Spectral Properties of the Stokes and Oseen Operator with Rotation Effect in L^q-spaces

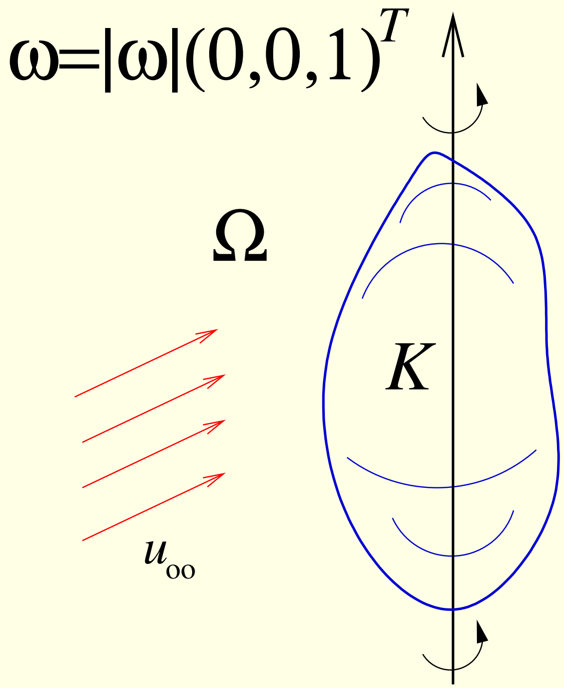
R. Farwig (TU Darmstadt) &
Š. Nečasová, J. Neustupa (Academy of Sciences, Prague)

International Workshop on Mathematical Fluid Dynamics
Waseda University, Tokyo, March 8-16, 2010

Spectral Properties of the Stokes and Oseen Operator with Rotation Effect in L^q-spaces

R. Farwig (TU Darmstadt) &
Š. Nečasová, J. Neustupa (Academy of Sciences, Prague)

International Workshop on Mathematical Fluid Dynamics
Waseda University, Tokyo, March 8-16, 2010



W.l.o.g. $|\omega| = 1$

$\Omega \rightsquigarrow \Omega(t) = O(t)\Omega$

$$O(t) = \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Navier-Stokes Equations

$$
\begin{aligned}
v_t - \nu \Delta v + v \cdot \nabla v + \nabla q &= f && \text{in } \Omega(t) \\
\operatorname{div} v &= 0 && \text{in } \Omega(t) \\
v &= \omega \wedge y && \text{on } \partial\Omega(t) \\
v &\to u_\infty && \text{at } \infty \\
v(0) &= a && \text{at } t = 0
\end{aligned}
$$

Main Problem: Time-dependent domain $\Omega(t)$

Global Coordinate Transformation

T. Hishida: $x = O^T(t)\, y,\ u(x,t) = O^T(t)(v(y,t) - u_\infty)$

$\Rightarrow$ modified Navier-Stokes system

$$
\begin{aligned}
u_t - \nu\Delta u + u\cdot\nabla u - (O^T(t)u_\infty)\cdot\nabla u - & \\
(\omega\wedge x)\cdot\nabla u + \omega\wedge u + \nabla p &= f \quad \text{in} \quad \Omega\times(0,\infty) \\
\operatorname{div} u &= 0 \quad \text{in} \quad \Omega\times(0,\infty) \\
u &\to 0 \quad \text{at} \quad \infty
\end{aligned}
$$

References

Hishida: Semigroup theory in L^2_σ, *non-analytic* C^0–semigroup

Geissert, Heck, Hieber L^q–semigroup theory

Galdi, Galdi-Silvestre, Galdi-Kyed: Strong L^2–solutions, stability, PR-solutions, decay estimates, wake behaviour

Hishida-Shibata: Stability, Oseen case, Oseen semigroup

Hansel: Oseen case when u_∞ not parallel to ω

F.– Hishida – D. Müller: $\boldsymbol{L^q}$–estimates, stationary case, $\boldsymbol{u_\infty = 0}$

