Steady solutions with finite kinetic energy for the Navier-Stokes equations in a three-dimensional exterior domain

Ana Leonor Silvestre

Dep. of Mathematics and C.E.M.A.T.

Instituto Superior Técnico, T.U. Lisbon

Outline

- Navier-Stokes equations around a rigid body

  flow region: $\Omega \subseteq \mathbb{R}^3$ *exterior domain*; *rigid body velocity*: $V(x) = \zeta + \omega \times x$

Outline

- Navier-Stokes equations around a rigid body

  flow region: $\Omega \subseteq \mathbb{R}^3$ *exterior domain*; *rigid body velocity*: $V(x) = \zeta + \omega \times x$

- Steady solutions with finite kinetic energy when the body translates without without spinning

  rigid body velocity: $V(x) = \zeta$

Outline

- Navier-Stokes equations around a rigid body

  flow region: $\Omega \subseteq \mathbb{R}^3$ *exterior domain*; *rigid body velocity*: $V(x) = \zeta + \omega \times x$

- Steady solutions with finite kinetic energy when the body translates without without spinning

  rigid body velocity: $V(x) = \zeta$

- Steady solutions with finite kinetic energy in the whole space

  flow region: $\Omega = \mathbb{R}^3$

Outline

- Navier-Stokes equations around a rigid body

  flow region: $\Omega \subseteq \mathbb{R}^3$ *exterior domain*; *rigid body velocity*: $V(x) = \zeta + \omega \times x$

- Steady solutions with finite kinetic energy when the body translates without without spinning

  rigid body velocity: $V(x) = \zeta$

- Steady solutions with finite kinetic energy in the whole space

  flow region: $\Omega = \mathbb{R}^3$

- Steady solutions with finite kinetic energy when the body translates and rotates

Equations of motion in an inertial frame

$$\left.\begin{aligned} &\partial_t u + u \cdot \nabla u = \nabla \cdot \mathcal{T}(u,\pi) + g \\ &\nabla \cdot u = 0 \end{aligned}\right\} \text{ in } \cup_{t \in \mathbb{R}} \tilde{\Omega}(t) \times \{t\}$$

$$u(y,t) = U(y,t) + u_*(y,t) \text{ on } \cup_{t \in \mathbb{R}} \tilde{\Sigma}(t) \times \{t\}$$

$$\lim_{|y| \to \infty} u(y,t) = 0,\ t \in \mathbb{R}$$

$\tilde{\mathcal{R}}(t)$ is the region occupied by the rigid body $\mathcal{R}$

$\tilde{\Omega}(t) = \mathbb{R}^3 \setminus \tilde{\mathcal{R}}(t)$ is the exterior domain occupied by the fluid

$$\mathcal{T}(u,\pi) := 2\nu D(u) - \pi Id = \nu(\nabla u + (\nabla u)^T) - \pi Id$$

$$\nabla \cdot \mathcal{T}(u,\pi) = \nu \Delta u - \nabla \pi$$

$$U(y,t) := \eta(t) + \varpi(t) \times (y - y_C(t))$$

Equations in a frame attached to $\mathcal{R}$

$$\left.\begin{aligned} &\partial_t v = \nabla \cdot T(v,p) + (V - v) \cdot \nabla v - \omega \times v + f \\ &\nabla \cdot v = 0 \end{aligned}\right\} \text{ in } \Omega \times \mathbb{R}$$

$$v = v_* + V \text{ on } \Sigma \times \mathbb{R}$$

$$\lim_{|x| \to \infty} v(x,t) = 0 \text{ in } \mathbb{R}$$

$\Omega = \mathbb{R}^3 \setminus \mathcal{R}$ is time independent

$$V(x,t) = \zeta(t) + \omega(t) \times x$$

$$(V \cdot \nabla v)(x,t) = \zeta(t) \cdot \nabla v(x,t) + \omega(t) \times x \cdot \nabla v(x,t)$$

Steady states (in the frame attached to $\mathcal{R}$)

