Criterion for stability of the stationary solution to Navier-Stokes equations in half-space

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International Workshop on Mathematical Fluid Dynamics, Waseda, March 10-13

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1-1. Problem

Perturbed Navier-Stokes equations

$$
\begin{cases}
\partial_t v - \Delta v + (u_s \cdot \nabla)v + (v \cdot \nabla)u_s + (v \cdot \nabla)v + \nabla\pi = 0 & \text{in } H, \\
\nabla \cdot v = 0 & \text{in } H, \\
v = 0 & \text{on } \partial H, \\
v(0,x) = v_0.
\end{cases}
$$

(PNS)

- $v = (v_1, \ldots, v_n)$: velocity field, π: pressure , [unknown]
- u_s ; solution to the stationary Navier-Stokes equations,
- $H = \{x = (x', x_n) \in \mathbb{R}^n \mid x_n > 0\}$.

1-2. Known Results (Stationary solutions)

$$-\Delta u_s + (u_s \cdot \nabla)u_s + \nabla \pi_s = f \qquad \text{in } \Omega,$$

For whole space case and exterior domain cases ($n \geq 3$)

When $f = \nabla \cdot F$ has the suitable decay rate for $|x| \gg 1$ and is sufficiently small, there exists a stationary solution satisfying

$$|u_s(x)| \leq C_F/|x|^{n-2}, \qquad |\nabla u_s| \leq C_F/|x|^{n-1} \qquad \text{for } |x| \gg 1.$$

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For half-space case $(n \geq 3)$

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For exterior domain cases ($n \geq 3$)

The stationary solution u_s is stable when v_0 and the following quantity is sufficiently small:

- Kozono-Ogawa(94) $\|u_s\|_n + \|\nabla u_s\|_{n/2}$,

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$$u_s \notin L^3(\Omega) = L^n(\Omega), \qquad \nabla u_s \notin L^{3/2}(\Omega) = L^{n/2}(\Omega)$$

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Goal of my talk

- In this talk, we assume that $(1 + x_n)u_s \in L^\infty(H)$.
- When $\|(1 + x_n)u_s\|_\infty$ is small enough for $n \geq 2$, the stationary solution u_s is stable.

2-1. Main result

Main result【Stability theorem for the stationary solution】

Let $n \geq 2$ and $v_0 \in J^n(H)$. Then there exist positive constant μ and δ such that if u_s and v_0 satisfy

$$\|(1+x_n)u_s\|_\infty \leq \mu, \quad \|v_0\|_n \leq \delta,$$

(PNS) admits a global strong solution v. Moreover the following asymptotic behaviors hold:

$$\begin{aligned} \|v(t)\|_p &= o(t^{-1/2+n/2p}) & n \leq p \leq \infty, \\ \|\nabla v(t)\|_p &= o(t^{-1+n/2p}) & n \leq p < \infty. \end{aligned}$$

as $t \to \infty$.

2-1. Remark on main result

Main result【Stability theorem for the stationary solution】

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$$\|(1+x_n)u_s\|_\infty \leq \mu, \quad \|v_0\|_n \leq \delta,$$

(PNS) admits a global strong solution v.

- Since the stationary solution u_s has decay properties:

  $$|u_s(x)| \leq C/|x|^{n-1}, \qquad |\nabla u_s| \leq C/|x|^n \qquad \text{for } |x| \gg 1,$$

  This solution satisfies this assumption.
- This case is the result corresponding to Borhcers-Miyakawa(95).
- For $n = 2$, the existence of stationary solutions satisfying this assumption is unknown.

