Strong well-posedness for a diffuse interface model for the two-phase flow of compressible viscous fluids

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Tokyo, March 2010

Diffuse interface models

Consider the flow of a binary mixture of macroscopically immiscible, viscous compressible Newtonian fluids filling a domain $\Omega \subset \mathbb{R}^3$.

In classical models: both fluids are separated by a sharp interface $\Gamma(t)$, across which certain jump conditions are prescribed. Problem: Topological transitions (e.g. due to droplet formation or coalescence) cannot be described.

This motivated the development of diffuse interface models: replace the sharp interface by a narrow transition layer across which the fluids may mix. See Anderson, McFadden, Wheeler (1998)

The following model is discussed in Lowengrub, Truskinovsky (1998), see also Abels, Feireisl (2008).

Model

- $c_j = \frac{M_j}{M}$ mass concentration of the fluid $j = 1, 2$,
  $M = M_1 + M_2$ total mass, $\Rightarrow c_1 + c_2 = 1$,
- $\rho_j = \frac{M_j}{V}$ apparent mass density of the fluid $j = 1, 2$,
- $\rho = \rho_1 + \rho_2$ total density,
- u_j velocity of the fluid $j = 1, 2$.

Mass balance for each component:

$$\partial_t \rho_j + \text{div}\,(\rho_j u_j) = 0.$$

Adding both equations gives

$$\partial_t \rho + \text{div}\,(\rho u) = 0,$$

where u is the mass-averaged velocity given by

$$\rho u = \rho_1 u_1 + \rho_2 u_2.$$

Let J_j be the mass flux of the fluid j relative to the mean velocity u, i.e.

$$\partial_t \rho_j + \text{div}\,(\rho_j u) + \text{div}\, J_j = 0.$$

Assume $J_1 + J_2 = 0$, to ensure conservation of mass.

Let $c = c_1 - c_2 = 2c_1 - 1$ (order parameter), and $J := J_1 - J_2 = 2J_1$. Since $\rho_j = \rho c_j$, we obtain

$$\partial_t(\rho c) + \text{div}\,(\rho c u) + \text{div}\, J = 0,$$

which, by conservation of mass, is equivalent to

$$\rho \partial_t c + \rho u \cdot \nabla c + \text{div}\, J = 0.$$

Conservation of momentum w.r.t. the mean velocity u:

$$\partial_t(\rho u) + \text{div}\,(\rho u \otimes u) - \text{div}\, \mathcal{T} = \rho g,$$

here: $\mathcal{T}$ is the stress tensor and g an external force. Using conservation of mass we obtain

$$\rho \partial_t u + \rho (u \cdot \nabla) u - \text{div}\, \mathcal{T} = \rho g.$$

Constitutive equations - the mass flux J

Assume that the relative motion of the fluids can be described by a diffusional model. Introduce the total free energy in the form

$$F = \int_\Omega \psi\Big(\rho(x), c(x), \nabla c(x)\Big)\, dx.$$

Then the chemical potential μ (”the driving force”) is defined as

$$\mu = \frac{1}{\rho}\frac{\delta F}{\delta c} = \frac{1}{\rho}\Big(\frac{\partial \psi}{\partial c} - \text{div}\, \frac{\partial \psi}{\partial \nabla c}\Big).$$

Generalized Fick’s law:

$$J = -\gamma \nabla \mu, \qquad (\gamma > 0 \text{ mobility}).$$

Assume (cf. [LowTru], where $0 < \varepsilon = \text{const}$)

$$\psi(\rho, c, \nabla c) = \rho\Big(\tilde{\psi}(\rho, c) + \frac{1}{2}\varepsilon(\rho, c)|\nabla c|^2\Big).$$

$$\Rightarrow \mu = \tilde{\psi}_c(\rho, c) + \frac{1}{2}\varepsilon_c(\rho, c)|\nabla c|^2 - \frac{1}{\rho}\text{div}\,\Big(\rho\varepsilon(\rho, c)\nabla c\Big).$$

$\hookrightarrow$ Cahn-Hilliard type equation for c.