<u>Problem</u>: given $V = \zeta + \omega \times x$, v_* and f, find v and p such that

$$\left.\begin{aligned} &\nabla \cdot T(v,p) = v \cdot \nabla v - \zeta \cdot \nabla v - \omega \times x \cdot \nabla v + \omega \times v + f \\ &\nabla \cdot v = 0 \end{aligned}\right\} \text{ in } \Omega$$

$$v = v_* + V \text{ on } \Sigma$$

$$\lim_{|x| \to \infty} v(x) = 0$$

Assume $\partial\Omega$ is of class C^2. If $f \in D_0^{-1,2}(\Omega) \cap L^2(\Omega)$ and $v_* \in W^{3/2,2}(\partial\Omega)$ the problem has a solution

$$v \in D_0^{1,2}(\Omega) \cap D^{2,2}(\Omega), \quad p \in D^{1,2}(\Omega) .$$

In general, $v \notin L^2(\Omega)$.

Steady solutions with finite kinetic energy

Our aim is to find a solution to

$$\left.\begin{aligned} &\nabla \cdot T(v,p) = v \cdot \nabla v - \zeta \cdot \nabla v - \omega \times x \cdot \nabla v + \omega \times v + f \\ &\nabla \cdot v = 0 \end{aligned}\right\} \text{ in } \Omega$$

$$v = v_* + V \text{ on } \Sigma$$

$$\lim_{|x|\to\infty} v(x) = 0$$

having finite kinetic energy, that is, $v \in L^2(\Omega)$.

Some references

1. Navier-Stokes equations around a "self-propelled body": Finn(1965), Galdi(1997), Pukhnacev (1989), S.(2002)
2. Navier-Stokes equations in the whole space with specific external forces: Bjorland and Schonbek (2009), S.(2009) $\rightarrow$ The idea is to solve the problem in the L^2-framework, avoiding potential theoretic methods.

When the body translates without spinning

If $\omega = 0$ the previous equations reduce to

$$\left.\begin{aligned} &\nabla \cdot T(v,p) = (v-\zeta)\cdot\nabla v + f \\ &\nabla \cdot v = 0 \end{aligned}\right\} \text{ in } \Omega$$

$$v = v_* + \zeta \text{ on } \Sigma$$

$$\lim_{|x|\to\infty} v(x) = 0$$

Assume f is of compact support. Then (Finn, Galdi)

$$\boxed{v \in L^2(\Omega)}$$

if and only if

$$\boxed{\int_\Sigma [T(v,p)\cdot n d\sigma(x) - (v_* + \zeta) v_* \cdot n d\sigma(x)] = \int_\Omega f dx}$$

When the fluid domain is the whole space

$$\left.\begin{array}{l}\nabla \cdot T(v,p) = v \cdot \nabla v - \zeta \cdot \nabla v - \omega \times x \cdot \nabla v + \omega \times v + F \\ \nabla \cdot v = 0\end{array}\right\} \text{ in } \mathbb{R}^3$$

$$\lim_{|x| \to \infty} v(x) = 0$$

When $\Omega = \mathbb{R}^3$ the previous compatibility condition reduces to a condition on the external force

$$\int_\Omega F = 0$$

We will see that, if $F \in L^2(\Omega)$ has compact support and null average (or satisfies a more general condition) and the data are sufficiently small, then the above Navier-Stokes system has a (unique) solution with finite kinetic energy

$$v \in L^2(\Omega)$$

A class of admissible forces

We denote by $\mathcal{A}$ the set of those functions $F \in L^2(\mathbb{R}^3) \cap D_0^{-1,2}(\mathbb{R}^3)$ that satisfy

$$\exists \alpha > 1/2,\ \varrho > 0 \text{ and } M > 0 :\ |\hat{F}(\xi)| \le M|\xi|^\alpha \text{ for a.a. } \xi \in B_\varrho.$$

- If $F \in L^2(\mathbb{R}^3) \cap D_0^{-1,2}(\mathbb{R}^3)$ and there exists $G \in L^1(\mathbb{R}^3)$ such that

$$F = \nabla \cdot G (= \frac{\partial G_{jk}}{\partial x_j} e_k)$$

  then $F \in \mathcal{A}$ with $\alpha = 1$, $M = \|G\|_1$ and any $\varrho > 0$.

