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Weak Neumann implies Stokes

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Outline

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Introduction
- The Helmholtz decomposition
- The Stokes operator

Main results

Proof of Main Results
- Localization
- Estimates in $\mathbb{R}^n_+$
- A solution formula

Outline

The Helmholtz projection

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- Let $1 < q < \infty$, $\Omega \subset \mathbb{R}^n$ be a domain.
- *Helmholtz decomposition* exists iff

$$L^q(\Omega)^n = L^q_\sigma(\Omega) \oplus G_q(\Omega),$$

where

$$G_q(\Omega) := \{g \in L^q(\Omega)^n : \exists h \in \widehat{W}^{1,q}(\Omega) \text{ such that } g = \nabla h\},$$
$$L^q_\sigma(\Omega) := \overline{\{\varphi \in C_c^\infty(\Omega)^n : \operatorname{div} \varphi = 0\}}^{L^q(\Omega)^n}.$$

In this case, there exists the *Helmholtz projection*

$$P_q : L^q(\Omega)^n \to L^q_\sigma(\Omega).$$

Existence of the Helmholtz projection

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- Let $1 < q < \infty$. Then the Helmholtz projection exists on $L^q(\Omega)^n$, where
  - $\Omega = \mathbb{R}^n$,
  - $\Omega = \mathbb{R}^n_+$,
  - Ω bounded with smooth boundary,
  - Ω exterior domain with smooth boundary,
  - Ω layer,
  - ...
- The Helmholtz projection exists $L^q(\Omega)^n \cap L^2(\Omega)^n$, $2 < q < \infty$, or $L^q(\Omega)^n + L^2(\Omega)^n$, $1 < q < 2$, Ω general unbounded domain (uniform C^1).

Contributors: Farwig, Fujiwara, Kozono, Miyakawa, Morimoto, Simader, Sohr, Thäter, von Wahl, Weyl, ...

Existence of the Helmholtz projection II

The Helmholtz projection exists on $L^q(\Omega)^n$, where

- $\Omega \subset \mathbb{R}^n$, bounded Lipschitz domain and $q \in (\frac{3}{2} - \varepsilon, 3 + \varepsilon)$, Fabes, Mendez and Mitrea.
- $\Omega \subset \mathbb{R}^2$ 'unbounded wedge' (smooth and non smooth), q depends on angle, Bogovskii.

Remark

The results above are sharp.

The weak Neumann problem

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Consider

$$\begin{cases} \Delta v &= \nabla \cdot g \quad \text{in } \Omega, \\ n \cdot \nabla v &= n \cdot g \quad \text{on } \partial\Omega. \end{cases} \qquad (\mathrm{WNP_q})$$

The weak Neumann problem

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Consider

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Proposition

(WNP_q) is uniquely solvable $\Leftrightarrow$ P_q exists.

Here: (WNP_q) is uniquely solvable $:\Leftrightarrow \forall g \in L^q(\Omega)^n \ \exists^1 v \in \widehat{W}^{1,q}(\Omega)$ s.t.

$$\int_\Omega \nabla v \nabla \varphi = \int_\Omega g \nabla \varphi, \quad \varphi \in \widehat{W}^{1,q'}(\Omega),$$

satisfying $\|v\|_{\widehat{W}^{1,q}(\Omega)} \leq C\|g\|_{L^q(\Omega)^n}$.

The Stokes operator

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Let $1 < q < \infty$ and $\Omega \subset \mathbb{R}^n$ be a domain such that the Helmholtz projection exists. Set

$$D(A_q) = W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega) \cap L_\sigma^q(\Omega)$$

and define the *Stokes operator*

$$A_q : \begin{cases} D(A_q) & \rightarrow & L_\sigma^q(\Omega), \\ u & \mapsto & P_q \Delta u. \end{cases}$$

Maximal Regularity

Let $I := [0, T]$. We say that A_q has *maximal L^p-regularity in $L^q_\sigma(\Omega)$* if for

$$f \in L^p(I; L^q_\sigma(\Omega))$$

there exists a unique

$$u \in W^{1,p}(I; L^q_\sigma(\Omega)) \cap L^p(I; D(A_q))$$

satisfying

$$\begin{aligned} u'(t) - A_q u(t) &= f(t), \quad t \in I, \\ u(0) &= 0. \end{aligned}$$

