Asymptotic behavior of solutions of the compressible Navier-Stokes equation around a parallel flow

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1. Introduction

- $\rho = \rho(x,t),\ v = (v^1(x,t), \cdots, v^n(x,t)),\ t \geq 0,\ x \in \mathbf{R}^n\ (n \geq 2)$.

$$\begin{cases} \partial_t \rho + \mathrm{div}\,(\rho v) = 0, \\ \rho(\partial_t v + v \cdot \nabla v) - \mu \Delta v - (\mu + \mu') \nabla \mathrm{div}\, v + \nabla p(\rho) = \rho \boldsymbol{f}. \end{cases}$$

- $p = p(\rho) = K\rho^a$: pressure $(K > 0,\ a > 1$: const's)

- $\mu,\ \mu'$: const's, $\mu > 0, \frac{2}{n}\mu + \mu' \geq 0$

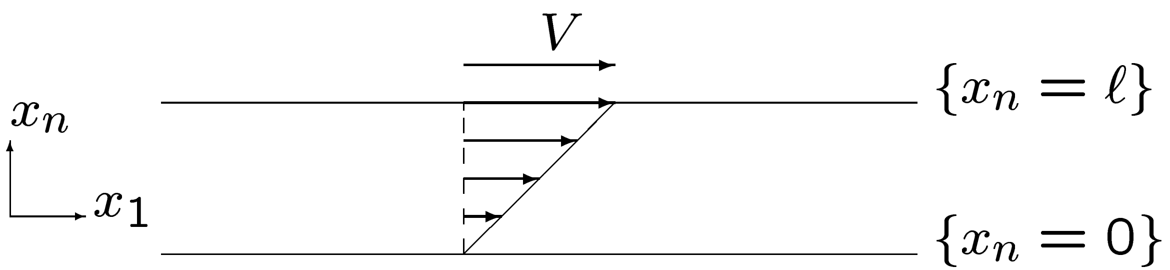
- $\Omega_\ell = \{x = (x', x_n); x' = (x_1, \cdots, x_{n-1}) \in \mathbf{R}^{n-1}, 0 < x_n < \ell\}$

$\{x_n = \ell\}$

$\{x_n = 0\}$

- Couette flow

- $\boldsymbol{f} = \mathbf{0}$, $v^1|_{x_n=\ell} = V$, $v^2|_{x_n=\ell} = \cdots = v^n|_{x_n=\ell} = 0$, $v|_{x_n=0} = 0$

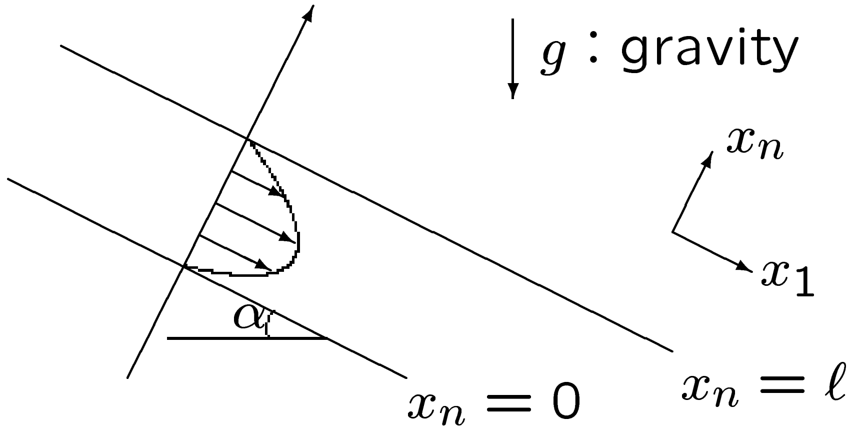


$$\rho_s = \rho_* > 0 \quad \text{const.},$$

$$v_s = \left(\frac{V}{\ell} x_n, 0, \cdots, 0\right).$$

- Poiseuille type flow

- $\boldsymbol{f} = (g \sin\alpha, \cdots, -g\cos\alpha)$, $g > 0$, α: const's; $v|_{x_n=0,\ell} = 0$