F.–Neustupa 2007: Spectrum in L^2 (Stokes and Oseen)

F.–Nečasová–Neustupa 2009: Spectrum in L^q (Stokes and Oseen)

The Spectral Problem I

Linearize, replace u_t by λu to get the spectral problem on Ω:

$$\begin{aligned} \lambda u - \Delta u + k\partial_3 u - (\omega \wedge x) \cdot \nabla u + \omega \wedge u + \nabla p &= f \\ \operatorname{div} u &= 0 \\ u &= 0 \text{ on } \partial\Omega \end{aligned}$$

The Spectral Problem I

Linearize, replace u_t by λu to get the spectral problem on Ω:

$$\begin{aligned} \lambda u - \Delta u + k\partial_3 u - (\omega \wedge x) \cdot \nabla u + \omega \wedge u + \nabla p &= f \\ \operatorname{div} u &= 0 \\ u &= 0 \text{ on } \partial\Omega \end{aligned}$$

Apply Helmholtz projection P on L^q, let $A_\omega = A_{q,\omega}$ be defined by

$$\mathcal{D}(A_\omega) = \{u \in W^{2,q} \cap W_0^{1,q} \cap L_\sigma^q : (\omega \wedge x) \cdot \nabla u \in L^q\},$$

$$A_\omega u = P\big(- \Delta u + k\partial_3 u - (\omega \wedge x) \cdot \nabla u + \omega \wedge u\big)$$

The Spectral Problem II

For $f \in L^q_\sigma(\Omega)$ and $\lambda \in \mathbb{C}$ consider the resolvent problem

$$\begin{aligned} \lambda u + A_\omega u &= f \\ \operatorname{div} u &= 0 \\ u &= 0 \text{ on } \partial\Omega \end{aligned}$$

The Spectral Problem II

For $f \in L^q_\sigma(\Omega)$ and $\lambda \in \mathbb{C}$ consider the resolvent problem

$$\begin{aligned} \lambda u + A_\omega u &= f \\ \operatorname{div} u &= 0 \\ u &= 0 \ \text{ on } \ \partial\Omega \end{aligned}$$

Question 1 Determine $\sigma(-A_\omega)$ for all $1 < q < \infty$, $k = 0$ (Stokes case) and $k \neq 0$ (Oseen case)

The Spectral Problem II

For $f \in L^q_\sigma(\Omega)$ and $\lambda \in \mathbb{C}$ consider the resolvent problem

$$\begin{aligned} \lambda u + A_\omega u &= f \\ \operatorname{div} u &= 0 \\ u &= 0 \ \text{ on } \ \partial\Omega \end{aligned}$$

Question 1 Determine $\sigma(-A_\omega)$ for all $1 < q < \infty$, $k = 0$ (Stokes case) and $k \neq 0$ (Oseen case)

Question 2 Determine the type of $\sigma(-A_\omega)$ for all $1 < q < \infty$

The Spectral Problem II

For $f \in L^q_\sigma(\Omega)$ and $\lambda \in \mathbb{C}$ consider the resolvent problem

$$\begin{aligned} \lambda u + A_\omega u &= f \\ \operatorname{div} u &= 0 \\ u &= 0 \ \text{ on } \ \partial\Omega \end{aligned}$$

Question 1 Determine $\sigma(-A_\omega)$ for all $1 < q < \infty$, $k = 0$ (Stokes case) and $k \neq 0$ (Oseen case)

Question 2 Determine the type of $\sigma(-A_\omega)$ for all $1 < q < \infty$
Recall: $-A_\omega$ generates a C^0-semigroup which is not analytic!
($k = 0$: Hishida 1999, Geissert, Heck, Hieber 2006)