3-1. Known results (Stokes operator)

Helmholtz decomposition

$$L^p(H) = J^p(H) \oplus G^p(H), \qquad\qquad 1 < p < \infty,$$

where

$$J^p(H) = \overline{\{u \mid u_j \in C_0^\infty(H),\ \nabla \cdot u = 0\}}^{\|\cdot\|_p},$$

$$G^p(H) = \{\nabla\pi \in L^p(H) \mid \pi \in L^p_{loc}(\overline{H})\}.$$

Analytic semigroup

$$P: \ L^p(H) \to J^p(H) \ \ \text{: a continuous projection}$$

The Stokes operator $A = -P\Delta$ is defined with dense domain

$$D(A) = \{u \in J^p(H) \cap W^{2,p}(H) \mid u|_{\partial H} = 0\}.$$

The Stokes operator $-A$ generates a bounded analytic semigroup $\{e^{-tA}\}_{t \geq 0}$ in $J^p(H)$

3-1. Known results (Stokes operator)

Some estimates for Stokes op.

Let $1 < p < \infty$.
(i) The following estimate holds.

$$\|\nabla^2 u\|_p \leq C\|Au\|_p, \qquad u \in D(A).$$

(ii) $D(A^{1/2}) = J^p(H) \cap W_0^{1,p}(H)$ and we have in particular

$$\|\nabla u\|_p \leq C\|A^{1/2}u\|_p, \qquad u \in D(A^{1/2})$$

(iii) If $u \in D(A^\alpha)$, $0 < \alpha < 1$ and if $0 < 1/q = 1/p - 2\alpha/n < 1$, then $u \in L^q$ and we have the estimate

$$\|u\|_q \leq C\|A^\alpha u\|_p, \qquad u \in D(A^\alpha)$$

3-2. Outline of the proof of Main theorem

$$\partial_t v - \Delta v + (u_s \cdot \nabla)v + (v \cdot \nabla)u_s + (v \cdot \nabla)v + \nabla\pi = 0 \qquad \text{in } H,$$

Let $1 < p < \infty$.
Let P be a Helmholtz projection from $L^p(H) \to J^p(H)$,

- $Av = -P\Delta v \quad v \in D(A) := W^{2,p} \cap W_0^{1,p} \cap J^p(H)$,

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- $Av = -P\Delta v \quad v \in D(A) := W^{2,p} \cap W_0^{1,p} \cap J^p(H)$,
- $Bv = P[(u_s \cdot \nabla)v + (v \cdot \nabla)u_s] \quad v \in D(B) := W_0^{1,p} \cap J^p(H)$,

3-2. Outline of the proof of Main theorem

$$\partial_t v - \Delta v + (u_s \cdot \nabla)v + (v \cdot \nabla)u_s + (v \cdot \nabla)v + \nabla\pi = 0 \qquad \text{in } H,$$

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- $L = A + B \quad D(L) = D(A)$.

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$$\partial_t v - \Delta v + (u_s \cdot \nabla)v + (v \cdot \nabla)u_s + (v \cdot \nabla)v + \nabla \pi = 0 \qquad \text{in } H,$$

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Applying Helmholtz projection P, we see

$$\partial_t v + Lv + P(v \cdot \nabla)v = 0$$

By Duhamel's principle, we can rewrite into the integral form:

$$v(t) = e^{-tL}v_0 - \int_0^t e^{-(t-s)L} P[(v \cdot \nabla)v](s)ds.$$

3-2. Outline of proof of Main Theorem

$$v(t) = e^{-tL}v_0 - \int_0^t e^{-(t-s)L} P[(v \cdot \nabla)v](s)ds.$$

By the contraction mapping principle and Key estimate (1), we can prove Main Theorem.

Key estimate(1) : $L^p - L^q$ estimate of the semigroup e^{-tL}

Let $n \geq 2$ and the stationary solution u_s satisfy Assumption. Then the following estimates hold: for $f \in J^p(H)$

$$\|e^{-tL}f\|_q \leq C_{p,q}\, t^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}\|f\|_p \qquad 1 < p \leq q \leq \infty,\ p \neq \infty,$$

$$\|\nabla e^{-tL}f\|_q \leq C_{p,q}\, t^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})-\frac{1}{2}}\|f\|_p \qquad 1 < p \leq q < \infty.$$

3-2. Outline of proof of Main Theorem

$$v(t) = e^{-tL}v_0 - \int_0^t e^{-(t-s)L}P[(v \cdot \nabla)v](s)ds.$$

If we set

$$H(u, v) = \int_0^t e^{-(t-s)L}P((u \cdot \nabla)v)(s)ds,$$
$$\Phi(v)(t) = e^{-tL}v_0 - H(v, v)(t),$$

then the integral equation is written in the form

$$v(t) = \Phi(v)(t).$$

To find the solution to the integral equation, we have to show that Φ is a contraction mapping of $\mathcal{I}_\varepsilon$ with suitable choice of ε and δ.