$\rho_s = \rho_s(x_n)$, $v_s = (v_s^1(x_n), 0, \cdots, 0)$

- Stability of stationary flow (ρ_s, v_s).
- Large time behavior of the perturbation :

$$(\phi(t), w(t)) = (\rho(t) - \rho_s, v(t) - v_s).$$

Incompressible case

- Reynolds number Re

$$\text{Re} \leq \exists\, \text{Re}_0 \quad \Rightarrow \quad |v(t) - v_s|_{L^2} \leq Ce^{-\delta_0 t}|v_0 - v_s|_{L^2}.$$

- Romanov ('73): Stablity of the plane Couette flow

$$\forall\, \text{Re},\ \exists \delta > 0 \text{ s.t.} |v_0 - v_s|_{H^1} << 1 \Rightarrow |v(t) - v_s|_{H^1} \leq Ce^{-\delta t}.$$

- Heck–Kim–Kozono ('09): Stablity of the plane Couette flow in L^n (periodic)

• Abe–Shibata ('03): stability of general parallel flow v_s in L^n

$$\mathrm{Re} \le \exists\, \mathrm{Re}_1, \quad |v_0 - v_s|_{L^n} << 1 \Rightarrow |v(t) - v_s|_{L^n} \le Ce^{-\delta_1 t}.$$

• Perturbation method for the analytic semigroup generated by the linearized operator at the motionless state $\rho_s = \rho_*$, $v_s = 0$.

Linearized Problem at (ρ_*, v_s):

$$\begin{cases} \rho_* \partial_t w - \mu \Delta w + \nabla P + (v_s \cdot \nabla w + w \cdot \nabla v_s) = F, \\ \mathrm{div}\, w = 0, \\ w|_{\partial\Omega} = 0 \end{cases}$$

2. Results.

- Non-dimensionalization

- $\Omega = \{x = (x', x_n); x' = (x_1, \cdots, x_{n-1}) \in \mathbf{R}^{n-1},\ 0 < x_n < 1\}$

$$\{x_n = 1\}$$

$$\{x_n = 0\}$$

- Couette flow: $\rho_s = 1$, $v_s = (x_n, 0, \cdots, 0)$

- Poiseuille type flow:

$$\omega = \frac{(a-1)g\ell \cos\alpha}{a\rho_*^{a-1}K}$$

$$\rho_s = (1 + \omega(1 - x_n))^{\frac{1}{a-1}} = 1 + O(\omega), \quad (|\omega| \ll 1)$$

$$v_s = \left(\frac{1}{2}x_n(1 - x_n) + O(\omega), 0, \cdots, 0\right)$$

- $u(t) = (\phi(t), w(t)) = (\gamma^2(\rho(t) - \rho_s), v(t) - v_s)$: perturbation

$$\partial_t \phi + v_s \cdot \nabla \phi + \gamma^2 \mathrm{div}\,(\rho_s w) = F^0, \tag{2.1}$$

$$\begin{aligned} &\partial_t w - \frac{\nu}{\rho_s}\Delta w - \frac{\tilde{\nu}}{\rho_s}\nabla \mathrm{div}\, w + \nabla\left(\frac{\tilde{p}'(\rho_s)}{\gamma^2\rho_s}\phi\right) \\ &\quad -\frac{\nu}{\gamma^2\rho_s}\phi\Delta v_s + v_s\cdot\nabla w + w\cdot\nabla v_s = G, \end{aligned} \tag{2.2}$$

$$w|_{x_n=0,1} = 0, \quad (\phi, w)|_{t=0} = (\phi_0, w_0), \tag{2.3}$$

- F^0, G : nonlinearities,

- $\nu = \frac{\mu}{\rho_*\ell V}$, $\tilde{\nu} = \frac{\mu+\mu'}{\rho_*\ell V}$, $\gamma^2 = \frac{p'(\rho_*)}{V^2} > 0$, ($V^2 = \frac{\rho_* g\ell\sin\alpha}{\mu}$ if Poiseuille).