The Case $\mathbb{R}^3$

Use cylindrical coordinates in x-space and in Fourier space $\Rrightarrow$ explicit solution ($k = 0$)

$$\begin{aligned}\hat{u}(\xi) &= \int_0^\infty e^{-(\lambda+|\xi|^2)t}\,\hat{f}(O(t)\xi)\,dt \\ &= \frac{1}{D(\xi)}\int_0^{2\pi} e^{-(\lambda+|\xi|^2)t}\,\hat{f}(O(t)\xi)\,dt\,,\end{aligned}$$

where

$$D(\xi) = 1 - e^{-2\pi(\lambda+|\xi|^2)}$$

The Case $\mathbb{R}^3$

Use cylindrical coordinates in x-space and in Fourier space $\Rrightarrow$ explicit solution ($k = 0$)

$$\begin{aligned}\hat{u}(\xi) &= \int_0^\infty e^{-(\lambda+|\xi|^2)t}\,\hat{f}(O(t)\xi)\,dt \\ &= \frac{1}{D(\xi)}\int_0^{2\pi} e^{-(\lambda+|\xi|^2)t}\,\hat{f}(O(t)\xi)\,dt\,,\end{aligned}$$

where

$$D(\xi) = 1 - e^{-2\pi(\lambda+|\xi|^2)}$$

Note:

$$\begin{aligned}D(\xi) \neq 0\ \forall\xi \quad &\Leftrightarrow \quad \mathrm{Re}\,\lambda > 0 \ \text{ or } \ \mathrm{Re}\lambda \leq 0,\ \mathrm{Im}\,\lambda \notin \mathbb{Z} \\ &\Leftrightarrow \quad \lambda \notin \mathcal{H}_\omega = \bigcup_{k\in\mathbb{Z}}\Big((-\infty,0] - ik\Big)\end{aligned}$$

Lemma 1 (N-F 2007, N-N-F 2007)

Let $1 < q < \infty$

- $\lambda \notin \mathcal{H}_\omega \Rightarrow \lambda \in \rho(-A_\omega)$
- $(A_{q,\omega})^* = A_{q',-\omega}, \quad \mathcal{D}((A_{q,\omega})^*) = \mathcal{D}(A_{q',\omega})$
- $\sigma(-A_\omega) = \mathcal{H}_\omega$

Lemma 1 (N-F 2007, N-N-F 2007)
Let $1 < q < \infty$

- $\lambda \notin \mathcal{H}_\omega \Rightarrow \lambda \in \rho(-A_\omega)$

- $(A_{q,\omega})^* = A_{q',-\omega}, \quad \mathcal{D}((A_{q,\omega})^*) = \mathcal{D}(A_{q',\omega})$

- $\sigma(-A_\omega) = \mathcal{H}_\omega = \sigma_{ess}(-A_\omega)$

- $q = 2,\ \Omega = \mathbb{R}^3 \ \Rrightarrow\ \sigma(-A_\omega) = \mathcal{H}_\omega = \sigma_c(-A_\omega)$

Proof: First assertion: Multiplier theory for $\mathbb{R}^3$

Question: What type of spectrum do we have?

Prove for $\Omega = \mathbb{R}^3$ that

$$\sigma(-A_\omega) = \sigma_c(-A_\omega) = \mathcal{H}_\omega \quad \text{for all} \quad q \in (1, \infty)$$

Prove for $\Omega = \mathbb{R}^3$ that

$$\sigma(-A_\omega) = \sigma_c(-A_\omega) = \mathcal{H}_\omega \quad \text{for all} \quad q \in (1,\infty)$$

Spectra of Δ, A_0, A_ω in $L^q(\mathbb{R}^n)$

Theorem 1 (N-N-F 2009) Consider $\Omega = \mathbb{R}^n$, $1 < q < \infty$

- $\sigma(\Delta) = \sigma_{ess}(\Delta) = (-\infty, 0]$ and $0 \in \sigma_c(\Delta)$

Spectra of Δ, A_0, A_ω in $L^q(\mathbb{R}^n)$

Theorem 1 (N-N-F 2009) Consider $\Omega = \mathbb{R}^n$, $1 < q < \infty$

- $\sigma(\Delta) = \sigma_{ess}(\Delta) = (-\infty, 0]$ and $0 \in \sigma_c(\Delta)$
- for $\lambda \in \mathcal{H}_\omega$ the range $\mathcal{R}(\lambda - \Delta)$ is not closed

