Temporal semi-discretization of the equations describing generalized Newtonian fluids

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System of equations

$\Omega \subset \mathbb{R}^d$ bounded domain with $C^{0,1}$-boundary, $0 < T < \infty$,

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\begin{aligned}
u_t - \mathit{div}\, S(Du) + (u \cdot \nabla)u + \nabla\pi &= f \quad \text{in } \Omega \times (0,T), \\
\mathit{div}\, u &= 0 \quad \text{in } \Omega \times (0,T), \\
u(\cdot, 0) &= u_0 \quad \text{in } \Omega, \\
u(\cdot, t) &= 0 \quad \text{on } \partial\Omega,
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- p-coercivity:
$$S(z) \cdot z \geq \nu_0 |z|^p$$
- growth condition:
$$|S(z)| \leq c\,(1 + |z|)^{p-1}$$
- strict monotonicity:
$$(S(z) - S(y)) \cdot (z - y) > 0, \quad z \neq y$$

Function spaces and operators

- spaces of solenoidal functions

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Operator formulation in Bochner space

Find $u \in L^p(0, T; V_p)$ with $u(0) = u_0$ in H_2, such that

$$u' + Au + Bu = f \quad \text{in } L^{r'}(0, T; V_r^*).$$

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- values $p > \frac{2d}{d+2}$ are necessary for weak theory
  $\longrightarrow$ Gelfand triple $V_p \overset{c}{\hookrightarrow} H_2 \hookrightarrow V_p^*$
- for $p > 1 + \frac{2d}{d+2}$, one can choose $r = p$
  $\longrightarrow$ usual theory of monotone operators

Approximation

- implicit Euler scheme (existence: Frehse, Málek, Steinhauer (2003))

$$\frac{u^n - u^{n-1}}{\Delta t} + Au^n + Bu^n = f^n$$
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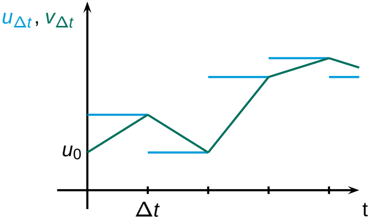
- piecewise polynomial prolongations
  - $u_{(\Delta t)_k}$ piecewise constant
  - $v_{(\Delta t)_k}$ piecewise linear, continuous

$$v'_{\Delta t} + Au_{\Delta t} + Bu_{\Delta t} = f_{\Delta t}$$



Convergence

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- a priori estimates: boundedness of $\{u_{\Delta t}\}$ and $\{v_{\Delta t}\}$ in $L^p(0,T;V_p)$ and $\{v'_{\Delta t}\}$ in $L^{r'}(0,T;V_r^*)$

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- weak limit u and weakly convergent subsequences

$$Au_{\Delta t} \rightharpoonup a \text{ in } L^{p'}(0,T;V_p^*), \qquad v'_{\Delta t} \rightharpoonup u' \text{ in } L^{r'}(0,T;V_r^*),$$
$$f_{\Delta t} \rightarrow f \text{ in } L^{p'}(0,T;V_p^*), \qquad Bu_{\Delta t} \rightarrow Bu \text{ in } L^{r'}(0,T;V_r^*)$$

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Identification of the limit

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- remains to show: $a = Au$
- usually Minty's monotonicity trick

Problem
lack of regularity, since time derivative belongs to large space $L^{r'}(0, T; V_r^*)$

Possible solution
construction of approximations to u with the so-called
Lipschitz truncation technique (see Diening, Růžička, Wolf (2008))

Lipschitz truncation technique

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Theorem (Lipschitz truncation technique)

Let

$$\{u_{\Delta t} - u\} \rightharpoonup 0 \quad \textit{in} \quad L^p(0,T;W^{1,p}) \cap N^{\sigma,q}(0,T;L^2),$$

then there exist truncations $T_{k,\Delta t}(u_{\Delta t} - u)$ *(different only on "small" sets) such that*

$$\limsup_{\Delta t \to 0} \|\nabla T_{k,\Delta t}(u_{\Delta t} - u)\|_{L^\infty(K)} \leq c\, 2^{2^{k+1}},$$

$$\lim_{\Delta t \to 0} \|T_{k,\Delta t}(u_{\Delta t} - u)\|_{L^\infty(K)} = 0$$

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Proof

- Hardy-Littlewood maximal functions
- Whitney covering of open sets with anisotropic metric
- adapted Chebyshev und Poincaré inequalities

Boundedness in Nikolskii spaces

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Let $0 < \sigma < 1$ and $1 < q < \infty$.
A function $v \in L^q(0,T;X)$ belongs to the Nikolskii space $N^{\sigma,q}(0,T;X)$, if there holds

$$\sup_{0<h<T} \int_0^{T-h} \left\| \frac{v(t+h) - v(t)}{h^\sigma} \right\|_X^q \mathrm{d}t \leq c.$$

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Lemma

$\{u_{\Delta t}\}$ is bounded in $N^{\sigma,2}(0,T;H_2)$ for $\sigma \leq \frac{1}{2} - \frac{\vartheta}{p}$, if

$$p > \frac{d + \sqrt{d^2 + 2d(d+2)}}{d+2} \quad \textit{and} \quad \vartheta = \frac{1}{2p}\left(\frac{1}{2} + \frac{1}{d} - \frac{1}{p}\right)^{-1}.$$

(for $d = 2$: $p > 1.62$ and for $d = 3$: $p > 1.85$)

Application to our problem

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- reconstruction of pressure functions
- estimate

$$\int_{Q\setminus E} \big(S(Du) - S(Du_{\Delta t})\big) : D(u - u_{\Delta t})\zeta \mathrm{d}(x,t) \leq c\, 2^{-k/p}$$

  to show $Du_{\Delta t} \to Du$ almost everywhere
- then follows

$$Au_{\Delta t} \rightharpoonup Au = a \quad \text{in } L^{p'}(0,T;V_p^*)$$

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Problem:
We did not find a proper estimate for the term

$$-\liminf_{\Delta t \to 0} \langle u' - v'_{\Delta t} + (\nabla p_{h,\Delta t})', T_{k,\Delta t}(u_{\Delta t} - u)\rangle.$$

Thank you for your attention.