On two-phase flows with surface viscosity

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The model under consideration (1/2)

- $\Omega = \Omega_1(t) \cup \Gamma(t) \cup \Omega_2(t) \subset \mathbb{R}^{n+1}$ bounded smooth domain, $\Gamma(t)$ smooth interface, ν_Γ unit normal, V_Γ normal velocity.
- Velocity u and pressure π satisfy the incompressible Navier-Stokes equations in $\Omega \setminus \Gamma(t)$:

$$\operatorname{div} u = 0, \quad \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u - S) = 0,$$

  $S = 2\mu D - \pi I$ stress tensor, $D = \frac{1}{2}(\nabla u + \nabla u^T)$,
  $\rho_{1,2} > 0$ constant densities in $\Omega_{1,2}$,
  $\mu_{1,2} > 0$ constant viscosities in $\Omega_{1,2}$.
- No interfacial mass and no phase transition on Γ:

$$[\![u]\!] := u_2|_\Gamma - u_1|_\Gamma = 0, \quad V_\Gamma = u \cdot \nu_\Gamma, \quad u_\Gamma = u.$$

The model under consideration (2/2)

- Interfacial stress balance: $-[\![S]\!]\nu_\Gamma = \mathrm{div}_\Gamma\, S_\Gamma$,
  surface stress tensor S_Γ defined by a constitutive equation:
- Surface tension: $S_\Gamma = \sigma P_\Gamma$, $\sigma > 0$ constant surface tension,
  $P_\Gamma = I - \nu \otimes \nu$ tangential projection, κ_Γ curvature.

$$\Rightarrow \quad -[\![S]\!]\nu_\Gamma = \sigma \kappa_\Gamma \nu_\Gamma \qquad \text{(Laplace-Young law).}$$

- This system was investigated by Tanaka (1996), Shibata & Shimizu (2007-2009), Prüss & Simonett (2009) and Köhne, Prüss & Wilke (2010).
- Surface tension and surface viscosity (Boussinesq-Scriven law):

$$S_\Gamma = \sigma P_\Gamma + [(\lambda_s - \mu_s)\,\mathrm{div}_\Gamma\, u]P_\Gamma + 2\mu_s D_\Gamma,$$
$$D_\Gamma = \tfrac{1}{2} P_\Gamma(\nabla_\Gamma u + \nabla_\Gamma u^T)P_\Gamma.$$

  $\lambda_s > 0$ dilatational surface viscosity,
  $\mu_s > 0$ surface shear viscosity, $\lambda_s > \mu_s$,
  div_Γ surface divergence, ∇_Γ surface gradient.

The nonlinear problem

Goal: Find solution (u, π, Γ) of the nonlinear problem

$$\begin{cases} \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u - S) = 0, & t > 0, \quad x \in \Omega \setminus \Gamma(t), \\ \operatorname{div} u = 0 & t > 0, \quad x \in \Omega \setminus \Gamma(t), \\ -\operatorname{div}_\Gamma S_\Gamma - [\![S]\!]\nu_\Gamma = 0, & t > 0, \quad x \in \Gamma(t), \\ [\![u]\!] = 0, & t > 0, \quad x \in \Gamma(t), \\ V_\Gamma - u \cdot \nu_\Gamma = 0, & t > 0, \quad x \in \Gamma(t), \\ u = 0, & t > 0, \quad x \in \partial\Omega, \end{cases}$$

with initial conditions $\Gamma(0) = \Gamma_0$, $u(0) = u_0$ in $\Omega \setminus \Gamma_0$.
This model was formally analyzed by Bothe & Prüss (2008). In particular, the linearization of the system and linear stability of equilibria are known.

The linearization at a state with zero velocity

Notation: $J = [0,T]$, $\Gamma = \mathbb{R}^n \times \{0\}$, $\dot{\mathbb{R}} = \mathbb{R} \setminus \{0\}$, $u = (v,w)$, Δ_Γ Laplace-Beltrami operator.

