Strong well-posedness for a diffuse interface model for the two-phase flow of compressible viscous fluids

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**Diffuse interface models**

Consider the flow of a binary mixture of macroscopically immiscible, viscous compressible Newtonian fluids filling a domain $\Omega \subset \mathbb{R}^3$.

In classical models: both fluids are separated by a sharp interface $\Gamma(t)$, across which certain jump conditions are prescribed. **Problem:** Topological transitions (e.g. due to droplet formation or coalescence) cannot be described.

This motivated the development of **diffuse interface models**: replace the sharp interface by a narrow transition layer across which the fluids may mix. See Anderson, McFadden, Wheeler (1998)

The following model is discussed in Lowengrub, Truskinovsky (1998), see also Abels, Feireisl (2008).
Model

- \( c_j = \frac{M_j}{M} \) mass concentration of the fluid \( j = 1, 2 \),
  \( M = M_1 + M_2 \) total mass, \( \Rightarrow c_1 + c_2 = 1 \),
- \( \rho_j = \frac{M_j}{V} \) apparent mass density of the fluid \( j = 1, 2 \),
- \( \rho = \rho_1 + \rho_2 \) total density,
- \( u_j \) velocity of the fluid \( j = 1, 2 \).

Mass balance for each component:

\[
\partial_t \rho_j + \text{div} (\rho_j u_j) = 0.
\]

Adding both equations gives

\[
\partial_t \rho + \text{div} (\rho u) = 0,
\]

where \( u \) is the mass-averaged velocity given by

\[
\rho u = \rho_1 u_1 + \rho_2 u_2.
\]
Let $J_j$ be the mass flux of the fluid $j$ relative to the mean velocity $u$, i.e.

$$\partial_t \rho_j + \text{div} (\rho_j u) + \text{div} J_j = 0.$$ 

Assume $J_1 + J_2 = 0$, to ensure conservation of mass.

Let $c = c_1 - c_2 = 2c_1 - 1$ (order parameter), and $J := J_1 - J_2 = 2J_1$. Since $\rho_j = \rho c_j$, we obtain

$$\partial_t (\rho c) + \text{div} (\rho c u) + \text{div} J = 0,$$

which, by conservation of mass, is equivalent to

$$\rho \partial_t c + \rho u \cdot \nabla c + \text{div} J = 0.$$

Conservation of momentum w.r.t. the mean velocity $u$:

$$\partial_t (\rho u) + \text{div} (\rho u \otimes u) - \text{div} T = \rho g,$$

here: $T$ is the stress tensor and $g$ an external force. Using conservation of mass we obtain

$$\rho \partial_t u + \rho (u \cdot \nabla) u - \text{div} T = \rho g.$$
Constitutive equations - the mass flux $J$

Assume that the relative motion of the fluids can be described by a diffusional model. Introduce the total free energy in the form

$$F = \int_{\Omega} \psi(\rho(x), c(x), \nabla c(x)) \, dx.$$  

Then the chemical potential $\mu$ ("the driving force") is defined as

$$\mu = \frac{1}{\rho} \frac{\delta F}{\delta c} = \frac{1}{\rho} \left( \frac{\partial \psi}{\partial c} - \text{div} \frac{\partial \psi}{\partial \nabla c} \right).$$

Generalized Fick’s law:

$$J = -\gamma \nabla \mu, \quad (\gamma > 0 \text{ mobility}).$$

Assume (cf. [LowTru], where $0 < \varepsilon = \text{const}$)

$$\psi(\rho, c, \nabla c) = \rho \left( \tilde{\psi}(\rho, c) + \frac{1}{2} \varepsilon(\rho, c)|\nabla c|^2 \right).$$

$$\Rightarrow \mu = \tilde{\psi}_c(\rho, c) + \frac{1}{2} \varepsilon_c(\rho, c)|\nabla c|^2 - \frac{1}{\rho} \text{div} \left( \rho \varepsilon(\rho, c) \nabla c \right).$$

$\rightarrow$ Cahn-Hilliard type equation for $c$.  