Known results on the Stokes operator

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- Let $1 < p, q < \infty$. Then A_q has maximal L^p-regularity in $L^q_\sigma(\Omega)$, where
  - $\Omega = \mathbb{R}^n$,
  - $\Omega = \mathbb{R}^n_+$,
  - Ω bounded with smooth boundary,
  - Ω exterior domain with smooth boundary,
  - Ω layer,
  - ...
- A_q has maximal L^p-regularity on $L^q_\sigma(\Omega) \cap L^2_\sigma(\Omega)$, $2 < q < \infty$, or $L^q\sigma(\Omega) + L^2_\sigma(\Omega)$, $1 < q < 2$, general unbounded domain (uniform C^2).

Contributors: Amann, Borchers, Desch, Farwig, Fujita, Fujiwara, Galdi, Giga, Grubb, Hieber, Hishida, Kato, Masuda, Miyakawa, Morimoto, Prüss, Shibata, Shimizu, Simader, Sohr, Solonnikov, Ukai, Varnhorn, Wiegner ...

Outline

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Main results

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Theorem

Assume that

- $\Omega \subset \mathbb{R}^n$ *has uniform* C^3*-boundary,*
- (WNP_q) *is uniquely solvable for some* $q \in (1, \infty)$*.*

Then the Stokes operator A_q *has maximal* L^p*-regularity in* $L^q_\sigma(\Omega)$ *for* $p \in (1, \infty)$*.*

Remark

In this case, A_q *generates an analytic semigroup on* $L^q_\sigma(\Omega)$*.*

The Navier-Stokes equations

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Proposition

Assume that

- $\Omega \subset \mathbb{R}^n$ *has uniform* C^3*-boundary,*
- (WNP_q) *is uniquely solvable for some* $q \in (1, \infty)$*.*

Then,

$$\|e^{tA_q} P_q f\|_{L^r(\Omega)} \le C t^{-\frac{n}{2}(\frac{1}{s}-\frac{1}{r})} \|f\|_{L^s_\sigma(\Omega)}, \quad \frac{1}{q} - \frac{2}{n} \le \frac{1}{r} \le \frac{1}{q} \le \frac{1}{s} \le \frac{1}{q} + \frac{2}{n}, \quad 0 < t < T,$$

$$\|\nabla e^{tA_q} P_q f\|_{L^r(\Omega)} \le C t^{-\frac{n}{2}(\frac{1}{s}-\frac{1}{r})-\frac{1}{2}} \|f\|_{L^s_\sigma(\Omega)}, \frac{1}{q} - \frac{1}{n} \le \frac{1}{r} \le \frac{1}{q} \le \frac{1}{s} \le \frac{1}{q} + \frac{1}{n}, \quad 0 < t < T,$$

$$\|e^{tA_q} P_q \operatorname{div} f\|_{L^r(\Omega)} \le C t^{-\frac{n}{2}(\frac{1}{s}-\frac{1}{r})-\frac{1}{2}} \|f\|_{L^s_\sigma(\Omega)}, \frac{1}{q} - \frac{1}{n} \le \frac{1}{r} \le \frac{1}{q} \le \frac{1}{s} \le \frac{1}{q} + \frac{1}{n}, \quad 0 < t < T.$$

The Navier-Stokes equations

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Consider

$$u(t) - A_q u(t) + P_q(u(t)\cdot\nabla)u(t) = 0 \quad t \in (0,T), \qquad u(0) = u_0. \tag{1}$$

Theorem

Assume that

- $\Omega \subset \mathbb{R}^n$ *has uniform* C^3*-boundary,*
- (WNP_q) *is uniquely solvable for some* $q > n$*,*
- $u_0 \in L^q_\sigma(\Omega)$*.*

Then $\exists T_0 > 0$ *and a unique mild solution* u *of* (1)*, i.e.* $u \in C([0,T); L^q_\sigma(\Omega))$ *and*

$$u(t) = e^{tA_q}u_0 - \int_0^t e^{(t-s)A_q} P_q \operatorname{div}\,(u(s)\otimes u(s))ds, \qquad 0 \le t < T.$$

Outline

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Idea of Proof

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Set

$$\tilde{u} := \sum_{j=1}^{N} \varphi_j u_j, \quad \tilde{\pi} = \sum_{j=1}^{N} \varphi_j \pi_j,$$

where φ_j are cut-off functions and (u_j, π_j) is the push-forward of the solution $(\hat{u}_j, \hat{\pi}_j)$ to

$$\begin{aligned} \lambda \hat{u}_j - \Delta \hat{u}_j + \nabla \hat{\pi}_j &= \hat{f}_j \quad \text{in } \mathbb{R}^n_+, \\ \operatorname{div} \hat{u}_j &= 0 \quad \text{in } \mathbb{R}^n_+, \\ \hat{u} &= 0 \quad \text{on } \partial\mathbb{R}^n_+, \end{aligned}$$

with a suitable right hand side $\hat{f}_j$.