Spectra of Δ, A_0, A_ω in $L^q(\mathbb{R}^n)$

Theorem 1 (N-N-F 2009) Consider $\Omega = \mathbb{R}^n$, $1 < q < \infty$

- $\sigma(\Delta) = \sigma_{ess}(\Delta) = (-\infty, 0]$ and $0 \in \sigma_c(\Delta)$
- for $\lambda \in \mathcal{H}_\omega$ the range $\mathcal{R}(\lambda - \Delta)$ is not closed
- $(-\infty, 0) \subset \begin{cases} \sigma_r(\Delta), & 1 < q < \frac{2n}{n+1} \\ \sigma_c(\Delta), & \frac{2n}{n+1} \le q \le \frac{2n}{n-1} \quad (\frac{3}{2} \le q \le 3,\ n = 3) \\ \sigma_p(\Delta), & \frac{2n}{n-1} < q \end{cases}$

Spectra of Δ, A_0, A_ω in $L^q(\mathbb{R}^n)$

Theorem 1 (N-N-F 2009) Consider $\Omega = \mathbb{R}^n$, $1 < q < \infty$

- $\sigma(\Delta) = \sigma_{ess}(\Delta) = (-\infty, 0]$ and $0 \in \sigma_c(\Delta)$
- for $\lambda \in \mathcal{H}_\omega$ the range $\mathcal{R}(\lambda - \Delta)$ is not closed
- $(-\infty, 0) \subset \begin{cases} \sigma_r(\Delta), & 1 < q < \frac{2n}{n+1} \\ \sigma_c(\Delta), & \frac{2n}{n+1} \le q \le \frac{2n}{n-1} \quad (\frac{3}{2} \le q \le 3,\ n = 3) \\ \sigma_p(\Delta), & \frac{2n}{n-1} < q \end{cases}$
- $\lambda \in \sigma_p(\Delta) \Rrightarrow \mathrm{mult}(\lambda) = \infty$
- $\lambda \in \sigma_r(\Delta) \Rrightarrow \mathrm{codim}\mathcal{R}(\lambda - \Delta) = \infty$

Spectra of Δ, A_0, A_ω in $L^q(\mathbb{R}^n)$

Theorem 1 (N-N-F 2009) Consider $\Omega = \mathbb{R}^n$, $1 < q < \infty$

- $\sigma(\Delta) = \sigma_{ess}(\Delta) = (-\infty, 0]$ and $0 \in \sigma_c(\Delta)$
- for $\lambda \in \mathcal{H}_\omega$ the range $\mathcal{R}(\lambda - \Delta)$ is not closed
- $(-\infty, 0) \subset \begin{cases} \sigma_r(\Delta), & 1 < q < \frac{2n}{n+1} \\ \sigma_c(\Delta), & \frac{2n}{n+1} \le q \le \frac{2n}{n-1} \quad (\frac{3}{2} \le q \le 3,\ n = 3) \\ \sigma_p(\Delta), & \frac{2n}{n-1} < q \end{cases}$
- $\lambda \in \sigma_p(\Delta) \Rrightarrow \mathrm{mult}(\lambda) = \infty$
- $\lambda \in \sigma_r(\Delta) \Rrightarrow \mathrm{codim}\mathcal{R}(\lambda - \Delta) = \infty$
- Same result for the Stokes operator $-A_0$ $(\omega = 0)$ and for $-A_\omega$ (with the set $\mathcal{H}_\omega$ instead of $(-\infty, 0]$)

Spectrum in $L^q(\Omega)$

Theorem 2 (N-N-F 2009) Consider an exterior domain $\Omega \subset \mathbb{R}^3$, $1 < q < \infty$