Constitutive equations - the stress tensor $\mathcal{T}$

Following [LowTru] we assume that

$$\mathcal{T} = \mathcal{S} + \mathcal{P},$$

where $\mathcal{S}$ (viscous stress tensor) and $\mathcal{P}$ (nonviscous contribution) have the form

$$\begin{aligned} \mathcal{S} &= \eta(\rho,c)\left(\nabla u + \nabla u^T\right) + \lambda(\rho,c)\mathrm{div}\, u\,\mathcal{I}, \\ \mathcal{P} &= -\rho^2 \frac{\partial \psi}{\partial \rho}\mathcal{I} - \rho \nabla c \otimes \frac{\partial \psi}{\partial \nabla c} \\ &= -\rho^2 \tilde{\psi}_\rho(\rho,c)\,\mathcal{I} - \frac{1}{2}\varepsilon_\rho(\rho,c)\rho^2|\nabla c|^2 - \rho\varepsilon\nabla c \otimes \nabla c. \end{aligned}$$

$\eta(\rho,c), \lambda(\rho,c)$ are the viscosity coefficients, $\pi = \rho^2\tilde{\psi}_\rho(\rho,c)$ is the pressure, $-\rho\nabla c \otimes \frac{\partial \psi}{\partial \nabla c}$ the Ericksen's stress.

These constitutive laws lead to a thermodynamically consistent model. The total energy is given by

$$E(t) = \int_\Omega \left(\frac{1}{2}\rho|u|^2 + \rho\psi(\rho,c,\nabla c)\right) dx.$$

Mathematical problem

Let $J = [0, T]$ and $\Omega \subset \mathbb{R}^n$ be a bounded domain with C^4 boundary Γ. Consider the (compressible) NSCH-system

$$
\begin{aligned}
\rho\partial_t u + \rho(u\cdot\nabla)u - \operatorname{div}\mathcal{S} - \operatorname{div}\mathcal{P} &= \rho g \quad (J\times\Omega),\\
\partial_t\rho + \operatorname{div}(\rho u) &= 0 \quad (J\times\Omega),\\
\rho\partial_t c + \rho u\cdot\nabla c - \operatorname{div}\Big(\gamma(\rho,c)\nabla\mu\Big) &= 0 \quad (J\times\Omega),\\
u = 0, \quad \partial_\nu c = \partial_\nu\mu &= 0 \quad (J\times\Gamma),\\
u|_{t=0} = u_0, \quad c|_{t=0} = c_0, \quad \rho|_{t=0} &= \rho_0 \quad (\Omega),
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{S} &= \eta(\rho,c)\Big(\nabla u + \nabla u^T\Big) + \lambda(\rho,c)\operatorname{div} u\,\mathcal{I},\\
\mathcal{P} &= -\pi\mathcal{I} - \frac{1}{2}\rho^2\varepsilon_\rho(\rho,c)|\nabla c|^2\mathcal{I} - \rho\varepsilon(\rho,c)\nabla c\otimes\nabla c,\\
\mu &= \psi_c(\rho,c,\nabla c) - \rho^{-1}\operatorname{div}\Big(\varepsilon(\rho,c)\rho\nabla c\Big),\\
\psi &= \tilde\psi(\rho,c) + \frac{1}{2}\varepsilon(\rho,c)|\nabla c|^2, \quad \pi = \rho^2\tilde\psi_\rho(\rho,c).
\end{aligned}
$$

Literature

- Abels and Feireisl (2008): Existence of global weak solutions for the NSCH-model in the case $\varepsilon = \frac{1}{\rho}$, which corresponds to the free energy

$$F = \int_\Omega \left(\rho \tilde{\psi}(\rho, c) + \frac{1}{2} |\nabla c|^2 \right) dx.$$

(see also Anderson, McFadden, Wheeler (1998) for this model, compare with Lowengrub, Truskinovsky (1998):

$$F = \int_\Omega \left(\rho \tilde{\psi}(\rho, c) + \frac{1}{2} \varepsilon \rho |\nabla c|^2 \right) dx.$$

Problem: energy estimates do not provide any bound for ∇c in vacuum zones

- Abels (2007, 2009): incompressible NSCH-model

- Solonnikov (1976): Existence and uniqueness of local strong solutions for the compressible Navier-Stokes equ., here $u \in H^1_p(J; L_p(\Omega)) \cap L_p(J; H^2_p(\Omega))$ and $\rho \in C^1(J; L_p(\Omega)) \cap C(J; H^1_p(\Omega))$, $p > n$.