Idea of Proof

Set

$$\tilde{u} := \sum_{j=1}^{N} \varphi_j u_j, \quad \tilde{\pi} = \sum_{j=1}^{N} \varphi_j \pi_j,$$

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with a suitable right hand side $\hat{f}_j$.

Problem:

$$\nabla \cdot \tilde{u} = \sum_j (\nabla \varphi_j) u_j \neq 0 \text{ in } \Omega.$$

Correction term: Bogovskiĭ operator

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For $g \in L^p(\Omega_0)$ with $\int_\Omega g = 0$ let $v := \mathrm{B}_{\Omega_0} g$ denote the solution of

$$\text{(DIV)} \qquad \begin{cases} \nabla \cdot v = g & \text{in } \Omega_0, \\ v = 0 & \text{on } \partial\Omega_0. \end{cases}$$

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Then,

- Regularity ✓
-

$$\begin{aligned} \nabla \cdot (g - \mathrm{B}_{\Omega_0} \nabla \cdot g) &= 0, && \text{in } \Omega_0, \\ \mathrm{B}_{\Omega_0} g &= 0, && \text{on } \partial\Omega_0. \end{aligned}$$

Correction term: Bogovskiĭ operator

For $g \in L^p(\Omega_0)$ with $\int_\Omega g = 0$ let $v := \mathrm{B}_{\Omega_0} g$ denote the solution of

$$\text{(DIV)} \qquad \begin{cases} \nabla \cdot v = g & \text{in } \Omega_0, \\ v = 0 & \text{on } \partial\Omega_0. \end{cases}$$

Then,

- Regularity ✓
- $$\nabla \cdot (g - \mathrm{B}_{\Omega_0}\, \nabla \cdot g) = 0, \qquad \text{in } \Omega_0,$$ $$\mathrm{B}_{\Omega_0}\, g = 0, \qquad \text{on } \partial\Omega_0.$$

Problem: Desired estimates for v known for *bounded* Lipschitz domains only.

Correction term: Weak Neumann Problem

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For $g \in L^p(\Omega)$ let $v \in \widehat{W}^{1,p}(\Omega)$ denote the weak solution of

$$\begin{cases} \Delta v & = \nabla \cdot g, \quad \text{in } \Omega, \\ n \cdot \nabla v & = g \cdot n, \quad \text{on } \partial\Omega. \end{cases}$$

Correction term: Weak Neumann Problem

For $g \in L^p(\Omega)$ let $v \in \widehat{W}^{1,p}(\Omega)$ denote the weak solution of

$$\begin{cases} \Delta v = \nabla \cdot g, & \text{in } \Omega, \\ n \cdot \nabla v = g \cdot n, & \text{on } \partial\Omega. \end{cases}$$

Then,

- Regularity ✓
-

$$\nabla \cdot (g - \nabla v) = \nabla \cdot g - \Delta v = 0.$$

Correction term: Weak Neumann Problem

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For $g \in L^p(\Omega)$ let $v \in \widehat{W}^{1,p}(\Omega)$ denote the weak solution of

$$\begin{cases} \Delta v &= \nabla \cdot g, \quad \text{in } \Omega, \\ n \cdot \nabla v &= g \cdot n, \quad \text{on } \partial\Omega. \end{cases}$$