  $\rightarrow$ The convective term can be written as $v \cdot \nabla v = \nabla \cdot (v \otimes v)$.

- If $F \in L^2(\mathbb{R}^3)$ has compact support and satisfies

$$\int_{\mathbb{R}^3} F(x) dx = 0.$$

  then $F \in \mathcal{A}$ with $\alpha = 1$, $M = 2\sqrt{\frac{\pi R(F)^5}{5}} \|F\|_2$, where $R(F)$ is such that $\mathrm{supp}(F) \subseteq B_{R(F)}$, and any $\varrho > 0$.

  $\rightarrow$ Recall that $\int_{\mathbb{R}^3} F(x) dx = 0 \Leftrightarrow \hat{F}(0) = 0$.

A linear problem in the whole space

Having in mind the application of a fixed point theorem, we first show

Theorem Let $F \in \mathcal{A}$. Then the problem

$$\left.\begin{aligned} &\nabla \cdot T(v,p) = -\zeta \cdot \nabla v + \omega \times v - \omega \times x \cdot \nabla v + F \\ &\nabla \cdot v = 0 \end{aligned}\right\} \text{ in } \mathbb{R}^3$$

$$\lim_{|x| \to \infty} v(x) = 0$$

has a solution $(v,p) \in W^{2,2}(\mathbb{R}^3) \times W^{1,2}(\mathbb{R}^3)$ satisfying the following estimates

$$\|v\|_{2,2} \leq \frac{1}{\nu} \left(\frac{2\sqrt{\pi} M \varrho^{\alpha - \frac{1}{2}}}{\sqrt{2\alpha - 1}} + \frac{1 + 2\varrho^2}{\varrho^2} \|F\|_2 + \|F\|_{-1,2} \right),$$

$$\|p\|_{1,2} \leq 2\|F\|_2 + \|F\|_{-1,2}.$$

Moreover, if $(v_1, p_1) \in W^{2,2}_{loc}(\mathbb{R}^3) \cap L^2(\mathbb{R}^3) \times W^{1,2}_{loc}(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$ is another solution to the problem then $(v,p) = (v_1, p_1)$.

Proof

- *Pressure*

$$\Delta p = -\nabla \cdot F \text{ in } \mathbb{R}^3,$$

$$\|p\|_2 \leq \|F\|_{-1,2}, \qquad |p|_{1,2} \leq 2\|F\|_2.$$

- *Velocity*

Let $\mathcal{F} := F + \nabla p$.

1. Using Galerkin method, we find $v \in D_0^{1,2}(\mathbb{R}^3)$ such that

$$\nu \int_{\mathbb{R}^3} \nabla v : \nabla \varphi dx = \int_{\mathbb{R}^3} \zeta \cdot \nabla v \cdot \varphi dx$$
$$+ \int_{\mathbb{R}^3} (\omega \times x \cdot \nabla v - \omega \times v) \cdot \varphi dx - \langle \mathcal{F}, \varphi \rangle, \ \forall \varphi \in C_0^\infty(\mathbb{R}^3).$$

2. Then we show that $\nabla \cdot v = 0$ in $\mathbb{R}^3$ and

$$|v|_{1,2} \leq \frac{1}{\nu} \|F\|_{-1,2}.$$

Proof

3. Applying the Fourier transform we get

$$-\nu|\xi|^2\hat{v}(\xi) = i\xi\cdot\zeta\hat{v}(\xi) + \omega\times\hat{v}(\xi) - \omega\times\xi\cdot\nabla\hat{v}(\xi) - i\xi\hat{p}(\xi) + \hat{F}(\xi),$$