Constitutive equations - the stress tensor $\mathcal{T}$

Following [LowTru] we assume that

$$\mathcal{T} = S + P,$$

where $S$ (viscous stress tensor) and $P$ (nonviscous contribution) have the form

$$S = \eta(\rho, c) \left( \nabla u + \nabla u^T \right) + \lambda(\rho, c) \text{div} u \mathcal{I},$$

$$P = -\rho^2 \frac{\partial \psi}{\partial \rho} \mathcal{I} - \rho \nabla c \otimes \frac{\partial \psi}{\partial \nabla c} = -\rho^2 \tilde{\psi}_\rho(\rho, c) \mathcal{I} - \frac{1}{2} \varepsilon(\rho, c) \rho^2 |\nabla c|^2 - \rho \varepsilon \nabla c \otimes \nabla c.$$

$\eta(\rho, c), \lambda(\rho, c)$ are the viscosity coefficients, $\pi = \rho^2 \tilde{\psi}_\rho(\rho, c)$ is the pressure, $-\rho \nabla c \otimes \frac{\partial \psi}{\partial \nabla c}$ the Ericksen’s stress.

These constitutive laws lead to a thermodynamically consistent model. The total energy is given by

$$E(t) = \int_{\Omega} \left( \frac{1}{2} \rho |u|^2 + \rho \psi(\rho, c, \nabla c) \right) \, dx.$$
Mathematical problem

Let $J = [0, T]$ and $\Omega \subset \mathbb{R}^n$ be a bounded domain with $C^4$ boundary $\Gamma$. Consider the (compressible) NSCH-system

\[
\begin{align*}
\rho \partial_t u + \rho (u \cdot \nabla) u - \text{div} S - \text{div} P &= \rho g \quad (J \times \Omega), \\
\partial_t \rho + \text{div} (\rho u) &= 0 \quad (J \times \Omega), \\
\rho \partial_t c + \rho u \cdot \nabla c - \text{div} \left( \gamma(\rho, c) \nabla \mu \right) &= 0 \quad (J \times \Omega), \\
u = 0, \quad \partial_\nu c = \partial_\nu \mu &= 0 \quad (J \times \Gamma), \\
\left. u \right|_{t=0} = u_0, \quad \left. c \right|_{t=0} = c_0, \quad \left. \rho \right|_{t=0} = \rho_0 \quad (\Omega),
\end{align*}
\]

where

\[
\begin{align*}
S &= \eta(\rho, c) \left( \nabla u + \nabla u^T \right) + \lambda(\rho, c) \text{div} u I, \\
P &= -\pi I - \frac{1}{2} \rho^2 \varepsilon_\rho(\rho, c) |\nabla c|^2 I - \rho \varepsilon(\rho, c) \nabla c \otimes \nabla c, \\
\mu &= \psi_c(\rho, c, \nabla c) - \rho^{-1} \text{div} \left( \varepsilon(\rho, c) \rho \nabla c \right), \\
\psi &= \tilde{\psi}(\rho, c) + \frac{1}{2} \varepsilon(\rho, c) |\nabla c|^2, \quad \pi = \rho^2 \tilde{\psi}_\rho(\rho, c).
\end{align*}
\]
Abels and Feireisl (2008): Existence of global weak solutions for the NSCH-model in the case $\varepsilon = \frac{1}{\rho}$, which corresponds to the free energy

$$F = \int_{\Omega} \left( \rho \tilde{\psi}(\rho, c) + \frac{1}{2} |\nabla c|^2 \right) \, dx.$$

(see also Anderson, McFadden, Wheeler (1998) for this model, compare with Lowengrub, Truskinovsky (1998):

$$F = \int_{\Omega} \left( \rho \tilde{\psi}(\rho, c) + \frac{1}{2} \varepsilon \rho |\nabla c|^2 \right) \, dx.$$

Problem: energy estimates do not provide any bound for $\nabla c$ in vacuum zones


Solonnikov (1976): Existence and uniqueness of local strong solutions for the compressible Navier-Stokes equ., here $u \in H^1_p(J; L^p(\Omega)) \cap L^p(J; H^2_p(\Omega))$ and $\rho \in C^1(J; L^p(\Omega)) \cap C(J; H^1_p(\Omega))$, $p > n$.