Then,

- Regularity ✓
-

$$\nabla \cdot (g - \nabla v) = \nabla \cdot g - \Delta v = 0.$$

Problem:

$$g - \nabla v = 0 \text{ on } \partial\Omega?$$

Estimates in $\mathbb{R}^n_+$: The case $(f, 0)$

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For $\hat{f} \in L^q_\sigma(\mathbb{R}^n_+)$ let $\hat{u} = \widehat{U}^1_\lambda \hat{f}$ and $\hat{\pi} = \widehat{\Pi}^1_\lambda \hat{f}$ denote the solution of

$$\begin{aligned} \lambda\hat{u} - \Delta\hat{u} + \nabla\hat{\pi} &= \hat{f} \quad \text{in } \mathbb{R}^n_+, \\ \operatorname{div} \hat{u} &= 0 \quad \text{in } \mathbb{R}^n_+, \\ \hat{u} &= 0 \quad \text{on } \partial\mathbb{R}^n_+. \end{aligned}$$

Estimates in $\mathbb{R}^n_+$: The case $(f, 0)$

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Lemma

Let $q \in (1, \infty)$, $\alpha \in (0, \frac{1}{2p'})$ and $X_n > 0$.

$$\begin{aligned} \mathcal{R}_{L^q_\sigma(\mathbb{R}^n_+) \to W^{k,q}(\mathbb{R}^n_+)}\{\lambda^{1-k/2}\widehat{U}^1_\lambda \; ; \; \lambda \in 1 + \Sigma_\theta\} &\le C, \qquad k = 0, 1, 2, \\ \mathcal{R}_{L^q_\sigma(\mathbb{R}^n_+) \to L^q(\mathbb{R}^n_+)}\{\nabla\widehat{\Pi}^1_\lambda \; ; \; \lambda \in 1 + \Sigma_\theta\} &\le C, \\ \mathcal{R}_{L^q_\sigma(\mathbb{R}^n_+) \to L^q(\mathbb{R}^{n-1}\times(0,X_n))}\{\lambda^\alpha\widehat{\Pi}^1_\lambda \; ; \; \lambda \in 1 + \Sigma_\theta\} &\le C. \end{aligned}$$

Estimates in $\mathbb{R}^n_+$: The case $(0, a)$

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For $\hat{a} \in W^{2-2/q,q}(\mathbb{R}^{n-1})$, $a \cdot \nu = 0$ let $(\hat{u},\hat{\pi})$ denote the solution of

$$\begin{aligned} \lambda \hat{u} - \Delta \hat{u} + \nabla \hat{\pi} &= 0 \quad \text{in } \mathbb{R}^n_+, \\ \operatorname{div} \hat{u} &= 0 \quad \text{in } \mathbb{R}^n_+, \\ \hat{u} &= \hat{a} \quad \text{on } \partial\mathbb{R}^n_+. \end{aligned}$$

Note that

$$\|u\|_{L^q(\mathbb{R}^n_+)} \leq C|\lambda|^{-\frac{1}{2q}} \|a\|_{L^q(\mathbb{R}^{n-1})}.$$

Estimates in $\mathbb{R}^n_+$: The case $(0, a)$
Representation for the solution $(\hat{u}, \hat{\pi})$

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We have $\hat{u} = \widehat{U}^2_\lambda(a, a)$, $\hat{\pi} = \widehat{\Pi}^2_\lambda(a, a)$, where

$$\widehat{\Pi}^2_\lambda : \begin{cases} \widehat{X}_{a,b} & \to & \widehat{W}^{1,q}(\mathbb{R}^n_+) \\ (a, b) & \mapsto & -(\sqrt{\lambda - \Delta'} + \sqrt{-\Delta'}) e^{-\sqrt{-\Delta'} x_n} \frac{\nabla' \cdot p(a,b)}{\sqrt{-\Delta'}}, \end{cases}$$

$$\widehat{U}^2_\lambda : \begin{cases} \widehat{X}_{a,b} & \to & W^{2,q}(\mathbb{R}^n_+) \cap L^q_\sigma(\mathbb{R}^n_+) \\ (a, b) & \mapsto & \begin{pmatrix} (\lambda - \Delta_D)^{-1}(-\nabla' \widehat{\Pi}^2_\lambda(a, b)) \\ (\lambda - \Delta_N)^{-1}(-\partial_n \Pi^2_\lambda(a, b)) \end{pmatrix} + e^{-\sqrt{\lambda - \Delta'} x_n} \begin{pmatrix} p(a, b) \\ \frac{\nabla' \cdot p(a,b)}{\sqrt{\lambda - \Delta'}} \end{pmatrix} \end{cases}$$