3-2. Outline of proof of Main Theorem

We set

$$\mathcal{I}_\varepsilon = \{v \in BC([0,\infty); J^n(H)) \mid |||v|||_t \le \varepsilon, \\ \lim_{t\to+0} \big([v(\cdot) - v_0]_{n,0,t} + [v]_{p,\mu(p),t} + [v]_{\infty,1/2,t} + [\nabla v]_{n,1/2,t} + [\nabla v]_{p,\mu'(p),t}\big) \\ = 0\}$$

with p a fixed number in (n,∞), ε is a small positive number determined later, and where

$$[v]_{p,\ell,t} = \sup_{0<s\le t} s^\ell \|u(\cdot,s)\|_{L^p},$$

$$|||v|||_t = [v(\cdot)]_{n,0,t} + [v]_{p,\mu(p),t} + [v]_{\infty,1/2,t} + [\nabla v]_{n,1/2,t} + [\nabla v]_{p,\mu'(p),t},$$

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$$\mu(p) = 1/2 - n/2p, \qquad \mu'(p) = 1 - n/2p.$$

To show that Φ is a contraction mapping of $\mathcal{I}_\varepsilon$ with suitable choice of ε and δ, we check these conditions.

3-2. Outline of proof of Main Theorem

$L^p - L^q$ estimates:

$$\|\nabla^k e^{-tL} f\|_q \le C_{p,q}\, t^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})-\frac{k}{2}} \|f\|_p, \qquad k = 0, 1.$$

we can prove some estimates for the nonlinear term $H(u, v)$: for example,

$$[H(u, v)]_{r,\mu(r),t} \le C[u]_{p,\mu(p),t}[\nabla v]_{n,1/2,t}, \qquad n \le r \le \infty.$$

nonlinear term:

$$H(u, v) = \int_0^t e^{-(t-s)L} P((u \cdot \nabla)v)(s)ds,$$

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nonlinear term:

$$H(u, v) = \int_0^t e^{-(t-s)L} P((u \cdot \nabla)v)(s)ds,$$

By standard argument (Kato's method), we can prove Φ is a contraction mapping of $\mathcal{I}_\varepsilon$.

3-3. Outline of the proof of key estimate (1)

Therefore we need to show that

- the operator $-L$ generates a bounded analytic semigroup $\{e^{-tL}\}_{t\geq 0}$,
- the semigroup e^{-tL} has the $L^p - L^q$ estimates:

$$\|\nabla^k e^{-tL} f\|_q \leq C_{p,q}\, t^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})-\frac{k}{2}}\|f\|_p, \qquad k = 0, 1.$$

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Key estimate (2) : resolvent estimate

Let $1 < p < \infty$. There exists a positive number μ such that if $\|(1 + x_n)u_s\|_\infty \leq \mu$, the following resolvent estimate holds:

$$\|(\lambda + L)^{-1}\|_{\mathcal{L}(J^p)} \leq \frac{C}{|\lambda|},$$

where $\lambda \in \Sigma_\varepsilon = \{\lambda \in \mathbb{C}\backslash\{0\} \mid |\arg \lambda| < \pi - \varepsilon\}$ $(\varepsilon \in (0, \pi/2))$.

3-3. Outline of proof of key estimate (1)

$$\|(\lambda + L)^{-1}\|_{\mathcal{L}(J^p)} \leq \frac{C}{|\lambda|},$$

By using the resolvent estimate and the representation formula of semigroup:

$$e^{-tL}f = \frac{1}{2\pi i}\int_\Gamma e^{t\lambda}(\lambda + L)^{-1} f d\lambda,$$

we can obtain

$$\|e^{-tL}f\|_p \leq C_p \ \|f\|_p,$$

for $1 < p < \infty$.