See also the monographs Feireisl (2004); Novotny, Straskraba (2004).
Setting

We are looking for strong solutions in the $L_p$-setting. Consider first the equation for $c$ in $L_p(J; L_p(\Omega))$:

$$\rho \partial_t c + \rho u \cdot \nabla c - \text{div} \left( \gamma(\rho, c) \nabla \mu \right) = 0,$$
$$\mu = \psi_c(\rho, c) - \rho^{-1} \text{div} \left( \varepsilon(\rho, c) \rho \nabla c \right).$$

The natural regularity class is

$$c \in H^1_p(J; L_p(\Omega)) \cap L_p(J; H^4_p(\Omega)).$$

**Problem:** third order term of $\rho$, i.e. we need at least $\rho \in L_p(J; H^3_p(\Omega))$

**Remark:** In the case $\varepsilon \rho = 1$ only $\rho \in L_p(J; H^2_p(\Omega))$ is required.
Since $\rho$ is governed by the hyperbolic equation

$$\partial_t \rho + \text{div} \ (\rho u) = 0,$$

we need $u \in L_p(J; H^4_p(\Omega; \mathbb{R}^n))$. To obtain this regularity for $u$, we need to consider the Navier-Stokes equation in $L_p(J; H^2_p(\Omega; \mathbb{R}^n))$:

$$\rho \partial_t u + \rho(u \cdot \nabla)u - \text{div} \ S - \text{div} \ P = \rho g,$$

$$P = -\pi I - \frac{1}{2} \rho^2 \varepsilon_\rho(\rho, c)|\nabla c|^2 I - \rho \varepsilon(\rho, c) \nabla c \otimes \nabla c.$$

Note that

$$c \in H^1_p(J; L_p(\Omega)) \cap L_p(J; H^4_p(\Omega)) \hookrightarrow H^2_p(J; H^2_p(\Omega)).$$

Therefore

$$\text{div} \ P \sim \partial_{x_i} \nabla c \in H^2_p(J; L_p(\Omega; \mathbb{R}^n)) \cap L_p(J; H^2_p(\Omega; \mathbb{R}^n)) =: X_1.$$
Taking \( X_1 \) as the base space for the Navier-Stokes equation one expects that

\[
\begin{aligned}
    u &\in H_p^{\frac{3}{2}}(J; L_p(\Omega; \mathbb{R}^n)) \cap H_p^1(J; H_p^2(\Omega; \mathbb{R}^n)) \cap L_p(J; H_p^4(\Omega; \mathbb{R}^n)).
\end{aligned}
\]

Using this regularity the continuity equation yields

\[
\begin{aligned}
    \rho &\in H_p^{2+\frac{1}{4}}(J; L_p(\Omega)) \cap C^1(J; H_p^2(\Omega)) \cap C(J; H_p^3(\Omega)).
\end{aligned}
\]

In fact, \( \partial_t^2 \rho = -\rho \partial_t(\nabla \cdot u) + \ldots \), and

\[
\begin{aligned}
    \partial_t u \in X_1 \Rightarrow \partial_t(\nabla \cdot u) \in H_p^{\frac{1}{4}}(J; L_p(\Omega)).
\end{aligned}
\]
Main result

**Theorem:** Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with $C^4$ boundary $\Gamma$, and let $\eta, \lambda, \gamma, \varepsilon$, and $\tilde{\psi}$ be (sufficiently) smooth. Let $p > p_* := \max\{4, n\}$, $p \neq 5$, and assume that

(i) $g \in H_p^{\frac{1}{2}}(J; L_p(\Omega; \mathbb{R}^n)) \cap L_p(J; H_p^2(\Omega; \mathbb{R}^n));$

(ii) $u_0 \in W_p^{4-\frac{2}{p}}(\Omega), c_0 \in W_p^{4-\frac{4}{p}}(\Omega), \rho_0 \in H_p^3(\Omega);$