- See also the monographs Feireisl (2004); Novotny, Straskraba (2004).

Setting

We are looking for strong solutions in the L_p-setting. Consider first the equation for c in $L_p(J; L_p(\Omega))$:

$$\rho\partial_t c + \rho u \cdot \nabla c - \mathrm{div}\Big(\gamma(\rho, c)\nabla\mu\Big) = 0,$$

$$\mu = \psi_c(\rho, c) - \rho^{-1}\mathrm{div}\Big(\varepsilon(\rho, c)\rho\nabla c\Big).$$

The natural regularity class is

$$c \in H^1_p(J; L_p(\Omega)) \cap L_p(J; H^4_p(\Omega)).$$

Problem: third order term of ρ, i.e. we need at least $\rho \in L_p(J; H^3_p(\Omega))$

Remark: In the case $\varepsilon\rho = 1$ only $\rho \in L_p(J; H^2_p(\Omega))$ is required.

Since ρ is governed by the hyperbolic equation

$$\partial_t \rho + \operatorname{div}(\rho u) = 0,$$

we need $u \in L_p(J; H_p^4(\Omega; \mathbb{R}^n))$. To obtain this regularity for u, we need to consider the Navier-Stokes equation in $L_p(J; H_p^2(\Omega; \mathbb{R}^n))$:

$$\rho \partial_t u + \rho(u \cdot \nabla)u - \operatorname{div} \mathcal{S} - \operatorname{div} \mathcal{P} = \rho g,$$

$$\mathcal{P} = -\pi \mathcal{I} - \frac{1}{2}\rho^2 \varepsilon_\rho(\rho, c)|\nabla c|^2 \mathcal{I} - \rho\varepsilon(\rho, c)\nabla c \otimes \nabla c.$$

Note that

$$c \in H_p^1(J; L_p(\Omega)) \cap L_p(J; H_p^4(\Omega)) \hookrightarrow H_p^{\frac{1}{2}}(J; H_p^2(\Omega)).$$

Therefore

$$\operatorname{div} \mathcal{P} \sim \partial_{x_i} \nabla c \in H_p^{\frac{1}{2}}(J; L_p(\Omega; \mathbb{R}^n)) \cap L_p(J; H_p^2(\Omega; \mathbb{R}^n)) =: X_1.$$

Taking X_1 as the base space for the Navier-Stokes equation one expects that

$$u \in H_p^{\frac{3}{2}}(J; L_p(\Omega; \mathbb{R}^n)) \cap H_p^1(J; H_p^2(\Omega; \mathbb{R}^n)) \cap L_p(J; H_p^4(\Omega; \mathbb{R}^n)).$$

Using this regularity the continuity equation yields

$$\rho \in H_p^{2+\frac{1}{4}}(J; L_p(\Omega)) \cap C^1(J; H_p^2(\Omega)) \cap C(J; H_p^3(\Omega)).$$

In fact, $\partial_t^2 \rho = -\rho\, \partial_t(\nabla \cdot u) + \ldots$, and

$$\partial_t u \in X_1 \Rightarrow \partial_t(\nabla \cdot u) \in H_p^{\frac{1}{4}}(J; L_p(\Omega)).$$

Main result

Theorem: Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with C^4 boundary Γ, and let $\eta, \lambda, \gamma, \varepsilon$, and $\tilde{\psi}$ be (sufficiently) smooth. Let $p > p_* := \max\{4, n\}$, $p \neq 5$, and assume that

(i) $g \in H_p^{\frac{1}{2}}(J; L_p(\Omega; \mathbb{R}^n)) \cap L_p(J; H_p^2(\Omega; \mathbb{R}^n))$;

