Strong solutions to the Euler equations with bounded initial data

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March 12, 2010, Waseda University

§1. Introduction Consider the Euler or NS eq. in $\mathbb{R}^n$, $n \geq 2$:

$$(\mathrm{P}_\mu) \quad \begin{cases} U_t - \mu\Delta U + (U,\nabla)U + \nabla P = F & \text{in } \mathbb{R}^n \times (0,T), \\ \nabla \cdot U = 0 & \text{in } \mathbb{R}^n \times (0,T), \\ U|_{t=0} = U_0 & \text{in } \mathbb{R}^n. \end{cases}$$

$U = U_{(\mu)} = (U^1(x,t), \ldots, U^n(x,t))$ velocity (unknown),

$P = P(x,t)$ pressure (unknown),

$U_0 = (U_0^1(x), \ldots, U_0^n(x))$ initial velocity (given),

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$\mu \geq 0$ constant; $\nabla \cdot U_0 = 0$, $\nabla \cdot F = 0$: compatibility cond.

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$\Rightarrow$ local existence of a unique strong solution U to (P_0);

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commutator type estimate, trajectory flow [Chemin, Vishik]

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Note: trajectory flow argument is not applicable for NS.

Initial Data $U_0(x) := u_0(x) + f(x)$

where $u_0 \in H^s_\sigma(\mathbb{R}^n)$, $s > n/2 + 1$;

$f \in W^{\infty,\infty}_\sigma(\mathbb{R}^n)$ s.t. $\|\nabla^k f\|_\infty \leq {}^\exists C$, ${}^\forall k \in \{0, \ldots, [s] + 1\}$.

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External Force Let $\tilde{F} := \tilde{F}_\mu := F + \mu\Delta f - (f, \nabla)f - \nabla\bar{P},$

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Moreover, $n = 2$ (or $n = 3$, axi-symmetry, no swirl) and

$F = 0 \quad \Rightarrow \quad f = C$. [Koch-Nadirashvili-Seregin-Šverák]

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Thm.1 $^\exists T_0 > 0$ and $^{\exists 1}(u,\tilde{P})$ solution to (P'_μ) in the class

$u \in C([0,T_0];H^s_\sigma) \cap C^1(0,T_0;H^{s-1})$ with $u(0) = u_0$ and

$$\partial_\ell\tilde{P} = \sum_{j,k=1}^n \left\{\partial_\ell R_j R_k u^j u^k + 2R_\ell R_j u^k \left(\partial_k f^j\right)\right\}.$$

Lemma [Kato-Ponce] If $n \in \mathbb{N}$, $s > 0$, $1 < p < \infty$, then $^{\exists}C$:

$$\|\Lambda^s(gh)\|_p \leq C\left(\|g\|_q\|\Lambda^s h\|_{\tilde{q}} + \|\Lambda^s g\|_r\|h\|_{\tilde{r}}\right),$$

$$\|\Lambda^s(gh) - g\Lambda^s h\|_p \leq C\left(\|\nabla g\|_q\|\Lambda^{s-1} h\|_{\tilde{q}} + \|\Lambda^s g\|_r\|h\|_{\tilde{r}}\right),$$

where $\Lambda := (-\Delta)^{\frac{1}{2}}$, $1/p = 1/q + 1/\tilde{q} = 1/r + 1/\tilde{r}$.

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$$\partial_t u_m + (u_{m-1} + f, \nabla) u_m + (u_m, \nabla) f + \nabla \tilde{P}_m = 0 \quad \text{in } \mathbb{R}^n \times (0,T)$$

for $m \ge 2$ with $\nabla \cdot u_m = 0$ and $u_m|_{t=0} = u_0$, $\quad u_1(x,t) := u_0(x)$.

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Multiplying $(\star)$ by u_m, integrating over $\mathbb{R}^n$, we get

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(2) blow-up criteria: let $n = 3$, then

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$\big(\Leftarrow$ e.g. $\Delta f \in H^s$, $\bar{P} = \sum_{j,k} R_j R_k f^j f^k$ and $F \in L^1(0,T; H^s_\sigma)\big)$.