$$\begin{cases} \rho\partial_t u - \mu\Delta u + \nabla\pi = f_u, & t \in J,\, x \in \mathbb{R}^n,\ y \in \dot{\mathbb{R}}, \\ \operatorname{div} u = f_d, & t \in J,\, x \in \mathbb{R}^n,\ y \in \dot{\mathbb{R}}, \\ -P_\Gamma \operatorname{div}_\Gamma S_\Gamma - P_\Gamma[\![S\nu]\!] = g_v, & t \in J,\, x \in \mathbb{R}^n, \\ -\sigma\Delta_\Gamma h - \nu\cdot[\![S\nu]\!] = g_w, & t \in J,\, x \in \mathbb{R}^n, \\ [\![u]\!] = 0, & t \in J,\, x \in \mathbb{R}^n, \\ \partial_t h - w|_\Gamma = g_h, & t \in J,\, x \in \mathbb{R}^n, \\ u(0) = u_0, & x \in \mathbb{R}^n,\ y \in \dot{\mathbb{R}}, \\ h(0) = h_0, & x \in \mathbb{R}^n. \end{cases}$$

Observe: $P_\Gamma \operatorname{div}_\Gamma S_\Gamma = \mu_s\Delta_\Gamma v|_\Gamma + \lambda_s\nabla_\Gamma \operatorname{div}_\Gamma v|_\Gamma$ is of second order!
$\Rightarrow$ We expect more spatial regularity for $v|_\Gamma$.

The result: Maximal regularity

There exist Banach spaces $\mathbb{E}$ and $\mathbb{F}$ such that the Stokes problem (SP) admits a unique solution $(u, \pi, h) \in \mathbb{E}$ if and only if $(f_u, f_d, g_v, g_w, g_h, u_0, h_0) \in \mathbb{F}$.
The solution operator $\mathcal{S} : \mathbb{F} \to \mathbb{E}$ is an isomorphism.

$$
\text{(SP)} \qquad \begin{cases}
\rho\partial_t u - \mu\Delta u + \nabla\pi = f_u, & t \in J,\, x \in \mathbb{R}^n,\ y \in \dot{\mathbb{R}}, \\
\operatorname{div} u = f_d, & t \in J,\, x \in \mathbb{R}^n,\ y \in \dot{\mathbb{R}}, \\
-P_\Gamma \operatorname{div}_\Gamma S_\Gamma - P_\Gamma [\![S\nu]\!] = g_v, & t \in J,\, x \in \mathbb{R}^n, \\
-\sigma\Delta_\Gamma h - \nu \cdot [\![S\nu]\!] = g_w, & t \in J,\, x \in \mathbb{R}^n, \\
[\![u]\!] = 0, & t \in J,\, x \in \mathbb{R}^n, \\
\partial_t h - w|_\Gamma = g_h, & t \in J,\, x \in \mathbb{R}^n, \\
u(0) = u_0, & x \in \mathbb{R}^n,\ y \in \dot{\mathbb{R}}, \\
h(0) = h_0, & x \in \mathbb{R}^n.
\end{cases}
$$

Regularity of the solution

Let $p \in (1,\infty) \setminus \{3/2, 3\}$. The solution space $\mathbb{E}$ consists of all functions $u = (v, w), \pi$ such that $[\![u]\!] = 0$ on $\mathbb{R}^n \times \{0\}$ and

$$\begin{aligned}
u \in \mathbb{E}_u &:= H^1_p(J; L_p(\mathbb{R}^{n+1}))^{n+1} \cap L_p(J; H^2_p(\dot{\mathbb{R}}^{n+1}))^{n+1},\\
v|_\Gamma \in \mathbb{E}_v &:= B^{1/2-1/2p}_{pp}(J; H^2_p(\mathbb{R}^n))^n \cap L_p(J; B^{3-1/p}_{pp}(\mathbb{R}^n))^n,\\
w|_\Gamma \in \mathbb{E}_w &:= H^1_p(J; \dot{B}^{-1/p}_{pp}(\mathbb{R}^n)),\\
\pi \in \mathbb{E}_\pi &:= L_p(J; \dot{H}^1_p(\dot{\mathbb{R}}^{n+1})),\\
h \in \mathbb{E}_h &:= B^{2-1/2p}_{pp}(J; L_p(\mathbb{R}^n)) \cap H^1_p(J; B^{2-1/p}_{pp}(\mathbb{R}^n))\\
&\quad \cap L_p(J; B^{3-1/p}_{pp}(\mathbb{R}^n)).
\end{aligned}$$

$\mathbb{E}$ has the norm of $\mathbb{E}_u \times \mathbb{E}_v \times \mathbb{E}_w \times \mathbb{E}_\pi \times \mathbb{E}_h$.