- $\mathrm{Re} = \dfrac{1}{\nu}$: Reynolds number, $\mathrm{Ma} = \dfrac{1}{\gamma}$: Mach number

Theorem 1. *Let* (ρ_s, v_s)*: Couette flow. Let* m *be an integer satisfying* $m \geq [n/2] + 2$*. Then* $\exists\ \nu_0 > 0$, $\gamma_0 > 0$ *such that if*

$$\nu \geq \nu_0, \qquad \frac{\gamma^2}{\nu + \tilde{\nu}} \geq \gamma_0^2$$

then the followings hold:

$u_0 = (\phi_0, w_0) \in H^m \cap L^1$*: small + compatibility condition,*
$\Rightarrow \exists!$ *global sol.* $u(t) = (\phi(t), w(t))$ *of (1.1)–(1.3):*

$$|\partial_{x'}^{\ell} u(t)|_{L^2} = O(t^{-\frac{n-1}{4}-\frac{\ell}{2}}) \quad (t \to \infty)$$

for $\ell = 0, 1$ *and*

$$|u(t) - G_t * \langle \phi_0 \rangle u^{(0)}|_{L^2} = O(t^{-\frac{n-1}{4}-\frac{1}{2}} L(t)) \quad (t \to \infty)$$

Here

$$G_t * \langle \phi_0 \rangle = \mathcal{F}^{-1}[e^{-(i\kappa_0 \xi_1 + \kappa_1 \xi_1^2 + \kappa_2 |\xi''|^2)t} \langle \hat{\phi}_0 \rangle],$$

$$\langle \hat{\phi}_0 \rangle = \int_0^1 \hat{\phi}_0(\xi', x_n)\, dx_n, \quad \xi' = (\xi_1, \xi''), \quad \xi'' = (\xi_2, \cdots, \xi_{n-1})$$

where $\kappa_j > 0$ $(j = 0, 1, 2)$*: constants;* $u^{(0)} = u^{(0)}(x_n)$*;* $L(t) = 1$ *when* $n \geq 3$ *and* $L(t) = \log(1 + t)$ *when* $n = 2$*.*

A similar result holds for Poiseuille type flow if $n \geq 3$ *and* $|\omega| \ll 1$ *with* $L(t)$ *given by* $L(t) = 1$ *when* $n \geq 4$ *and* $L(t) = \log(1 + t)$ *when* $n = 3$

3. Proof

Theorem 1 is proved by

(a) A variant of Matsumura–Nishida's Energy Method

(b) Decay Estimates for Linearized Problem

- Poiseuille flow: $\omega = 0$; $\rho_s = 1$, $v_s^1 = \frac{1}{2}x_n(1 - x_n)$

- <u>3-1 Decay Estimates for Linearized Problem</u>

  - $S(t)u_0 = (\phi(t), w(t))$: sol. of

$$
\begin{cases}
\partial_t \phi + v_s^1 \partial_{x_1} \phi + \gamma^2 \mathrm{div}\, w = 0, \\
\partial_t w - \nu \Delta w - \tilde{\nu} \nabla \mathrm{div}\, w + \nabla \phi \\
\qquad\qquad - \frac{\nu}{\gamma^2} \phi \boldsymbol{e}_1 + v_s^1 \partial_{x_1} w + \partial_{x_n} v_s^1 w^n \boldsymbol{e}_1 = 0, \\
w|_{x_n = 0,1} = 0; \quad (\phi, w)|_{t=0} = (\phi_0, w_0).
\end{cases}
$$

Theorem 2. $n \geq 2$.