(ii) $u_0 \in W_p^{4-\frac{2}{p}}(\Omega)$, $c_0 \in W_p^{4-\frac{4}{p}}(\Omega)$, $\rho_0 \in H_p^3(\Omega)$;

(iii) $u_0 = 0$ on Γ; $-\operatorname{div} \mathcal{S}(\rho_0, c_0, u_0) - \operatorname{div} \mathcal{P}(\rho_0, c_0) = \rho_0 g|_{t=0}$ on Γ;

$\partial_\nu c_0 = 0$ on Γ; $\partial_\nu \mu(\rho_0, c_0) = 0$ on Γ, if $p > 5$;

(iv) $(\eta, 2\eta + \lambda, \gamma, \varepsilon)|_{\rho=\rho_0, c=c_0} \in (0, \infty)^4$ in $\bar{\Omega}$; $\rho_0 > 0$ in $\bar{\Omega}$.

Then there exists $T > 0$ such that the NSCH-system has a unique solution $(u, c, \rho) \in Z_1 \times Z_2 \times Z_3$ on $J = [0, T]$ where

$$Z_1 = H_p^{\frac{3}{2}}(J; L_p(\Omega; \mathbb{R}^n)) \cap H_p^1(J; H_p^2(\Omega; \mathbb{R}^n)) \cap L_p(J; H_p^4(\Omega; \mathbb{R}^n)),$$
$$Z_2 = H_p^1(J; L_p(\Omega)) \cap L_p(J; H_p^4(\Omega)),$$
$$Z_3 = H_p^{2+\frac{1}{4}}(J; L_p(\Omega)) \cap C^1(J; H_p^2(\Omega)) \cap C(J; H_p^3(\Omega)).$$

Basic ideas of the proof

1. Given ρ_0 and u with $u = 0$ on Γ, solve the continuity equation (e.g. by the method of characterictics). $\hookrightarrow \rho = \Phi(u)$.
2. Insert $\rho = \Phi(u)$ into the equations for u and c.
3. Fixed point formulation for (u, c):

$$
\begin{aligned}
\rho_0 \partial_t u - \operatorname{div} \mathcal{S}(\rho_0, c_0, u) + M(\rho_0, c_0)\nabla^2 c &= F_1(u, c, \Phi(u)) \quad (J \times \Omega), \\
\kappa_0\, \partial_t c + \nabla \cdot (\varepsilon_0 \nabla \nabla \cdot (\varepsilon_0 \nabla c)) &= F_2(u, c, \Phi(u)) \quad (J \times \Omega), \\
u = 0, \quad \partial_\nu c = 0, \quad \partial_\nu \nabla \cdot (\varepsilon_0 \nabla c) &= \partial_\nu h(c, \Phi(u)) \quad (J \times \Gamma), \\
u|_{t=0} = u_0, \quad c|_{t=0} &= c_0 \quad (\Omega),
\end{aligned}
$$

where $\varepsilon_0 := \varepsilon(\rho_0, c_0)$, $\gamma_0 := \gamma(\rho_0, c_0)$, $\kappa_0 := \frac{\varepsilon_0 \rho_0}{\gamma_0}$. More abstractly,

$$
\begin{aligned}
\mathcal{L}_{11} u + \mathcal{L}_{12} c &= F_1(u, c, \Phi(u)), \\
\mathcal{L}_{22} c &= F_2(u, c, \Phi(u)).
\end{aligned}
$$

Fixed point argument

Define reference functions $\tilde{u} \in Z_1^{T_0}$, $\tilde{c} \in Z_2^{T_0}$ by means of

$$\mathcal{L}(\tilde{u}, \tilde{c}) = F(u_0, c_0, \rho_0),$$
$$(\tilde{u}, \tilde{c})|_{t=0} = (u_0, c_0).$$