Here,

$$\widehat{X}_{a,b} := \{a \in W^{1-1/q,q}(\partial\mathbb{R}^n_+)^n : a \cdot \nu = 0\} \times \{b \in W^{2-1/q,q}(\partial\mathbb{R}^n_+) : b \cdot \nu = 0\},$$

$$p(a, b) := \left(\frac{\lambda}{\lambda - \Delta'} a - \frac{\Delta'}{\lambda - \Delta'} b \right)$$

Estimates in $\mathbb{R}^n_+$: The case $(0, a)$

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Lemma

For $\alpha \in (0, \frac{1}{2q'})$ *and* $X_n > 0$ *there exists* $C > 0$ *such that*

$$\mathcal{R}_{\widehat{X}_{a,b}\to L^q(\mathbb{R}^{n-1}\times(0,X_n))}\left\{\lambda^\alpha \widehat{\Pi}^2_\lambda \widetilde{K}^{-1}_\lambda : \lambda \in 1+\Sigma_\theta\right\} \le C,$$
$$\mathcal{R}_{\widehat{X}_{a,b}\to L^q(\mathbb{R}^n_+)}\left\{\nabla \widehat{\Pi}^2_\lambda \widetilde{K}^{-1}_\lambda : \lambda \in 1+\Sigma_\theta\right\} \le C,$$
$$\mathcal{R}_{\widehat{X}_{a,b}\to W^{k,q}(\mathbb{R}^n_+)}\left\{\lambda^{\frac{2-k}{2}} \widehat{U}^2_\lambda \widetilde{K}^{-1}_\lambda : \lambda \in 1+\Sigma_\theta\right\} \le C, \quad k = 0, 1, 2.$$

Here,

$$\widetilde{K}_\lambda := \begin{pmatrix} \lambda^{1-\frac{1}{3q}} & 0 \\ 0 & 1 \end{pmatrix},$$

Estimates in $\mathbb{R}^n_+$: The case (f, a)

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For $X_n > 0$ we set

$$\widehat{X} := L^q_\sigma(\mathbb{R}^n_+) \times \widehat{X}_{a,b},$$

$$\widehat{\Pi}_\lambda := \begin{cases} \widehat{X} & \to L^q(\mathbb{R}^{n-1} \times (0, X_n)) \cap \widehat{W}^{1,q}(\mathbb{R}^n_+) \\ (f, a, b) & \mapsto \widehat{\Pi}^1_\lambda(f) + \Pi^2_\lambda(a, b) \end{cases}$$

$$\widehat{U}_\lambda := \begin{cases} \widehat{X} & \to W^{2,q}(\mathbb{R}^n_+) \cap L^p_\sigma(\mathbb{R}^n_+) \\ (f, a, b) & \mapsto \widehat{U}^1_\lambda(f) + \widehat{U}^2_\lambda(a, b) \end{cases}$$

Estimates in $\mathbb{R}^n_+$: The case (f, a)

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Lemma

For $X_n > 0$ and $\alpha \in (0, \frac{1}{2p'})$ there exists $C > 0$ such that

$$\mathcal{R}_{\widehat{X} \to L^p(\mathbb{R}^{n-1} \times (0, X_n))} \left\{ \lambda^\alpha \widehat{\Pi}_\lambda K_\lambda^{-1} : \lambda \in 1 + \Sigma_\theta \right\} \le C,$$

$$\mathcal{R}_{\widehat{X} \to L^p(\mathbb{R}^n_+)} \left\{ \nabla \widehat{\Pi}_\lambda K_\lambda^{-1} : \lambda \in 1 + \Sigma_\theta \right\} \le C,$$

$$\mathcal{R}_{\widehat{X} \to W^{k,p}(\mathbb{R}^n_+)} \left\{ \lambda^{\frac{k-2}{2}} \widehat{U}_\lambda K_\lambda^{-1} : \lambda \in 1 + \Sigma_\theta \right\} \le C, \quad k = 0, 1, 2.$$

Here:

$$K_\lambda := \begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda^{1 - \frac{1}{3q}} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Pull-back of f

For

$$f \in L^q_\sigma(\Omega)$$

we define

$$f_j := \psi_j f - B_{\Omega_j}\Big((\nabla\psi_j)f\Big)$$
$$\widehat{f_j} := S^1_j f = \mathcal{G}_{\sigma,j} f_j$$

Then,

$$\|S_j f\|_{L^q(\mathbb{R}^n_+)^n} \leq C\|f\|_{L^q(\Omega_j)^n}.$$

Pull-back of (a, b)