(i) $u_0 = (\phi_0, w_0) \in H^1 \times L^2$, $\partial_{x'} w_0 \in L^2$

$$\Rightarrow \quad |S(t)u_0|_{L^2} \leq |u_0|_{L^2} \quad (t \in [0,1]),$$

$$|S(t)u_0|_{H^1} \leq Ct^{-\frac{1}{2}}(|u_0|_{H^1 \times L^2} + |\partial_{x'} w_0|_{L^2}) \quad (t \in (0,1]).$$

(ii) *If* $\nu \geq \nu_0$, $\dfrac{\gamma^2}{\nu + \tilde{\nu}} \geq \gamma_0^2$,

$$\Rightarrow S(t)u_0 = S^{(0)}(t)u_0 + S^{(\infty)}(t)u_0,$$

(ii-1) $S^{(0)}(t)u_0 \ \sim \ \mathcal{F}^{-1}[e^{-(i\kappa_0\xi_1+\kappa_1\xi_1^2+\kappa_2|\xi''|^2)t}\,\langle\widehat{\phi}_0\rangle]u^{(0)}$,

$$\langle\widehat{\phi}_0\rangle = \textstyle\int_0^1 \widehat{\phi}_0(\xi', x_n)\,dx_n, \quad \xi' = (\xi_1, \xi''), \quad \xi'' = (\xi_2, \cdots, \xi_{n-1}),$$

$$u^{(0)} = (\phi^{(0)}, w^{(0),1}\boldsymbol{e}_1),\ \phi^{(0)} = 1,\ w^{(0),1} = \tfrac{1}{2\gamma^2}x_n(1 - x_n)$$

(ii-2) $|S^{(\infty)}(t)u_0|_{H^1} \leq Ce^{-ct}|u_0|_{H^1}$

- <u>Spectrum of the Linearized Operator</u>

- Fourier transform in $x' = (x_1, \cdots, x_{n-1}) \in \mathbf{R}^{n-1}$: $\partial_{x'} \;\to\; i\xi'$

$$\begin{cases} \partial_t \hat{u} + \widehat{L}_{\xi'} \hat{u} &= 0, \\ \hat{u}|_{t=0} &= \hat{u}_0 \end{cases}$$

- $\exists\, r > 0$ s.t.
    - $|\xi'| \le r \;\;\Rightarrow\;\; \sigma(-\widehat{L}_{\xi'}) \subset \{\lambda_0(\xi')\} \cup \{\mathrm{Re}\,\lambda \le -\Lambda_0\}$,

$$\lambda_0(\xi') \sim -i\kappa_0\xi_1 - \kappa_1\xi_1^2 - \kappa_2|\xi''|^2,$$

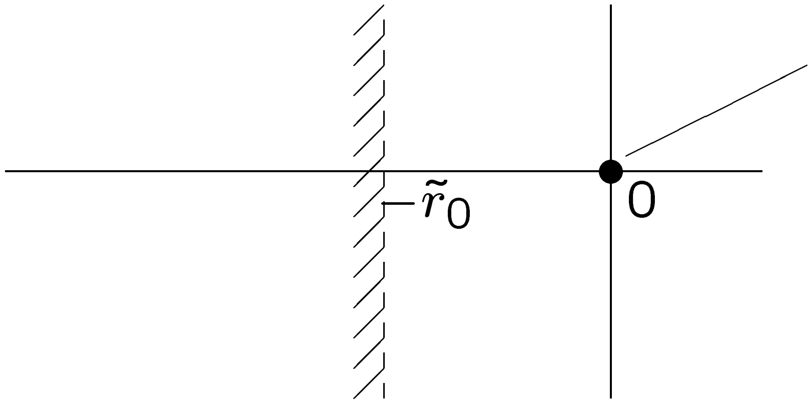
$$\xi' = (\xi_1, \xi''), \quad \xi'' = (\xi_2 \cdots, \xi_{n-1}).$$

    - $|\xi'| \ge r \;\;\Rightarrow\;\; \sigma(-\widehat{L}_{\xi'}) \subset \{\mathrm{Re}\,\lambda \le -\Lambda_1\}$.