(iii) $u_0 = 0$ on $\Gamma; -\text{div} \mathcal{S}(\rho_0, c_0, u_0) - \text{div} \mathcal{P}(\rho_0, c_0) = \rho_0 g|_{t=0}$ on $\Gamma;$

\[
\partial_\nu c_0 = 0 \text{ on } \Gamma; \partial_\nu \mu(\rho_0, c_0) = 0 \text{ on } \Gamma, \text{ if } p > 5;
\]

(iv) $\left(\eta, 2\eta + \lambda, \gamma, \varepsilon\right)|_{\rho=\rho_0, c=c_0} \in (0, \infty)^4$ in $\bar{\Omega}; \rho_0 > 0$ in $\bar{\Omega}.$

Then there exists $T > 0$ such that the NSCH-system has a unique solution $(u, c, \rho) \in Z_1 \times Z_2 \times Z_3$ on $J = [0, T]$ where

\[
Z_1 = H_p^{\frac{3}{2}}(J; L_p(\Omega; \mathbb{R}^n)) \cap H_p^1(J; H_p^2(\Omega; \mathbb{R}^n)) \cap L_p(J; H_p^4(\Omega; \mathbb{R}^n)),
\]

\[
Z_2 = H_p^1(J; L_p(\Omega)) \cap L_p(J; H_p^4(\Omega)),
\]

\[
Z_3 = H_p^{2+\frac{1}{4}}(J; L_p(\Omega)) \cap C^1(J; H_p^2(\Omega)) \cap C(J; H_p^3(\Omega)).
\]
Basic ideas of the proof

1. Given $\rho_0$ and $u$ with $u = 0$ on $\Gamma$, solve the continuity equation (e.g. by the method of characteristics). $\rightarrow \rho = \Phi(u)$.

2. Insert $\rho = \Phi(u)$ into the equations for $u$ and $c$.

3. Fixed point formulation for $(u, c)$:

$$
\rho_0 \partial_t u - \text{div} S(\rho_0, c_0, u) + M(\rho_0, c_0) \nabla^2 c = F_1(u, c, \Phi(u)) \quad (J \times \Omega),
$$

$$
\kappa_0 \partial_t c + \nabla \cdot (\varepsilon_0 \nabla \nabla \cdot (\varepsilon_0 \nabla c)) = F_2(u, c, \Phi(u)) \quad (J \times \Omega),
$$

$$
u = 0, \quad \partial_\nu c = 0, \quad \partial_\nu \nabla \cdot (\varepsilon_0 \nabla c) = \partial_\nu h(c, \Phi(u)) \quad (J \times \Gamma),
$$

$$
u \big|_{t=0} = u_0, \quad c \big|_{t=0} = c_0 \quad (\Omega),
$$

where $\varepsilon_0 := \varepsilon(\rho_0, c_0)$, $\gamma_0 := \gamma(\rho_0, c_0)$, $\kappa_0 := \frac{\varepsilon_0 \rho_0}{\gamma_0}$. More abstractly,

$$
\mathcal{L}_{11} u + \mathcal{L}_{12} c = F_1(u, c, \Phi(u)),
$$

$$
\mathcal{L}_{22} c = F_2(u, c, \Phi(u)).
$$
Fixed point argument

Define reference functions \( \tilde{u} \in Z_1^{T_0} \), \( \tilde{c} \in Z_2^{T_0} \) by means of

\[
\mathcal{L}(\tilde{u}, \tilde{c}) = F(u_0, c_0, \rho_0),
\]

\[
(\tilde{u}, \tilde{c})|_{t=0} = (u_0, c_0).
\]

For \( T \in (0, T_0] \) and \( r \in (0, r_0] \) let \( \Sigma_r^T := \{(u, c) \in Z_1^T \times Z_2^T : u = 0 \text{ on } J \times \Gamma, (u, c)|_{t=0} = (u_0, c_0), \text{ and } |(u, c) - (\tilde{u}, \tilde{c})|_{Z_1^T \times Z_2^T} \leq r\} \).