- $\sigma_{ess}(-A_\omega) = \mathcal{H}_\omega$

- Ω axially symmetric $\Rrightarrow \sigma(-A_\omega) = \sigma_{ess}(-A_\omega) = \mathcal{H}_\omega$

Spectrum in $L^q(\Omega)$

Theorem 2 (N-N-F 2009) Consider an exterior domain $\Omega \subset \mathbb{R}^3$, $1 < q < \infty$

- $\sigma_{ess}(-A_\omega) = \mathcal{H}_\omega$
- Ω axially symmetric $\Rrightarrow \sigma(-A_\omega) = \sigma_{ess}(-A_\omega) = \mathcal{H}_\omega$
- Ω not axially symmetric $\Rrightarrow \sigma(-A_\omega) \setminus \mathcal{H}_\omega$ may contain isolated eigenvalues of finite multiplicities in the open left half plane
- Such eigenvalues, if they do exist, are independent of $q \in (1, \infty)$, their multiplicity is independent of q, and the corresponding eigenfunctions lie in $\bigcap_{1<q<\infty} \mathcal{D}(A_{q,\omega})$

First Ideas

- Reduce $\lambda \in \mathcal{H}_\omega$ with $\mathrm{Im}\,\lambda = \mathrm{k}$, $k \in \mathbb{Z}$, to $k = 0$:

$$(\lambda + A_\omega)(iR'_1 + R'_2)^k = (iR'_1 + R'_2)^k(\lambda + ik + A_\omega)$$

with the partial Riesz transforms $R'_1, R'_2 \Rrightarrow (iR'_1 + R'_2)^k \sim e^{-ik\varphi}$

First Ideas

- Reduce $\lambda \in \mathcal{H}_\omega$ with $\mathrm{Im}\,\lambda = \mathrm{k}$, $k \in \mathbb{Z}$, to $k = 0$:

$$(\lambda + A_\omega)(iR_1' + R_2')^k = (iR_1' + R_2')^k(\lambda + ik + A_\omega)$$

  with the partial Riesz transforms $R_1', R_2' \Rrightarrow (iR_1' + R_2')^k \sim e^{-ik\varphi}$

- If $1 < q \leq 2$ and $\lambda < 0$, then $\lambda \notin \sigma_p$. By analogy, $\lambda \notin \sigma_r$ for $q \geq 2$
  Proof: Assume $(\lambda + |\xi|^2)\hat{u} = 0$. Since $q \leq 2$, $\hat{u} \in L^{q'}(\mathbb{R}^n)$
  $\Rrightarrow \hat{u}(\xi) = 0$ a.e. $\Rrightarrow u = 0$.

First Ideas

- Reduce $\lambda \in \mathcal{H}_\omega$ with $\mathrm{Im}\,\lambda = \mathrm{k}$, $k \in \mathbb{Z}$, to $k = 0$:

$$(\lambda + A_\omega)(iR_1' + R_2')^k = (iR_1' + R_2')^k(\lambda + ik + A_\omega)$$

  with the partial Riesz transforms $R_1', R_2' \Rrightarrow (iR_1' + R_2')^k \sim e^{-ik\varphi}$

- If $1 < q \leq 2$ and $\lambda < 0$, then $\lambda \notin \sigma_p$. By analogy, $\lambda \notin \sigma_r$ for $q \geq 2$
  Proof: Assume $(\lambda + |\xi|^2)\hat{u} = 0$. Since $q \leq 2$, $\hat{u} \in L^{q'}(\mathbb{R}^n)$
  $\Rrightarrow \hat{u}(\xi) = 0$ a.e. $\Rrightarrow u = 0$.