- $\underline{\sigma(-\hat{L}_{\xi_1})\ (n=2)}$

  $\underline{\xi_1 = 0}$:

  $\underline{\sigma(-\hat{L}_0) \subset \{0\} \cup \{\mathrm{Re}\,\lambda \le -\tilde{r}_0\}}$:



0 : simple eigenvalue

$\hat{\Pi}_0$: eigenprojection

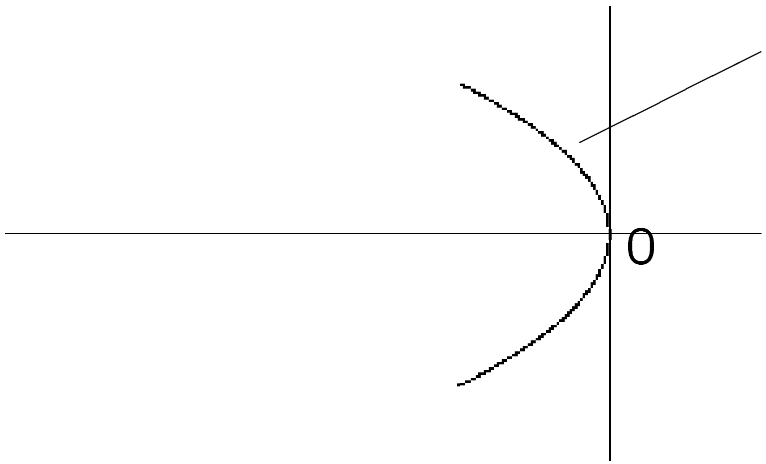
$\hat{\Pi}_0 u = \langle \phi \rangle u^{(0)}, \quad u = (\phi, w)$

$\langle \phi \rangle = \int_0^1 \phi(x_n)\, dx_n$

$$-\hat{L}_0 u^{(0)} = 0,\ u^{(0)} = (\phi^{(0)}, w^{(0),1}, 0 \cdots, 0),$$

$$\phi^{(0)} = 1,\ w^{(0),1} = \frac{1}{2\gamma^2} x_n(1 - x_n) = O\left(\frac{1}{\gamma^2}\right)$$

- $\sigma(-\widehat{L}_{\xi_1})$ for $|\xi_1| << 1$:



$$\lambda_0(\xi_1) \sim -i\kappa_0\xi_1 - \kappa_1\xi_1^2$$

$$\widehat{\Pi}_0(\xi_1) = \widehat{\Pi}_0 + O(|\xi_1|)$$

3-2. Global Existence

$$\partial_t u + Lu = \boldsymbol{F}, \quad u = {}^T(\phi, w),$$

- $L = A + B + C,$

$$A = \begin{pmatrix} 0 & 0 \\ 0 & -\nu\Delta I_n - \tilde{\nu}\nabla \mathrm{div} \end{pmatrix}, \; B = \begin{pmatrix} v_s^1 \partial_{x_1} & \gamma^2 \mathrm{div} \\ \nabla & v_s^1 \partial_{x_1} I_n \end{pmatrix},$$

$$C = \begin{pmatrix} 0 & 0 \\ -\frac{\nu}{\gamma^2}\boldsymbol{e}_1 & (\partial_{x_n} v_s^1)\boldsymbol{e}_1 \otimes \boldsymbol{e}_n \end{pmatrix}.$$

- $(Cu, u) = -\frac{\nu}{\gamma^2}(\phi, w^1) + ((\partial_{x_n} v_s^1) w^n, w^1)$

- $u^{(0)} = u^{(0)}(x_n) \neq 0 \; : \; Lu^{(0)} = \widehat{L}_0 u^{(0)} = 0.$

- $\widehat{L}_0 = \widehat{A}_0 + \widehat{B}_0 + C,$

$$\widehat{A}_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\nu\partial_{x_n}^2 I_{n-1} & 0 \\ 0 & 0 & -(\nu+\tilde{\nu})\partial_{x_n}^2 \end{pmatrix}, \; \widehat{B}_0 = \begin{pmatrix} 0 & 0 & \gamma^2\partial_{x_n} \\ 0 & 0 & 0 \\ \partial_{x_n} & 0 & 0 \end{pmatrix}.$$