For $T \in (0, T_0]$ and $r \in (0, r_0]$ let $\Sigma_r^T := \{(u, c) \in Z_1^T \times Z_2^T : u = 0$ on $J \times \Gamma$, $(u, c)|_{t=0} = (u_0, c_0)$, and $|(u, c) - (\tilde{u}, \tilde{c})|_{Z_1^T \times Z_2^T} \leq r\}$.
Define the mapping $\Lambda : \Sigma_r^T \to Z_1^T \times Z_2^T$ by $\Lambda(u, c) := (\hat{u}, \hat{c})$ where

$$\mathcal{L}(\hat{u}, \hat{c}) = F(u, c, \Phi(u)),$$
$$(\hat{u}, \hat{c})|_{t=0} = (u_0, c_0).$$

Show that for sufficiently small T and r: (i) Λ leaves Σ_r^T invariant, (ii) Λ is a strict contraction in the space $Y_1^T \times Y_2^T$ (weaker topology!) with

$$Y_1^T = H_p^1(J; L_p(\Omega; \mathbb{R}^n)) \cap L_p(J; H_p^2(\Omega; \mathbb{R}^n)),$$
$$Y_2^T = H_p^{\frac{1}{2}}(J; L_p(\Omega)) \cap L_p(J; H_p^2(\Omega)).$$

Σ_r^T is closed in $Y_1^T \times Y_2^T$, so the contraction mapping principle is applicable.

Some auxiliary results

(i) For the Cahn-Hilliard equation we need maximal L_p-regularity ($c \in Z_2$) for the problem

$$\begin{aligned} \kappa_0\, \partial_t c + \nabla \cdot (\varepsilon_0 \nabla \nabla \cdot (\varepsilon_0 \nabla c)) = f \quad (J \times \Omega), \\ \partial_\nu c = 0, \quad \partial_\nu \nabla \cdot (\varepsilon_0 \nabla c) = \varphi \quad (J \times \Gamma), \\ c|_{t=0} = c_0 \quad (\Omega). \end{aligned}$$

Natural regularity class for boundary data:

$$\varphi \in W_p^{\frac{1}{4}-\frac{1}{4p}}(J; L_p(\Gamma)) \cap L_p(J; W_p^{1-\frac{1}{p}}(\Gamma)).$$

See e.g. Prüss, Racke, Zheng (2006); Prüss, Wilke (2006); Denk, Hieber, Prüss (2007).

(ii) Maximal L_p-regularity and higher regularity with base space X_1, i.e.
$u \in Z_1 = H_p^{\frac{3}{2}}(J; L_p(\Omega;\mathbb{R}^n)) \cap H_p^1(J; H_p^2(\Omega;\mathbb{R}^n)) \cap L_p(J; H_p^4(\Omega;\mathbb{R}^n))$,
for the parabolic system

$$\rho_0 \partial_t u - \mathrm{div}\, \mathcal{S}(\rho_0, c_0, u) = f \quad (J \times \Omega),$$
$$u = 0 \quad (J \times \Gamma), \quad u|_{t=0} = u_0 \quad (\Omega).$$

In case of constant coefficients the PDE reads

$$\rho_0 \partial_t u - \eta_0 \Delta u - (\lambda_0 + \eta_0) \nabla \nabla \cdot u = f.$$

Assumptions: $\rho_0, \eta_0, 2\eta_0 + \lambda_0 > 0$. Symbol of $-\eta_0 \Delta - (\lambda_0 + \eta_0) \nabla\, \mathrm{div}$:
$A(\xi) = \eta_0 |\xi|^2 + (\lambda_0 + \eta_0) \xi \otimes \xi$. For $\xi \neq 0$ the eigenvalues are η_0, $2\eta_0 + \lambda_0$.
$\hookrightarrow$ ellipticity in the sense of Denk, Hieber, Prüss (2003), (2007). Further, condition (LS) is satisfied. $\Rightarrow$ max. L_p-regularity. See also Solonnikov (1965).

Higher regularity: reduction to model problems (full and half space) by localization and perturbation arguments, Fourier transform w.r.t. tangential variables, Newton polygon trace theory, natural regularity class for Dirichlet data:
$W_p^{\frac{3}{2} - \frac{1}{4p}}(J; L_p(\Gamma)) \cap H_p^1(J; W_p^{2-\frac{1}{p}}(\Gamma)) \cap L_p(J; W_p^{4-\frac{1}{p}}(\Gamma))$.