- If $1 < q \leq 2$ and $\lambda < 0$, then $\lambda - \Delta$ is not surjective
  Proof: Assume $\lambda = -1$. Choose $\hat{f} \in C_0^\infty$ equal to 1 near the unit surface $|\xi|^2 = 1$ and let $(-1 - \Delta)u = f$
  $\Rrightarrow (-1 + |\xi|^2)\hat{u} = \hat{f}$ (in $L^{q'}$)
  $\Rrightarrow |\hat{u}(\xi)| \geq \frac{1}{2(1-|\xi|)}$ for $|\xi| \sim 1 \Rrightarrow \hat{u} \notin L^{q'}$

Eigenvalues

Let $\hat{j}_n = \chi_{\partial B_1(0)} \Rightarrow (-1 + |\xi|^2)\hat{j}_n = 0$, $(-1 - \Delta) j_n = 0 \Rightarrow$
$j_n(x) = c r^{(2-n)/2} J_{(n-2)/2}(r)$ with the Bessel function

$$J_\mu(x) = \sum_{m=0}^{\infty} (-1)^m \frac{(r/2)^{(\mu+m)}}{m!\,\Gamma(\mu + m + 1)}$$

Example: $n = 3 \Rightarrow j_3(x) = c\frac{\sin r}{r}$

$$-1 \text{ is eigenvalue } \Leftrightarrow j_n \in L^q(\mathbb{R}^n) \Leftrightarrow q > \frac{2n}{n-1}$$

Eigenvalues

Let $\hat{j}_n = \chi_{\partial B_1(0)} \Rrightarrow (-1 + |\xi|^2)\hat{j}_n = 0$, $(-1 - \Delta) j_n = 0 \Rrightarrow$
$j_n(x) = c r^{(2-n)/2} J_{(n-2)/2}(r)$ with the Bessel function

$$J_\mu(x) = \sum_{m=0}^{\infty} (-1)^m \frac{(r/2)^{(\mu+m)}}{m!\,\Gamma(\mu+m+1)}$$

Example: $n = 3 \Rrightarrow j_3(x) = c\frac{\sin r}{r}$

$$-1 \text{ is eigenvalue } \Leftrightarrow j_n \in L^q(\mathbb{R}^n) \Leftrightarrow q > \frac{2n}{n-1}$$

The functions $j_n, \partial_1 j_n, \ldots, \partial_1^k j_n$ are linearly independent eigenfunctions $\Rrightarrow \mathrm{mult}(-1) = \infty \Rrightarrow$

$$(-\infty, 0) = \sigma_p \text{ for } q > \frac{2n}{n-1} \quad \text{and} \quad (-\infty, 0) = \sigma_r \text{ for } 1 < q < \frac{2n}{n+1}$$

Continuous Spectrum

Assertion Let $\frac{2n}{n+1} \leq q \leq \frac{2n}{n-1}$. Then $-1 \in \sigma_c(\Delta)$

Continuous Spectrum

Assertion Let $\frac{2n}{n+1} \leq q \leq \frac{2n}{n-1}$. Then $-1 \in \sigma_c(\Delta)$

Proof Consider $f \in L^{q'}(\mathbb{R}^n)$ with

$$0 = \langle(-1-\Delta)u, f\rangle \quad \forall u \in \mathcal{D}(\Delta)$$

$\Rightarrow \operatorname{supp}\hat{f} \subset \partial B_1$

Show that $f = 0$

If $\hat{f} = c\chi_{\partial B_1}$, i.e., $f = cj_n \Rightarrow c = 0$ since $j_n \notin L^{q'}(\mathbb{R}^n)$ for $\frac{2n}{n+1} \leq q' \leq \frac{2n}{n-1}$.

Continuous Spectrum

Assertion Let $\frac{2n}{n+1} \leq q \leq \frac{2n}{n-1}$. Then $-1 \in \sigma_c(\Delta)$

Proof Consider $f \in L^{q'}(\mathbb{R}^n)$ with

$$0 = \langle(-1 - \Delta)u, f\rangle \quad \forall u \in \mathcal{D}(\Delta)$$

$\Rightarrow \operatorname{supp}\hat{f} \subset \partial B_1$

Show that $f = 0$

If $\hat{f} = c\chi_{\partial B_1}$, i.e., $f = cj_n \Rightarrow c = 0$ since $j_n \notin L^{q'}(\mathbb{R}^n)$ for $\frac{2n}{n+1} \leq q' \leq \frac{2n}{n-1}$.
However, f may not be radially symmetric (e.g. $f = \partial_1 j_n$)
(... take radial averages ... and ... Lebesgue's Differentiation Theorem to prove $f = 0$...)
$\Rightarrow \mathcal{R}(-1 - \Delta)$ is dense in $L^q \Rightarrow -1 \in \sigma_c(\Delta)$

Exterior Domains Ω

No explicit construction!