- $Lu^{(0)} = \widehat{L}_0 u^{(0)} = 0$

$u^{(0)} = u^{(0)}(x_n) = (\phi^{(0)}(x_n), w^{(0)}(x_n))$, $w^{(0)} = w^{(0),1}(x_n)\boldsymbol{e}_1$

$$\phi^{(0)} = 1, \quad w^{(0),1} = O\left(\frac{1}{\gamma^2}\right)$$

- <u>P_0: Projection</u>

$$\widehat{P_0 u}(\xi', x_n) = \left\langle \widehat{\phi}(\xi') \right\rangle u^{(0)}(x_n) \quad \text{for } u = (\phi, w),$$

where $\xi' = (\xi_1, \cdots, \xi_{n-1})$,

$$\left\langle \widehat{\phi}(\xi') \right\rangle = \chi_0(\xi') \int_0^1 \widehat{\phi}(\xi', x_n)\, dx_n, \quad \chi_0(\xi') = \begin{cases} 1 & (|\xi'| \leq r) \\ 0 & (|\xi'| > r) \end{cases}$$

- $P_1 = I - P_0$

- Decomposition

$$u = P_0 u + P_1 u \equiv \sigma u^{(0)} + u_1,$$

$$\sigma = \sigma(x', t), \quad \hat{\sigma} = \left\langle \chi_0 \widehat{\phi}(\xi', \cdot, t) \right\rangle$$

$$u_1 = (\phi_1, w_1)$$

- $|w_1|_2 \leq |\partial_x w_1|_2$, $|\phi_1|_2 \leq C|\partial_x \phi_1|_2$

- $P_0 L_0 = L_0 P_0 = O$

- $Lu = Lu_1 + \mathcal{M}(\sigma u^{(0)})$, $\mathcal{M} = L - \hat{L}_0$

$$\partial_t u + L u_1 + \mathcal{M}(\sigma u^{(0)}) = \boldsymbol{F}, \quad \mathcal{M} = L - L_0$$

$\Longleftrightarrow$

$$\begin{cases} \partial_t(\sigma u^{(0)}) + P_0 \mathcal{M}(\sigma u^{(0)} + u_1) = P_0 \boldsymbol{F}, \\ \partial_t u_1 + L u_1 + \mathcal{M}(\sigma u^{(0)}) - P_0 \mathcal{M}(\sigma u^{(0)} + u_1) = P_1 \boldsymbol{F} \end{cases}$$

$\Longleftrightarrow$

$$\begin{cases} \partial_t \sigma + \left\langle \mathcal{M}(\sigma u^{(0)} + u_1) \right\rangle = \langle \boldsymbol{F} \rangle, \\ \partial_t u_1 + L u_1 + \mathcal{M}(\sigma u^{(0)}) - \left\langle \mathcal{M}(\sigma u^{(0)} + u_1) \right\rangle u^{(0)} = P_1 \boldsymbol{F} \end{cases}$$

$$\begin{cases} \partial_t \sigma + \left\langle v_s^1 \partial_{x_1} (\sigma \phi^{(0)} + \phi_1) \right\rangle + \gamma^2 \left\langle \nabla' \cdot (\sigma w^{(0)} + w_1) \right\rangle = \langle \boldsymbol{F} \rangle \,, \\ \partial_t u_1 + L u_1 + \mathcal{M}(\sigma u^{(0)}) - \left\langle \mathcal{M}(\sigma u^{(0)} + u_1) \right\rangle u^{(0)} = P_1 \boldsymbol{F} \end{cases}$$

where $\sigma u^{(0)} = \sigma(\phi^{(0)}, w^{(0)}),\ u_1 = (\phi_1, w_1)$,

$$\left\langle \mathcal{M}(\sigma u^{(0)} + u_1) \right\rangle = \left\langle v_s^1 \partial_{x_1} (\sigma \phi^{(0)} + \phi_1) \right\rangle + \gamma^2 \left\langle \nabla' \cdot (\sigma w^{(0)} + w_1) \right\rangle ,$$

$$\langle \boldsymbol{F} \rangle = - \langle \nabla' \cdot (\phi w) \rangle = - \left\langle \nabla' \cdot (\sigma \phi^{(0)} + \phi_1)(\sigma w^{(0)} + w_1) \right\rangle ,$$

$$\nabla' = {}^T(\partial_{x_1}, \cdots, \partial_{x_{n-1}}, 0)$$

Proposition.