3-3. Outline of proof of key estimate (1)

$$\|\nabla(\lambda + L)^{-1}\|_{\mathcal{L}(J^p)} \leq \frac{C}{|\lambda|^{1/2}},$$

By using the resolvent estimate and the representation formula of semigroup:

$$e^{-tL}f = \frac{1}{2\pi i}\int_\Gamma e^{t\lambda}(\lambda + L)^{-1} f d\lambda,$$

we can obtain

$$\|e^{-tL}f\|_p \leq C_p \, \|f\|_p, \qquad \|\nabla e^{-tL}f\|_p \leq C_p \, t^{-1/2}\|f\|_p$$

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we can obtain

$$\|e^{-tL}f\|_p \leq C_p \|f\|_p, \qquad \|\nabla e^{-tL}f\|_p \leq C_p\, t^{-1/2}\|f\|_p$$

for $1 < p < \infty$. By these estimates and Sobolev's embedding theorem, we can prove $L^p - L^q$ estimates.

3-4. Proof of Key estimate (2)

Key estimate (2) : resolvent estimate

$$\|(1+x_n)u_s\|_\infty \le \exists\mu \qquad \Rightarrow \qquad \|(\lambda+L)^{-1}\|_{\mathcal{L}(J^p)} \le \frac{C}{|\lambda|},$$

We shall prove key estimate (2) by following the method due to Kozono-Yamazaki(98). We notice

$$\begin{aligned}\lambda+L &= \lambda+A+B \\ &= (\lambda+A)^{1/2}\left[1+(\lambda+A)^{-1/2}B(\lambda+A)^{-1/2}\right](\lambda+A)^{1/2}.\end{aligned}$$

Therefore we see

$$(\lambda+L)^{-1} = (\lambda+A)^{-1/2}\left[1+(\lambda+A)^{-1/2}B(\lambda+A)^{-1/2}\right]^{-1}(\lambda+A)^{-1/2}.$$

In order to use the standard argument of Neumann series, we need

$$\|(\lambda+A)^{-1/2}B(\lambda+A)^{-1/2}\|_{\mathcal{L}(J^p)} \le 1/2.$$

We see

$$\begin{aligned}
&\langle(\lambda+A)^{-1/2}B(\lambda+A)^{-1/2}f,\phi\rangle\\
&=\langle(u_s\cdot\nabla)(\lambda+A)^{-1/2}f+((\lambda+A)^{-1/2}f\cdot\nabla)u_s,(\overline{\lambda}+A^*)^{-1/2}\phi\rangle\\
&=-\langle(\lambda+A)^{-1/2}f,u_s\cdot\nabla(\overline{\lambda}+A^*)^{-1/2}\phi\rangle\\
&\qquad\qquad-\langle u_s,((\lambda+A)^{-1/2}f\cdot\nabla)(\overline{\lambda}+A^*)^{-1/2}\phi\rangle
\end{aligned}$$

for $\phi\in C_{0,\sigma}^\infty(H)$, $f\in J^p(H)$. Therefore we obtain

$$\begin{aligned}
&|\langle(\lambda+A)^{-1/2}B(\lambda+A)^{-1/2}f,\phi\rangle|\\
&\le\|\frac{1}{1+x_n}(\lambda+A)^{-1/2}f\|_p\|(1+x_n)u_s\|_\infty\|\nabla(\overline{\lambda}+A^*)^{-1/2}\phi\|_{p'}\\
&\qquad+\|(1+x_n)u_s\|_\infty\|\frac{1}{1+x_n}(\lambda+A)^{-1/2}f\|_p\|\nabla(\overline{\lambda}+A^*)^{-1/2}\phi\|_{p'}\\
&=2\|(1+x_n)u_s\|_\infty\|\frac{1}{1+x_n}(\lambda+A)^{-1/2}f\|_p\|\nabla(\overline{\lambda}+A^*)^{-1/2}\phi\|_{p'}
\end{aligned}$$