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For

$$(a, b) \in X_{a,b} := \{a \in W^{1-1/q,q}(\partial\Omega)^n : a \cdot \nu = 0\} \times \\ \times \{b \in W^{2-1/q,q}(\partial\Omega)^n : b \cdot \nu = 0\},$$

we define

$$a_j := \Psi_j a, \quad \widehat{a_j} := \mathcal{G}_{j,\sigma} \Psi_j a$$
$$b_j := \Psi_j b, \quad \widehat{b_j} := \mathcal{G}_{j,\sigma} \Psi_j b.$$

$$S_j^2(a, b) := (\widehat{a_j}, \widehat{b_j}).$$

Ansatz

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$$U_\lambda(f, a, b) := \sum_{j \in \mathbb{N}} \varphi_j \mathcal{G}_{j,\sigma}^{-1} \widehat{U}_\lambda S_j(f, a, b) - \nabla N \left(\sum_{j \in \mathbb{N}} \varphi_j \mathcal{G}_{j,\sigma}^{-1} \widehat{U}_\lambda S_j(f, a, b) \right),$$

where N is the solution operator of the weak Neumann problem and $S_j(f, a, b) := (S_j^1 f, S_j^2(a, b))$.

Error terms

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Setting $u := U_\lambda(f, a, a)$, we calculate

$$\begin{aligned} \lambda u - P_\Omega \Delta u &= f + \mathcal{T}_\lambda^1(f, a, a) && \text{in } \Omega \\ \nabla \cdot u &= 0 && \text{in } \Omega \\ u &= a + \mathcal{T}_\lambda^2(f, a, a) && \text{in } \partial\Omega \end{aligned}$$

where

$$\begin{aligned} (\mathcal{T}_\lambda^1(f, a, b), \mathcal{T}_\lambda^2(f, a, b), \mathcal{T}_\lambda^3(f, a, b)) &= \mathcal{T}_\lambda(f, a, b) \\ &:= T_{1,\lambda}(f, a, b) + \cdots + T_{6,\lambda}(f, a, b). \end{aligned}$$

Error terms in Ω

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$$T_{1,\lambda}(f,a,b) := (P_\Omega \sum_{j=1}^{\infty} \varphi_j [\nabla, \mathcal{G}_j^{-1}] \widehat{\Pi}_\lambda S_j(f,a,b), 0, 0)$$

$$T_{2,\lambda}(f,a,b) := (P_\Omega \sum_{j=1}^{\infty} (\nabla \varphi_j) \mathcal{G}_j^{-1} \widehat{\Pi}_\lambda S_j(f,a,b), 0, 0)$$

$$T_{3,\lambda}(f,a,b) := -(P_\Omega \sum_{j=1}^{\infty} (\Delta \varphi_j) \mathcal{G}_{j,\sigma}^{-1} \widehat{U}_\lambda S_j(f,a,b),$$

$$T_{4,\lambda}(f,a,b) := -(2P_\Omega \sum_{j=1}^{\infty} (\nabla \varphi_j) \nabla \mathcal{G}_{j,\sigma}^{-1} \widehat{U}_\lambda S_j(f,a,b), 0, 0)$$

$$T_{5,\lambda}(f,a,b) := -(P_\Omega \sum_{j=1}^{\infty} \varphi_j [\Delta, \mathcal{G}_{j,\sigma}^{-1}] \widehat{U}_\lambda S_j(f,a,b), 0, 0),$$

Error terms on $\partial\Omega$

$$T_{6,\lambda}(f,a,b) := -\Big(0, \nabla N\left(\sum_{j\in\mathbb{N}} \varphi_j \mathcal{G}_{j,\sigma}^{-1} \widehat{U}_\lambda S_j(f,a,b)\right)|_{\partial\Omega},$$
$$\nabla N\left(\sum_{j\in\mathbb{N}} \varphi_j \mathcal{G}_{j,\sigma}^{-1} \widehat{U}_\lambda S_j(f,a,b)\right)|_{\partial\Omega}\Big).$$

A formal solution formula

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- Formally:

$$R(\lambda)f := U_\lambda(1+\mathcal{T}_\lambda)^{-1}(f,0,0) = U_\lambda \sum_{n\in\mathbb{N}_0} \mathcal{T}_\lambda^n(f,0,0).$$