Estimates for the continuity equation

Let $T \in (0, T_0]$, $r \in (0, r_0]$ and $(u, c) \in \Sigma_r^T$. Then $u = 0$ on $[0, T] \times \Gamma$ and

$$|u|_{Z_1^T} \le |u - \tilde{u}|_{Z_1^T} + |\tilde{u}|_{Z_1^T} \le r + |\tilde{u}|_{Z_1^{T_0}} \le r_0 + |\tilde{u}|_{Z_1^{T_0}}.$$

Suppose $\rho_0 \in H_p^3(\Omega)$, $p > p_*$, $\rho_0 > 0$ in $\bar{\Omega}$. Then the continuity equation, together with $\rho|_{t=0} = \rho_0$, has a unique positive solution $\rho = \Phi(u) \in Z_3^T$, and we have the a priori estimate

$$|\rho|_{Z_3^T} \le C_0,$$

where C_0 is independent of T, r, u.

For the contraction estimate we use that ($J = [0, T]$)

$$|\Phi(u_1) - \Phi(u_2)|_{H_2^1(J;L_p(\Omega)) \cap C(J;H_p^1(\Omega))} \le C_1(T)|u_1 - u_2|_{Z_1^T},$$

for all $(u_i, c_i) \in \Sigma_r^T$, $i = 1, 2$, where $C_1(T) \to 0$ as $T \to 0$ and C_1 is independent of r.

A weak estimate for Cahn-Hilliard

Recall the fixed point formulation for (CH):

$$\begin{aligned} \kappa_0\,\partial_t\hat{c} + \nabla\cdot(\varepsilon_0\nabla\nabla\cdot(\varepsilon_0\nabla\hat{c})) &= F_2(u,c,\Phi(u)) \quad (J\times\Omega),\\ \partial_\nu\hat{c} = 0,\quad \partial_\nu\nabla\cdot(\varepsilon_0\nabla\hat{c}) &= \partial_\nu h(c,\Phi(u)) \quad (J\times\Gamma),\\ \hat{c}|_{t=0} &= c_0 \quad (\Omega). \end{aligned}$$

Here $h(c,\rho) = \nabla\cdot([\varepsilon(\rho,c)-\varepsilon_0]\nabla c) + \rho^{-1}\varepsilon(\rho,c)\nabla\rho\cdot\nabla c - \partial_c\psi(\rho,c,\nabla c)$.

Recall $Y_2^T := H_p^{1/2}(J;L_p(\Omega))\cap L_p(J;H_p^2(\Omega))$.

For $(u_i,c_i)\in\Sigma_r^T$, $\rho_i=\Phi(u_i)$, $i=1,2$, we can show that

$$\begin{aligned} |\hat{c}_1-\hat{c}_2|_{Y_2^T} \le C_2\Big(&|h(c_1,\rho_1)-h(c_2,\rho_2)|_{L_p(L_p)} + |\Theta|_{H_p^{1/2}(L_p)}\\ &+ |\tilde{\Theta}|_{L_p(L_p)} + |\gamma_0-\gamma_1|_{C(C^1)}|c_1-c_2|_{L_p(H_p^2)} + T^{1/4}|\gamma_1-\gamma_2|_{C(H_p^1)}\Big), \end{aligned}$$

where $\gamma_i=\gamma(\rho_i,c_i)$, $\Theta = (\rho_0-\rho_1)(c_1-c_2)-(\rho_1-\rho_2)c_2$, and
$\tilde{\Theta} = c_1\rho_1u_1 - c_2\rho_2u_2$.

Pf.: Uses the divergence structure, duality relations, and max. reg. methods.

Final remarks

- The local solution can be extended to a maximally defined solution.

- The main result can be generalized to cover other boundary conditions like the pure slip condition:

$$u \cdot \nu = 0, \quad (\mathcal{I} - \nu \otimes \nu)\mathcal{S}\nu = 0 \quad \text{on } [0, T] \times \Gamma.$$