$$\xi\cdot\hat{v}(\xi) = 0\,.$$

Since

$$\hat{\mathcal{F}}(\xi) = \hat{F}(\xi) - i\xi\hat{p}(\xi) = \hat{F}(\xi) - \frac{\xi}{|\xi|^2}\xi\cdot\hat{F}(\xi) = \left(\hat{F}(\xi)\times\frac{\xi}{|\xi|}\right)\times\frac{\xi}{|\xi|},$$

we can study the summability properties of the velocity field based on the following relation

$$-\nu|\xi|^2\hat{v}(\xi) = i\xi\cdot\zeta\hat{v}(\xi) + \omega\times\hat{v}(\xi) - \omega\times\xi\cdot\nabla\hat{v}(\xi) + \hat{\mathcal{F}}(\xi)\,.$$

From $\nabla v \in L^2(\mathbb{R}^3) \Longleftrightarrow |\xi|\hat{v} \in L^2(\mathbb{R}^3;\mathbb{C})$ it follows

$$\hat{v} \in L^2(B^\epsilon;\mathbb{C}),\ \forall\epsilon > 0,$$

$$|\xi|^2\hat{v},\ (\omega\times\hat{v} - \omega\times\xi\cdot\nabla\hat{v}) \in L^2(B_R;\mathbb{C}),\ \forall R > 0.$$

Proof

Since

$$\int_{\mathbb{R}^3} \nu^2 |\xi|^4 |\hat{v}(\xi)|^2 d\xi \leq \int_{\mathbb{R}^3} |\hat{F}(\xi)|^2 d\xi \,,$$

we conclude, by Plancherel Theorem, that $v \in D^{2,2}(\mathbb{R}^3)$ with

$$|v|_{2,2} \leq \frac{2}{\nu} \|F\|_2.$$

As a consequence, $\lim_{|x|\to\infty} v(x) = 0$ uniformly pointwise.

Consequently,

$$\omega \times \hat{v} - \omega \times \xi \cdot \nabla \hat{v} \in L^2(\mathbb{R}^3; \mathbb{C})$$

which implies that

$$\omega \times v - \omega \times x \cdot \nabla v \in L^2(\mathbb{R}^3).$$

Proof

Now we show that $v \in L^2(\mathbb{R}^3)$. We know that $\hat{v} \in L^2(B^\varrho;\mathbb{C})$, and we have

$$\nu^2|\hat{v}(\xi)|^2 + \nu\frac{1}{|\xi|^2}\omega \times \xi \cdot \nabla(|\hat{v}|^2)(\xi) \leq |\hat{F}(\xi)| \text{ in } B^\varrho\,.$$

Since

$$\int_{B_R^\varrho} \frac{1}{|\xi|^2}\omega \times \xi \cdot \nabla(|\hat{v}|^2)(\xi)d\xi = 0,\ \forall R > \varrho\,,$$

we get

$$\|\hat{v}\|_{L^2(B^\varrho;\mathbb{C})} \leq \frac{1}{\nu}\left\|\frac{\hat{F}}{|\xi|^2}\right\|_{L^2(B^\varrho;\mathbb{C})} \leq \frac{1}{\nu\varrho^2}\|F\|_2\,.$$

Proof

From the hypotheses on F

$$\nu^2|\hat{v}(\xi)|^2 + \nu\frac{1}{|\xi|^2}\omega \times \xi \cdot \nabla(|\hat{v}|^2)(\xi) \leq M^2|\xi|^{2\alpha-4} \text{ in } B_\varrho\,,$$

and

$$\int_{B_\varrho^\epsilon} \frac{1}{|\xi|^2}\omega \times \xi \cdot \nabla(|\hat{v}|^2)d\xi = 0 \quad (0 < \epsilon < \varrho).$$

Integrating in B_ϱ^ϵ and then $\epsilon \to 0$

$$\nu^2\|\hat{v}\|^2_{L^2(B_\varrho;\mathbb{C})} \leq 4\pi M^2\frac{\varrho^{2\alpha-1}}{2\alpha-1}.$$