Define the mapping \( \Lambda : \Sigma_r^T \rightarrow Z_1^T \times Z_2^T \) by \( \Lambda(u, c) := (\hat{u}, \hat{c}) \) where

\[
\mathcal{L}(\hat{u}, \hat{c}) = F(u, c, \Phi(u)),
\]

\[
(\hat{u}, \hat{c})|_{t=0} = (u_0, c_0).
\]

Show that for sufficiently small \( T \) and \( r \): (i) \( \Lambda \) leaves \( \Sigma_r^T \) invariant, (ii) \( \Lambda \) is a strict contraction in the space \( Y_1^T \times Y_2^T \) (weaker topology!) with

\[
Y_1^T = H^1_p(J; L_p(\Omega; \mathbb{R}^n)) \cap L_p(J; H^2_p(\Omega; \mathbb{R}^n)),
\]

\[
Y_2^T = H^1_p(J; L_p(\Omega)) \cap L_p(J; H^2_p(\Omega)).
\]

\( \Sigma_r^T \) is closed in \( Y_1^T \times Y_2^T \), so the contraction mapping principle is applicable.
Some auxiliary results

(i) For the Cahn-Hilliard equation we need maximal \(L_p\)-regularity \((c \in \mathbb{Z}_2)\) for the problem

\[
\kappa_0 \partial_t c + \nabla \cdot (\varepsilon_0 \nabla \nabla \cdot (\varepsilon_0 \nabla c)) = f \quad (J \times \Omega),
\]

\[
\partial_\nu c = 0, \quad \partial_\nu \nabla \cdot (\varepsilon_0 \nabla c) = \varphi \quad (J \times \Gamma),
\]

\[
c \big|_{t=0} = c_0 \quad (\Omega).
\]

Natural regularity class for boundary data:

\[
\varphi \in W_p^{\frac{1}{4} - \frac{1}{4p}}(J; L_p(\Gamma)) \cap L_p(J; W_p^{\frac{1}{4} - \frac{1}{p}}(\Gamma)).
\]

See e.g. Prüss, Racke, Zheng (2006); Prüss, Wilke (2006); Denk, Hieber, Prüss (2007).
(ii) Maximal $L_p$-regularity and higher regularity with base space $X_1$, i.e. 
\[ u \in Z_1 = H^{3/2}_p(J; L_p(\Omega; \mathbb{R}^n)) \cap H^1_p(J; H^2_p(\Omega; \mathbb{R}^n)) \cap L_p(J; H^4_p(\Omega; \mathbb{R}^n)), \]

for the parabolic system 
\[
\rho_0 \partial_t u - \text{div} \, S(\rho_0, c_0, u) = f \quad (J \times \Omega), \\
\quad u = 0 \quad (J \times \Gamma), \quad u|_{t=0} = u_0 \quad (\Omega).
\]

In case of constant coefficients the PDE reads 
\[
\rho_0 \partial_t u - \eta_0 \Delta u - (\lambda_0 + \eta_0) \nabla \nabla \cdot u = f.
\]

Assumptions: $\rho_0, \eta_0, 2\eta_0 + \lambda_0 > 0$. Symbol of $-\eta_0 \Delta - (\lambda_0 + \eta_0) \nabla \text{div}$: 
\[ A(\xi) = \eta_0 |\xi|^2 + (\lambda_0 + \eta_0) \xi \otimes \xi. \]
For $\xi \neq 0$ the eigenvalues are $\eta_0, 2\eta_0 + \lambda_0$. $\Leftarrow$ ellipticity in the sense of Denk, Hieber, Prüss (2003), (2007). Further, condition (LS) is satisfied. $\Rightarrow$ max. $L_p$-regularity. See also Solonnikov (1965).