$n \geq 3, \quad s \geq [n/2] + 1$

$\Rightarrow \quad \exists!\, u(t) = \sigma(t)u^{(0)} + (\phi_1(t), w_1(t))$: global sol.

$$|\sigma(t)|^2_{H^s} + |u(t)|^2_{H^s} + \int_0^t (|\partial_{x'}\sigma|^2_{H^s} + |\partial_x \phi_1|^2_{H^{s-1}} + |\partial_x w_1|^2_{H^s})\, d\tau$$

$$\leq C|u_0|^2_{H^s},$$

provided that

$$|u_0|_{H^s} \ll 1, \quad \nu \geq \nu_0, \quad \frac{\gamma^2}{\nu + \tilde{\nu}} \geq \gamma_0^2.$$

Take L^2 inner product:

$$\begin{cases} \partial_t \sigma + \left\langle \mathcal{M}(\sigma u^{(0)} + u_1) \right\rangle = \langle \boldsymbol{F} \rangle \,, \\ \partial_t u_1 + L u_1 + \mathcal{M}(\sigma u^{(0)}) - \left\langle \mathcal{M}(\sigma u^{(0)} + u_1) \right\rangle u^{(0)} = P_1 \boldsymbol{F} \end{cases}$$

- $$\begin{aligned} |(C u_1, u_1)| &= \left| -\tfrac{\nu}{\gamma^2}(\phi_1, w_1^1) + ((\partial_{x_n} v_s^1) w_1^n, w_1^1) \right| \\ &\le \tfrac{\nu}{\gamma^2} |\partial_x \phi_1|_2 |\partial_x w_1|_2 + |\partial_x w_1|_2^2 \end{aligned}$$

- $\nu \geq \nu_0 > 0 \Longrightarrow$

$$\begin{aligned} &|\sigma(t)|_2^2 + |u_1(t)|_2^2 + \int_0^t \nu |\nabla w_1|_2^2 + \tilde{\nu} |\mathrm{div} w_1|_2^2 \, d\tau \\ &\quad \le |\sigma(0)|_2^2 + |u_1(0)|_2^2 + C \left(\frac{\nu + \tilde{\nu}}{\gamma^4} + \frac{1}{\gamma^2} \right) \int_0^t \left(|\partial_{x'} \sigma|_2^2 + |\partial_x \phi_1|_2^2 \right) \, d\tau \\ &\qquad + h.o.t. \end{aligned}$$

- Matsumura-Nishida's Energy Method for $u_1 = (\phi_1, w_1)$

$$\partial_t u_1 + L u_1 + \mathcal{M}(\sigma u^{(0)}) - \left\langle \mathcal{M}(\sigma u^{(0)} + u_1) \right\rangle u^{(0)} = P_1 \boldsymbol{F}$$

$\Longrightarrow$

$$|\sigma(t)|_{H^1}^2 + |u_1(t)|_{H^1}^2 + \int_0^t \frac{1}{\nu + \tilde{\nu}} |\partial_x \phi_1|_2^2 + \nu |\partial_x w_1|_{H^1}^2 \, d\tau$$

$$\leq |\sigma(0)|_{H^1}^2 + |u_1(0)|_{H^1}^2 + C \left(\frac{\nu + \tilde{\nu}}{\gamma^2} + \frac{1}{\gamma^2} \right) \int_0^t |\partial_{x'} \sigma|_2^2 \, d\tau$$

$$+ h.o.t.$$

- Estimate for $\int_0^t |\partial_{x'}\sigma|_2^2 \, d\tau$

$$\partial_t \phi + v_s^1 \partial_{x_1} \phi + \gamma^2 \mathrm{div}\, w = -\mathrm{div}\,(\phi w)$$