Combining with the previous estimate allows to conclude that $v \in L^2(\mathbb{R}^3)$ and

$$\|v\|_2 = \|\hat{v}\|_{L^2(\mathbb{R}^3;\mathbb{C})} \leq \frac{1}{\nu}\left(2\sqrt{\pi}M\frac{\varrho^{\alpha-\frac{1}{2}}}{\sqrt{2\alpha-1}} + \frac{1}{\varrho^2}\|F\|_2\right).$$

The steady problem in the whole space

Theorem Let $F \in \mathcal{A}$ and let

$$\mathbb{F}_1 := \frac{2\sqrt{\pi} M \varrho^{\alpha - 1/2}}{\sqrt{2\alpha - 1}} + \frac{1 + 2\varrho^2}{\varrho^2} \|F\|_2 + \|F\|_{-1,2},$$

$$\mathbb{F}_2 := 2\|F\|_2 + \|F\|_{-1,2}.$$

There exists a positive absolute constant C_0 such that if $\mathbb{F}_1 < C_0 \nu^2$ then:

1. The non-linear problem has a solution $(v, p) \in W^{2,2}(\mathbb{R}^3) \times W^{1,2}(\mathbb{R}^3)$ that satisfies the estimates

$$\|v\|_{2,2} \le \frac{2\mathbb{F}_1}{\nu}, \qquad \|p\|_{1,2} \le C_1 \mathbb{F}_1 + \mathbb{F}_2,$$

   where C_1 is a positive constant independent of the data, and the energy equation

$$2\nu \|D(v)\|_2^2 = -\int_{\mathbb{R}^3} F \cdot v \, .$$

2. If $(v_1, p_1) \in W^{2,2}_{loc}(\mathbb{R}^3) \cap L^2(\mathbb{R}^3) \cap D^{1,2}(\mathbb{R}^3) \times W^{1,2}_{loc}(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$ is another solution then $(v, p) = (v_1, p_1)$.

Forces with compact support and null average

Theorem Let $F \in L^2(\mathbb{R}^3)$ have compact support and satisfy $\int_{\mathbb{R}^3} F(x)dx = 0$. Let $R(F)$ be such that $B_{R(F)} \supseteq$ *supp*(F). There exists a positive constant $C_0' = C_0'(R(F))$ such that if $\|F\|_2 < C_0'\nu^2$ then the following hold:

1. The non-linear problem has a solution $(v,p) \in W^{2,2}(\mathbb{R}^3) \times W^{1,2}(\mathbb{R}^3)$ that satisfes the estimates
$$\|v\|_{2,2} \leq \frac{C_1'}{\nu}\|F\|_2,$$
$$\|p\|_{1,2} \leq C_2'\|F\|_2,$$
for positive constants $C_j' = C_j'(R(F))$ ($j = 1, 2$), and the energy equation
$$2\nu\|D(v)\|_2^2 = -\int_{\mathbb{R}^3} F \cdot v\,.$$

2. If $(v_1, p_1) \in W^{2,2}_{loc}(\mathbb{R}^3) \cap L^2(\mathbb{R}^3) \cap D^{1,2}(\mathbb{R}^3) \times W^{1,2}_{loc}(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$ is another solution then $(v,p) = (v_1, p_1)$.

When the body translates and rotates

The existence of a solution with finite kinetic energy to

$$\left.\begin{aligned} &\nabla \cdot T(v,p) = v \cdot \nabla v - \zeta \cdot \nabla v - \omega \times x \cdot \nabla v + \omega \times v + f \\ &\nabla \cdot v = 0 \end{aligned}\right\} \text{ in } \Omega$$

$$v = v_* + V \text{ on } \Sigma$$

$$\lim_{|x|\to\infty} v(x) = 0$$

requires a compatibility condition involving the data and the solution itself. In order to fulfill that compatibility condition, we also have to introduce an additional unknown in the equations. We will assume that $v_* = \lambda \in \mathbb{R}^3$ is not given.