A formal solution formula

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- Formally:

$$R(\lambda)f := U_\lambda(1+\mathcal{T}_\lambda)^{-1}(f,0,0) = U_\lambda \sum_{n\in\mathbb{N}_0} \mathcal{T}_\lambda^n(f,0,0).$$

- Write:

$$\begin{aligned} R(\lambda)f &= U_\lambda \sum_{n\in\mathbb{N}_0} \mathcal{T}_\lambda^n(f,0,0) = U_\lambda K_\lambda^{-1} \sum_{n\in\mathbb{N}_0} (K_\lambda \mathcal{T}_\lambda K_\lambda^{-1})^n K_\lambda(f,0,0) \\ &= U_\lambda K_\lambda^{-1} \sum_{n\in\mathbb{N}_0} (K_\lambda \mathcal{T}_\lambda K_\lambda^{-1})^n (f,0,0). \end{aligned}$$

A formal solution formula

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- Formally:

$$R(\lambda)f := U_\lambda(1+\mathcal{T}_\lambda)^{-1}(f,0,0) = U_\lambda \sum_{n\in\mathbb{N}_0} \mathcal{T}_\lambda^n (f,0,0).$$

- Write:

$$\begin{aligned} R(\lambda)f &= U_\lambda \sum_{n\in\mathbb{N}_0} \mathcal{T}_\lambda^n (f,0,0) = U_\lambda K_\lambda^{-1} \sum_{n\in\mathbb{N}_0} (K_\lambda \mathcal{T}_\lambda K_\lambda^{-1})^n K_\lambda (f,0,0) \\ &= U_\lambda K_\lambda^{-1} \sum_{n\in\mathbb{N}_0} (K_\lambda \mathcal{T}_\lambda K_\lambda^{-1})^n (f,0,0). \end{aligned}$$

- Show: $\exists \lambda_0 \in \mathbb{R}$ s.t.

$$\mathcal{R}_{L^p_\sigma(\Omega)}\{K_\lambda \mathcal{T}_\lambda K_\lambda^{-1} : \lambda \in \lambda_0 + \Sigma_\theta\} < 1,$$
$$\mathcal{R}_{L^p_\sigma(\Omega)}\{\lambda U_\lambda K_\lambda^{-1} : \lambda \in \lambda_0 + \Sigma_\theta\} \le C.$$

Estimates for $T_{6,\lambda}$

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- Recall:

$$T_{6,\lambda}^2 = \nabla N \left(\sum_{j\in\mathbb{N}} \varphi_j \mathcal{G}_{j,\sigma}^{-1} \widehat{U}_\lambda S_j(f, a, b) \right)|_{\partial\Omega}$$

Estimates for $T_{6,\lambda}$

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- Recall:

$$T_{6,\lambda}^2 = \nabla N \left(\sum_{j\in\mathbb{N}} \varphi_j \mathcal{G}_{j,\sigma}^{-1} \widehat{U}_\lambda S_j(f, a, b) \right)|_{\partial\Omega}$$

- Since $(\varphi_j)_{j\in\mathbb{N}}$ is a uniformly locally finite cover, we obtain

$$\begin{aligned}
&\mathcal{R}_{L^q(\Omega)\to W^{1-1/q,q}(\Omega)}\{\lambda^{1-\frac{1}{3q}} T_{6,\lambda}^2 : \lambda \in \lambda_0 + \Sigma_\theta\} \\
&\quad\le C\mathcal{R}_{L^q(\Omega)\to L^q(\Omega)}\{\lambda^{1-\frac{1}{3q}} T_{6,\lambda}^2 : \lambda \in \lambda_0 + \Sigma_\theta\} \\
&\quad\le C\mathcal{R}_{L^q(\mathbb{R}_+^n)\to L^q(\mathbb{R}_+^n)}\{\lambda^{1-\frac{1}{3q}} \widehat{U}_\lambda : \lambda \in \lambda_0 + \Sigma_\theta\} \\
&\quad\le C\lambda_0^{-\frac{1}{3q}} \mathcal{R}_{L^q(\mathbb{R}_+^n)\to L^q(\mathbb{R}_+^n)}\{\lambda \widehat{U}_\lambda : \lambda \in \lambda_0 + \Sigma_\theta\} \\
&\quad\le C\lambda_0^{-\frac{1}{3q}}.
\end{aligned}$$