i}{dt} = -\frac{\partial}{\partial x_i} H
\]

\[
(q, p) \iff (x, y)
\]

and

**The total phase space volume is finite for the system confined in a finite space.**

Phase space volume = \((\pi R^2)^{2N}\)

→ 2D point vortex system has a negative temperature state.
3. Numerical results

Massive numerical simulation using a special-purpose supercomputer, MDGRAPE-3

FIG. 3.1 Hardware accelerator: MDGRAPE-3 PCI-X board
Approx. 300 times faster than a normal PC
Density of state

Direct calculation of the density of state as functions of the system energy $E$ and inertia $I = \sum_i^{2N} \Gamma_i |r_i|^2$

random sampling of states based on microcanonical statistics

number of vortices $= 6724$

number of total states sampled $= 10^8$

FIG. 3.2 Density of state

FIG. 3.3 Density of state (I=0)
**Equilibrium distribution**

Time-asymptotic equilibrium distributions are obtained by time-development simulations. System energy is controlled by the initial distribution of the point vortices.

\[ \beta > 0 \quad \beta < 0 \]

**FIG. 3.4** Time-asymptotic equilibrium states

- red: positive point vortex
- blue: negative point vortex
4. Analytical result

The vorticity equation has a microscopic solution.

The 2D inviscid vorticity equation has the EXACT point vortex solution:

\[
\frac{\partial \omega_z}{\partial t} + u \cdot \nabla \omega_z = 0
\]

\[
\omega_z(r, t) = \sum_i \Omega_i \delta(r - r_i(t))
\]

This is a rare case that a macroscopic equation has a microscopic, particle solution.

There should be an effective viscous effect in the microscopic particle system.
♦ Effective viscous effect
due to "collison" between the point vortices

To identify microscopic and macroscopic physical quantities, new notations are introduced:

Macroscopic vorticity field:

\[ \omega_z(r, t) \equiv \langle \hat{\omega}_z(r, t) \rangle = \left\langle \sum_i \Omega_i \delta(r - r_i(t)) \right\rangle_{SE} \]

Microscopic vorticity field:

\[ \hat{\omega}_z(r, t) = \sum_i \Omega_i \delta(r - r_i(t)) = \langle \hat{\omega}_z(r, t) \rangle_{SE} + \delta\omega_z(r, t) \]

A hat "^" means a microscopic variable.
The microscopic vorticity consists of a macroscopic part and a fluctuation part.

Operator \( \langle \cdot \rangle_E \) is an ensemble average and operator \( \langle \cdot \rangle_S \) is an space average:

\[ \langle \hat{\omega}_z(r, t) \rangle_S = \frac{1}{|\Lambda|} \int_{\Lambda(r)} d\mathbf{r} \hat{\omega}_z(r', t) \]
Evaluation of the diffusion term

The space-averaged vorticity equation:
\[
\frac{\partial}{\partial t} \omega_z(r, t) + \nabla \cdot \left[ u(r, t) \omega_z(r, t) \right] = -\nabla \cdot \left\langle \delta u(r, t) \delta \omega_z(r, t) \right\rangle_{S.E}
\]

The right hand side of this Eq. is the effective diffusion term.

To obtain an explicit formula of the diffusion term, use the linearized vorticity equation:
\[
\frac{\partial}{\partial t} \delta \omega_z(r, t) + u(r, t) \cdot \nabla \delta \omega_z(r, t) = -\delta u(r, t) \cdot \nabla \omega_z(r, t)
\]

Assuming the macroscopic variables approximately constant in the microscopic scale, a formal solution is obtained:
\[
\delta \omega_z(r, t) = -\int_{-\infty}^{t} d\tau \delta u(r - (t - \tau)u(r, t), \tau) \cdot \nabla \omega_z(r, t)
\]

Note that velocity correlation time is sufficiently short.
Using this result, explicit diffusion term is written as:

\[ \frac{\partial}{\partial t} \omega_z(r, t) + \nabla \cdot \left[ u(r, t) \omega_z(r, t) \right] = -\nabla \cdot \left\langle \delta u(r, t) \delta \omega_z(r, t) \right\rangle_{S.E} \]

\[ = -\nabla \cdot \left\langle \delta u(r, t) \delta \omega_z(r, t) \right\rangle_{S.E} = -\nabla \cdot \left( \tilde{\eta} \cdot \nabla \omega_z \right) \]

\[ \tilde{\eta} = \left\langle \int_{-\infty}^{t} d\tau \delta u(r, t) \delta u(r - (t - \tau)u, \tau) \right\rangle_{S.E} \]

The result is analogous to the well-known Kubo formula.
5. Conclusion

- The existence of the negative temperature state in two-dimensional point vortex system is confirmed through the massive numerical simulations.

- In the negative temperature state, point vortices of the same sign make a clump, while in the positive temperature state, both sign vortices mix up with each other and form a uniform distribution.

- Effective diffusion coefficient is derived by using the Klimontovich formalism.

- The diffusion term arises from the discreteness of the vorticity.

- Our result is an extension of the well-known Kubo formula under a presence of flow.