Compatibility conditions

Each solution $(u, \pi, h) \in \mathbb{E}$ to (SP) satisfies

$$\begin{aligned} [\![u_0]\!] &= 0, && \text{(all } p\text{)}, \\ \operatorname{div} u_0 &= f_d(0), && \text{(all } p\text{)}, \\ -P_\Gamma \operatorname{div}_\Gamma S_\Gamma(v_0) - P_\Gamma [\![S(u_0)\nu]\!] &= g_v(0), && (p > 3). \end{aligned}$$

$$(\mathrm{SP}) \quad \left\{ \begin{aligned} \rho\partial_t u - \mu\Delta u + \nabla\pi &= f_u, && t \in J,\, x \in \mathbb{R}^n,\ y \in \dot{\mathbb{R}}, \\ \operatorname{div} u &= f_d, && t \in J,\, x \in \mathbb{R}^n,\ y \in \dot{\mathbb{R}}, \\ -P_\Gamma \operatorname{div}_\Gamma S_\Gamma - P_\Gamma [\![S\nu]\!] &= g_v, && t \in J,\, x \in \mathbb{R}^n, \\ -\sigma\Delta_\Gamma h - \nu \cdot [\![S\nu]\!] &= g_w, && t \in J,\, x \in \mathbb{R}^n, \\ [\![u]\!] &= 0, && t \in J,\, x \in \mathbb{R}^n, \\ \partial_t h - w|_\Gamma &= g_h, && t \in J,\, x \in \mathbb{R}^n, \\ u(0) &= u_0, && x \in \mathbb{R}^n,\ y \in \dot{\mathbb{R}}, \\ h(0) &= h_0, && x \in \mathbb{R}^n. \end{aligned} \right.$$

Regularity of the data

The space of data $\mathbb{F}$ consists of all $data = (f_u, f_d, g_v, g_w, g_h, u_0, h_0)$, $u_0 = (v_0, w_0)$, which satisfy the compatibility conditions and

$$
\begin{aligned}
f_u &\in \mathbb{F}_u := L_p(J \times \mathbb{R}^{n+1})^{n+1},\\
f_d &\in \mathbb{F}_d := H^1_p(J; \dot{H}^{-1}_p(\dot{\mathbb{R}}^{n+1})) \cap L_p(J; H^1_p(\dot{\mathbb{R}}^{n+1})),\\
g_v &\in \mathbb{G}_v := B^{1/2-1/2p}_{pp}(J; L_p(\mathbb{R}^n))^n \cap L_p(J; B^{1-1/p}_{pp}(\mathbb{R}^n))^n,\\
g_w &\in \mathbb{G}_w := L_p(J; \dot{B}^{1-1/p}_{pp}(\mathbb{R}^n)),\\
g_h &\in \mathbb{G}_h := B^{1-1/2p}_{pp}(J; L_p(\mathbb{R}^n))^n \cap L_p(J; B^{2-1/p}_{pp}(\mathbb{R}^n))^n,\\
u_0 &\in \mathrm{tr}_t\, \mathbb{E}_u = B^{2-2/p}_{pp}(\dot{\mathbb{R}}^{n+1})^{n+1},\\
h_0 &\in \mathrm{tr}_t\, \mathbb{E}_h = B^{3-2/p}_{pp}(\mathbb{R}^n),\\
v_0|_\Gamma &\in \mathrm{tr}_t\, \mathbb{E}_v = \begin{cases} B^{4-6/p}_{pp}(\mathbb{R}^n)^n & \text{if } p \in (3/2, 3),\\ B^{3-3/p}_{pp}(\mathbb{R}^n)^n & \text{if } p \in (3, \infty). \end{cases}
\end{aligned}
$$

$\mathbb{F}$ has the norm of the corresponding product space.

Strategy of the proof

For given *data* $\in \mathbb{F}$ we construct bounded linear operators

$$\mathcal{S}_i : \mathbb{F} \to \mathbb{E}, \quad \textit{data} \mapsto (u_i, \pi_i, h_i),$$

such that (u_i, π_i, h_i) solve auxiliary problems (P_i) and $(u, \pi, h) = \sum_i (u_i, \pi_i, h_i)$ solves (SP).

(P_1) Parabolic problem: Only $f_u, u_0, h_0 \neq 0$.