We consider the extended functions $\tilde{v} : \mathbb{R}^3 \to \mathbb{R}^3$ and $\tilde{p} : \mathbb{R}^3 \to \mathbb{R}$

$$\tilde{v} = \begin{cases} V + \lambda & \text{in } \mathcal{R} \\ v & \text{in } \Omega \end{cases} \qquad \tilde{p} = \begin{cases} 0 & \text{in } \mathcal{R} \\ p & \text{in } \Omega \end{cases}$$

The whole space problem for $\tilde{v}$ and $\tilde{p}$

$$\left.\begin{aligned} &\nabla\cdot T(\tilde{v},\tilde{p}) = \tilde{v}\cdot\nabla\tilde{v} - \zeta\cdot\nabla\tilde{v} - \omega\times x\cdot\nabla\tilde{v} + \omega\times\tilde{v} \\ &\qquad\qquad + f - T(v,p)\cdot n\delta_{\partial\Omega} - \omega\times(V+2\lambda)\delta_{\mathcal{R}} \\ &\nabla\cdot\tilde{v} = 0 \end{aligned}\right\} \text{ in } \mathbb{R}^3$$

$$\lim_{|x|\to\infty}\tilde{v}(x) = 0$$

$$F := f - T(\tilde{v},\tilde{p})\cdot n\delta_{\partial\Omega} - \omega\times(V+2\lambda)\delta_{\mathcal{R}} \in \mathcal{E}'(\mathbb{R}^3)$$

Comparing with the previous whole space problem, we obtain the following condition on F

$$\hat{F}(0) = 0 \Longleftrightarrow \int_\Omega f\,dx - \int_{\partial\Omega} T(v,p)\cdot n\,d\sigma(x) - \int_{\mathcal{R}} \omega\times(V+2\lambda)\,dx = 0$$

$$\Longleftrightarrow \int_{\partial\Omega} T(v,p)\cdot n\,d\sigma(x) + |\mathcal{R}|\omega\times(\zeta+2\lambda) = \int_\Omega f\,dx$$

Formulation of the problem

Compatibility condition:

$$\int_{\partial\Omega} T(v,p)\cdot n\, d\sigma(x) + |\mathcal{R}|\omega \times (\zeta + 2\lambda) = \int_{\Omega} f\, dx$$

Problem: Given V and f, find v, p and λ such that

$$\left.\begin{aligned} &\nabla \cdot T(v,p) = v\cdot\nabla v - \zeta\cdot\nabla v - \omega\times x\cdot\nabla v + \omega\times v + f \\ &\nabla\cdot v = 0 \end{aligned}\right\} \text{ in } \Omega$$

$v = \lambda + V$ on Σ

$$\lim_{|x|\to\infty} v(x) = 0$$

$$\int_{\partial\Omega} T(v,p)\cdot n\, d\sigma(x) + |\mathcal{R}|\omega \times (\zeta + 2\lambda) = \int_{\Omega} f\, dx$$

The main result

Theorem Assume $\partial\Omega$ is of class C^2. Let $\zeta,\ \omega \in \mathbb{R}^3$ and $f \in L^2(\mathbb{R}^3)$ with compact support. There exists a positive constant $C_0 = C_0(\partial\Omega, \nu)$ such that if $|\zeta| + |\omega| + \|f\|_2 < C_0$ then the following hold:

1. The non-linear problem has a solution $(v, p, \lambda) \in W^{2,2}(\Omega) \times W^{1,2}(\Omega) \times \mathbb{R}^3$ that satisfes the energy equation

$$2\nu\|D(v)\|_2^2 + \frac{1}{2}\int_{\partial\Omega} |V + \lambda|^2 V \cdot n - \int_{\partial\Omega} (V + \lambda) \cdot T(v, p) \cdot n = -\int_{\Omega} f \cdot v\,.$$

2. If $(v_1, p_1, \lambda_1) \in W^{2,2}_{loc}(\Omega) \cap L^2(\Omega) \cap D^{1,2}(\Omega) \times W^{1,2}_{loc}(\Omega) \cap L^2(\Omega) \times \mathbb{R}^3$ is another solution then $(v, p, \lambda) = (v_1, p_1, \lambda_1)$.