Hardy type inequality

$$\left\| \frac{g}{1+x_n} \right\|_p \leq C \|\nabla g\|_p \qquad \text{for } g \in W_0^{1,p}(H).$$

We use Hardy's inequality and Assumption: $\|(1+x_n)u_s\|_\infty \leq \mu$.

$$\begin{aligned}
&2\|(1+x_n)u_s\|_{L^\infty} \left\| \frac{1}{1+x_n}(\lambda+A)^{-1/2}f \right\|_{L^p} \left\|\nabla(\overline{\lambda}+A^*)^{-1/2}\phi\right\|_{L^{p'}} \\
&\leq 2\mu \|\nabla(\lambda+A)^{-1/2}f\|_{L^p} \left\|\nabla(\overline{\lambda}+A^*)^{-1/2}\phi\right\|_{L^{p'}} \\
&\leq 2\mu C \|A^{1/2}(\lambda+A)^{-1/2}f\|_{L^p} \left\|A^{1/2}(\overline{\lambda}+A^*)^{-1/2}f\right\|_{L^{p'}} \\
&\leq 2\mu C \|f\|_{L^p} \|\phi\|_{L^{p'}}.
\end{aligned}$$

By duality argument, we obtain

$$\|(\lambda+A)^{-1/2}B(\lambda+A)^{-1/2}f\|_{L^p} \leq 2C\mu \|f\|_{L^p}.$$

Therefore choosing μ sufficiently small, we have

$$\|(\lambda + A)^{-1/2} B (\lambda + A)^{-1/2}\|_{\mathcal{L}(J^p)} \le \frac{1}{2}.$$

It follows from the standard theory of Neumann series that

$$\begin{aligned}
&\|(\lambda + L)^{-1}\|_{\mathcal{L}(J^p)} \\
&= \|(\lambda + A + B)^{-1}\|_{\mathcal{L}(J^p)} \\
&= \left\|(\lambda + A)^{-1/2}[1 + (\lambda + A)^{-1/2} B (\lambda + A)^{-1/2}]^{-1} (\lambda + A)^{-1/2}\right\|_{\mathcal{L}(J^p)} \\
&\le \|(\lambda + A)^{-1/2}\|^2_{\mathcal{L}(J^p)} \left(\sum_{k=0}^{\infty} \|(\lambda + A)^{-1/2} B (\lambda + A)^{-1/2}\|^k_{\mathcal{L}(J^p)} \right) \\
&\le C|\lambda|^{-1}
\end{aligned}$$

Conclusion

Main result【Stability theorem for the stationary solution】

Let $n \geq 2$ and $v_0 \in J^n(H)$. Then there exist positive constant μ and δ such that if u_s and v_0 satisfy

$$\|(1 + x_n)u_s\|_\infty \leq \mu, \quad \|v_0\|_n \leq \delta,$$

(PNS) admits a global strong solution v.

- Main results is proved by $L^p - L^q$ estimates of certain semigroup.
- $L^p - L^q$ estimates are obtained by resolvent estimate.
- Resolvent estimate is shown by using Hardy type inequality and duality argument.

Appendix: Proof of Hardy type inequality

Hardy type inequality

$$\left\| \frac{g}{1+x_n} \right\|_p \leq C \|\nabla g\|_p \qquad \text{for } g \in W_0^{1,p}(H).$$

By $g \in W_0^{1,p}(H)$, we see

$$\begin{aligned}
\left\| \frac{g(x)}{1+x_n} \right\|_p &\leq \left\| \frac{g(x', x_n) - g(x', 0)}{1+x_n} \right\|_p \\
&\leq \left\| \frac{1}{x_n} \int_0^{x_n} \partial_{y_n} g(x', y_n) dy_n \right\|_p \\
&\leq C_p \|\partial_{x_n} g(x', x_n)\|_p \\
&\leq C_p \|\nabla g\|_p.
\end{aligned}$$