**Higher regularity:** reduction to model problems (full and half space) by localization and perturbation arguments, Fourier transform w.r.t. tangential variables, Newton polygon trace theory, natural regularity class for Dirichlet data: 
\[ W^{3/4, 1}_{4p} (J; L_p(\Gamma)) \cap H^{1/4}_{p} (J; W^{2/3}_{4p, p}(\Gamma)) \cap L_p (J; W^{4/5}_{p, p}(\Gamma)). \]
Estimates for the continuity equation

Let $T \in (0, T_0]$, $r \in (0, r_0]$ and $(u, c) \in \Sigma^T_r$. Then $u = 0$ on $[0, T] \times \Gamma$ and

$$|u|_{Z^1_T} \leq |u - \tilde{u}|_{Z^1_T} + |\tilde{u}|_{Z^1_T} \leq r + |\tilde{u}|_{Z^1_{T_0}} \leq r_0 + |\tilde{u}|_{Z^1_{T_0}}.$$

Suppose $\rho_0 \in H^3_p(\Omega)$, $p > p_*, \rho_0 > 0$ in $\bar{\Omega}$. Then the continuity equation, together with $\rho|_{t=0} = \rho_0$, has a unique positive solution $\rho = \Phi(u) \in Z^T_3$, and we have the a priori estimate

$$|\rho|_{Z^T_3} \leq C_0,$$

where $C_0$ is independent of $T, r, u$.

For the contraction estimate we use that ($J = [0, T]$)

$$|\Phi(u_1) - \Phi(u_2)|_{H^1_2(J;L^p(\Omega)) \cap C(J;H^1_p(\Omega))} \leq C_1(T)|u_1 - u_2|_{Z^1_T},$$

for all $(u_i, c_i) \in \Sigma^T_r$, $i = 1, 2$, where $C_1(T) \to 0$ as $T \to 0$ and $C_1$ is independent of $r$. 
A weak estimate for Cahn-Hilliard

Recall the fixed point formulation for (CH):

\[
\kappa_0 \frac{\partial}{\partial t} \hat{c} + \nabla \cdot (\varepsilon_0 \nabla \nabla \cdot (\varepsilon_0 \nabla \hat{c})) = F_2(u, c, \Phi(u)) \quad (J \times \Omega),
\]

\[
\frac{\partial}{\partial \nu} \hat{c} = 0, \quad \frac{\partial}{\partial \nu} \nabla \cdot (\varepsilon_0 \nabla \hat{c}) = \frac{\partial}{\partial \nu} h(c, \Phi(u)) \quad (J \times \Gamma),
\]

\[
\hat{c}|_{t=0} = c_0 \quad (\Omega).
\]

Here \( h(c, \rho) = \nabla \cdot ([\varepsilon(\rho, c) - \varepsilon_0] \nabla c) + \rho^{-1} \varepsilon(\rho, c) \nabla \rho \cdot \nabla c - \partial_c \psi(\rho, c, \nabla c) \).

Recall \( Y_2^T := H_p^{1/2}(J; L_p(\Omega)) \cap L_p(J; H_p^2(\Omega)). \)

For \((u_i, c_i) \in \Sigma_r^T, \rho_i = \Phi(u_i), i = 1, 2,\) we can show that

\[
|\hat{c}_1 - \hat{c}_2|_{Y_2^T} \leq C_2 \left( |h(c_1, \rho_1) - h(c_2, \rho_2)|_{L_p(L_p)} + |\Theta|_{H_p^{1/2}(L_p)} \right.
\]

\[
+ |\tilde{\Theta}|_{L_p(L_p)} + |\gamma_0 - \gamma_1|_{C(\tilde{C}^1)} |c_1 - c_2|_{L_p(H_p^2)} + T^{1/4} |\gamma_1 - \gamma_2|_{C(H_p^1)}) \right),
\]

where \( \gamma_i = \gamma(\rho_i, c_i), \Theta = (\rho_0 - \rho_1)(c_1 - c_2) - (\rho_1 - \rho_2)c_2, \) and \( \tilde{\Theta} = c_1 \rho_1 u_1 - c_2 \rho_2 u_2. \)

Pf.: Uses the divergence structure, duality relations, and max. reg. methods.
Final remarks

- The local solution can be extended to a maximally defined solution.

- The main result can be generalized to cover other boundary conditions like the pure slip condition:

\[ u \cdot \nu = 0, \quad (I - \nu \otimes \nu) S \nu = 0 \quad \text{on} \quad [0, T] \times \Gamma. \]