Note : $\lambda \in \sigma_{ess} \Leftrightarrow \mathrm{nul}'(\lambda + A_\omega) = \infty$ and

$$\mathrm{def}'(\lambda + A_\omega) := \mathrm{nul}'(\lambda + (A_\omega)') = \infty$$

Note : $\mathrm{nul}'(\lambda + A_\omega) = \infty \Leftrightarrow$

$$\exists\ (v_m) \subset \mathcal{D}(A_\omega) \text{ noncompact} : \|v_m\|_q = 1,\ \|(\lambda + A_\omega)v_m\|_q \to 0$$

Exterior Domains Ω

No explicit construction!
Note : $\lambda \in \sigma_{ess} \Leftrightarrow \mathrm{nul}'(\lambda + A_\omega) = \infty$ and

$$\mathrm{def}'(\lambda + A_\omega) := \mathrm{nul}'(\lambda + (A_\omega)') = \infty$$

Note : $\mathrm{nul}'(\lambda + A_\omega) = \infty \Leftrightarrow$

$$\exists\ (v_m) \subset \mathcal{D}(A_\omega) \text{ noncompact} : \|v_m\|_q = 1,\ \|(\lambda + A_\omega)v_m\|_q \to 0$$

By cut-off and Bogovskii's operator we transfer such a sequence from Ω to $\mathbb{R}^3$ and vice versa.
Analogous procedure for the condition $\mathrm{def}'(\lambda + A_\omega) = \infty$

Exterior Domains Ω

No explicit construction!

Note : $\lambda \in \sigma_{ess} \Leftrightarrow \text{nul}'(\lambda + A_\omega) = \infty$ and

$$\text{def}'(\lambda + A_\omega) := \text{nul}'(\lambda + (A_\omega)') = \infty$$

Note : $\text{nul}'(\lambda + A_\omega) = \infty \Leftrightarrow$

$$\exists\ (v_m) \subset \mathcal{D}(A_\omega) \text{ noncompact} : \|v_m\|_q = 1,\ \|(\lambda + A_\omega)v_m\|_q \to 0$$

By cut-off and Bogovskii's operator we transfer such a sequence from Ω to $\mathbb{R}^3$ and vice versa.

Analogous procedure for the condition $\text{def}'(\lambda + A_\omega) = \infty$

Theorem For $1 < q < \infty$ we get $\sigma_{ess}(A_\omega) = \mathcal{H}_\omega$

Non-/Existence of additional eigenvalues is open.

Oseen Case

Theorem (Neustupa-F. 2009) Let $1 < q < \infty$.

- Let $\Omega = \mathbb{R}^3$. Then $\sigma(-A_{\omega,k}) = \sigma_c(-A_{\omega,k})$ consists of an infinite set, $\mathcal{P}_{\omega,k}$, of parabola in the left half plane (replacing $(-\infty, 0] + ik,\ k \in \mathbb{Z}$)

- Let Ω be an exterior domain. Then $\sigma_{ess}(-A_{\omega,k}) = \mathcal{P}_{\omega,k}$

Idea of Proof $u \in L^q_\sigma(\mathbb{R}^3)$ be an eigenfunction for $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda < 0$. Then supp $\hat{u}$ is a union of finitely many circles in $\mathbb{R}^3$ parallel to the ξ_2, ξ_3−plane $\Rrightarrow$ ... $\Rrightarrow$ $u \notin L^q(\mathbb{R}^3)$

Thank you very much for your attention!