(P_2) Divergence condition: Only $f_d \neq 0$.

(P_3) Stress condition: Only $g_v, g_w \neq 0$.

(P_4) Free boundary condition: Only $g_h \neq 0$.

(P_3) Stress condition (1/2)

Goal: Solve the "Neumann" problem $A_N(u,\pi) = (g_v, g_w)$, i. e.

$$(P_3) \quad \begin{cases} \rho\partial_t u - \mu\Delta u + \nabla\pi = 0, & t \in J,\, x \in \mathbb{R}^n,\ y \in \dot{\mathbb{R}}, \\ \operatorname{div} u = 0, & t \in J,\, x \in \mathbb{R}^n,\ y \in \dot{\mathbb{R}}, \\ -\operatorname{div}_\Gamma S_\Gamma - P_\Gamma[\![S\nu]\!] = g_v, & t \in J,\, x \in \mathbb{R}^n, \\ -\nu \cdot [\![S\nu]\!] = g_w, & t \in J,\, x \in \mathbb{R}^n, \\ [\![u]\!] = 0, & t \in J,\, x \in \mathbb{R}^n, \\ u(0) = 0, & x \in \mathbb{R}^n,\ y \in \dot{\mathbb{R}}. \end{cases}$$

Consider also the Dirichlet problem $A_D(u,\pi) = (g_v, g_w)$, i. e.

$$(P_{3'}) \quad \begin{cases} \rho\partial_t u - \mu\Delta u + \nabla\pi = 0, & t \in J,\, x \in \mathbb{R}^n,\ y \in \dot{\mathbb{R}}, \\ \operatorname{div} u = 0, & t \in J,\, x \in \mathbb{R}^n,\ y \in \dot{\mathbb{R}}, \\ u|_\Gamma = u_b, & t \in J,\, x \in \mathbb{R}^n, \\ u(0) = 0, & x \in \mathbb{R}^n,\ y \in \dot{\mathbb{R}}, \end{cases}$$

(P_3) Stress condition (2/2)

- The Dirichlet problem has maximal regularity, i. e. $A_D : (u, \pi) \mapsto u_b$ is an isomorphism [Prüss, Simonett 2009].
- Define Dirichlet-Neumann operator $\mathcal{N}$ by

$$\mathcal{N} = A_N A_D^{-1} : u_b \mapsto (g_v, g_w).$$

- Show that $\mathcal{N}$ is an isomorphism:
  - Fourier-Laplace transform $\rightsquigarrow$ symbol $(n_{ij})(\lambda, |\xi|)$, $(\lambda, |\xi| \in \mathbb{C})$.
  - Invertibility of n: [Bothe, Prüss 2008].
  - Asymptotic behaviour of n, n^{-1} for $|\lambda| \to 0, \infty$, $|\xi| \to 0, \infty$.
  - Joint functional calculus: $\mathcal{N} = n(\partial_t, \sqrt{-\Delta})$.
- Then $\mathcal{S}_3 := A_N^{-1} = A_D^{-1}\mathcal{N}^{-1} : (g_v, g_w) \mapsto (u_3, \pi_3)$ bounded.

(P_4) Free boundary condition

- Free boundary condition: $\partial_t h + w|_\Gamma = g_h$.
- Apply Dirichlet-Neumann operator to $(g_v, g_w) = (0, \sigma\Delta_\Gamma h)$.

$$\Rightarrow \quad w|_\Gamma = (\mathcal{N}^{-1})_{n+1,n+1}\, \sigma\Delta_\Gamma h.$$

- Define the boundary operator A_h by

$$A_h = \partial_t + (\mathcal{N}^{-1})_{n+1,n+1}\sigma\Delta_\Gamma : h \mapsto g_h.$$

- Prove that A_h is an isomorphism:
  - Fourier-Laplace transform $\rightsquigarrow$ boundary symbol $s(\lambda, |\xi|)$.
  - Invertibility the boundary symbol:

$$c(|\lambda| + |\tau|) \leq s(\lambda, \tau) \leq C(|\lambda| + |\tau|).$$

  - Joint functional calculus: $A_h = s(\partial_t, \sqrt{-\Delta})$.
- Then $\mathcal{S}_4 : (g_h) \mapsto (u_4, \pi_4, h_4)$ bounded.