$\phi = \sigma\phi^{(0)} + \phi_1,\ w = \sigma w^{(0)} + w_1,\ \int_0^1 \cdots \, dx_n,$

$$\partial_t \sigma + \left\langle v_s^1 \partial_{x_1}(\sigma\phi^{(0)} + \phi_1)\right\rangle + \gamma^2 \left\langle \nabla' \cdot (\sigma w^{(0)'} + w_1')\right\rangle = \langle \boldsymbol{F} \rangle$$

Fourier transform in x':

$$\partial_t \widehat{\sigma} + \left\langle v_s^1 \phi^{(0)} \right\rangle i\xi_1 \widehat{\sigma} + \gamma^2 \left\langle i\xi' \cdot \widehat{w}_1' \right\rangle = \left\langle \widehat{h} \right\rangle$$

- $\partial_t w' - \nu \Delta w' - \tilde{\nu} \nabla' \mathsf{div}\, w + \nabla' \phi = h'$

$\phi = \sigma \phi^{(0)} + \phi_1,\ \phi^{(0)} = \text{const.} \equiv 1,\ w' = \sigma w^{(0),1} \boldsymbol{e}_1' + w_1'$

$\Longrightarrow$

$$-\nu \Delta w_1' = -\nabla' \sigma + h_1'$$

$\Longrightarrow$

$$(|\xi'|^2 - \partial_{x_n}^2)\widehat{w}_1' = -\tfrac{1}{\nu}\, i\xi' \widehat{\sigma} + \tfrac{1}{\nu} \widehat{h}_1'$$

$\Longrightarrow$

$$\widehat{w}_1' = -\tfrac{1}{\nu}\, i\xi'\, \widehat{\sigma}\, (|\xi'|^2 - \partial_{x_n}^2)^{-1} \cdot 1 + \tfrac{1}{\nu}(|\xi'|^2 - \partial_{x_n}^2)^{-1}[\widehat{h}_1']$$

$$\partial_t \widehat{\sigma} + \left\langle v_s^1 \phi^{(0)} \right\rangle i\xi_1 \widehat{\sigma} + \gamma^2 \left\langle i\xi' \cdot \widehat{w}_1' \right\rangle = \left\langle \widehat{h} \right\rangle$$

$$\widehat{w}_1' = -\frac{1}{\nu} i\xi' \, \widehat{\sigma} \, (|\xi'|^2 - \partial_{x_n}^2)^{-1} \cdot 1 + \frac{1}{\nu} (|\xi'|^2 - \partial_{x_n}^2)^{-1} [\widehat{h}_1']$$

$\Longrightarrow$

$$\partial_t \widehat{\sigma} + \left\langle v_s^1 \phi^{(0)} \right\rangle i\xi_1 \widehat{\sigma} + \frac{\gamma^2}{\nu} \left\langle (|\xi'|^2 - \partial_{x_n}^2)^{-1} \cdot 1 \right\rangle |\xi'|^2 \widehat{\sigma} = \left\langle \widehat{H} \right\rangle$$

$$\frac{1}{2} \frac{d}{dt} |\widehat{\sigma}|^2 + \frac{\gamma^2}{\nu} \left\langle (|\xi'|^2 - \partial_{x_n}^2)^{-1} \cdot 1 \right\rangle |\xi'|^2 |\widehat{\sigma}|^2 = \Re \left[\left\langle \widehat{H} \right\rangle \overline{\widehat{\sigma}} \right]$$

$$\begin{aligned} \left\langle (|\xi'|^2 - \partial_{x_n}^2)^{-1} \cdot 1 \right\rangle &= |(|\xi'|^2 - \partial_{x_n}^2)^{-1/2} \cdot 1|_{L^2(0,1)}^2 \\ &\geq c(r) > 0 \quad (|\xi'| \leq r) \end{aligned}$$

$$\frac{1}{2} \frac{d}{dt} |\widehat{\sigma}|^2 + \frac{\gamma^2}{\nu} c(r) |\xi'|^2 |\widehat{\sigma}|^2 \leq \Re \left[\left\langle \widehat{H} \right\rangle \overline{\widehat{\sigma}} \right] \quad (|\xi'| \leq r)$$