Hamilton-Pontryagin Principle on Lie groups and Applications to Incompressible Ideal Fluids

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References for Ideal Fluids and Dirac Structures

- Marsden and Weinstein [1983], Physica D. **7**.
- Cendra and Marsden [1987], Physica D. **27**.
- Holm, Marsden and Ratiu [1998], Adv. in Math. **137**.
- Yoshimura and Marsden [2006], Part I, II, J. Geo. Phys. **57**.
- Yoshimura and Marsden [2007], Rep. Math. Phys. **60**(3).

Contents

Variational Principle for Ideal Fluids

- It was **Arnold [1966]** who first derived motion of incompressible ideal fluids as the **geodesic spray of the group of volume preserving diffeomorphisms** of an oriented Riemaniann manifold $\mathfrak{D}$ with respect to the right invariant metric

$$T(\mathbf{v}) = \int_{\mathfrak{D}} \frac{1}{2} \|\mathbf{v}\|^2 \, d^3x.$$

- It was shown by **Marsden and Ratiu [1999]** that Euler equations for incompressible ideal fluids can be derived in the context of **Euler-Poincaré reduction**:

$$\delta \int_{t_0}^{t_1} l(\xi(t)) \, dt = 0,$$

with $\delta\xi = \dot{\eta} - [\xi, \eta]$, which yields **Euler-Poincaré equations**

$$\frac{d}{dt}\frac{\partial l}{\partial \xi} = -\mathrm{ad}^*_{\xi}\frac{\partial l}{\partial \xi}.$$

- **Cendra and Marsden [1987]** showed that a variational principle using **Clebsch potentials** (see Marsden and Weinstein [1983]) enables us to incorporates the so-called "**Lin constraints**".
- Based on **Lagrangian semi-direct product theory**, it was shown by **Holm, Marsden and Ratiu [1998]** that reduced Hamilton's principle for a parameter dependent Lagrangian $L_{a_0} : TG \times V^* \to \mathbb{R}$,

$$\delta \int_{t_0}^{t_1} l(\xi(t), a(t))\, dt = 0,$$

with $\delta\xi = \dot{\eta} - [\xi, \eta]$ and $\delta a = -\eta a$, yields **Euler-Poincaré equations with advected parameters** such that

$$\frac{d}{dt}\frac{\partial l}{\partial \xi} = -\mathrm{ad}^*_{\xi}\frac{\partial l}{\partial \xi} + \frac{\partial l}{\partial a} \diamond a,$$

where the diamond operator is defined as

$$\diamond : V \times V^* \to \mathfrak{g}^*; \quad (v, w) \mapsto v \diamond w := \rho_v^*(w)$$

by using a linear map $\rho_v : \mathfrak{g} \to V;\ \ \xi \mapsto \rho_v(\xi) := v\xi$.

Dirac Structures

- The notion of **Dirac structures** was developed by **Courant and Weinstein** [1989] and **Dorfmann** [1987] as a generalized idea of unifying **pre-symplectic and Poisson structures**.
- For a smooth manifold P, the duality paring between TP and T^*P is given by $\langle \cdot, \cdot \rangle$.

  The **Pontryagin bundle** $TP \oplus T^*P$ is endowed with a non-degenerate symmetric paring:

$$\langle\langle (X, \alpha), (\bar{X}, \bar{\alpha}) \rangle\rangle := \langle \alpha, \bar{X} \rangle + \langle \bar{\alpha}, X \rangle$$

  for all $X, \bar{X} \in TP$ and $\alpha, \bar{\alpha} \in T^*P$.
- A **Dirac structure** D_P on P is a subbundle

$$D_P \subset TP \oplus T^*P$$

  such that $D_P = D_P^{\perp}$, where, for each $x \in P$,

$$D_P^{\perp}(x) = \{(u_x, \beta_x) \in T_xP \times T_x^*P \,|\, \alpha_x(u_x) + \beta_x(v_x) = 0, \quad \text{for all } \ (v_x, \alpha_x) \in D_P(x)\}.$$

Lagrange-Dirac Systems on Lie Groups

- Recall the **canonical Dirac structure** on $P = T^*G$ is given by

$$D_P = \operatorname{graph} \Omega^\flat \quad \text{or} \quad D_P = \operatorname{graph} B^\sharp,$$

and an **implicit Lagrangian system** (E, D_P, X) satisfies

$$(X, \mathbf{d}E|_{TP}) \in D_P,$$

where $X : TQ \oplus T^*Q \to TT^*G$ is a partial vector field,

$$E = \langle p, v \rangle - L(g, v)$$

is the generalized energy on $TQ \oplus T^*Q$, and $P = \mathbb{F}L(TG) \subset T^*G$.

- One obtains **implicit Euler-Lagrange equations**:

$$\dot{g} = v, \quad \dot{p} = \frac{\partial L}{\partial g}, \quad p = \frac{\partial L}{\partial v}.$$

- A **solution curve** is a curve in $TG \oplus T^*G$,

$$(g(t), v(t), p(t)), \quad t_1 \leq t \leq t_2,$$

such that $(g(t), p(t) = (\partial L/\partial v)(t))$ is an **integral curve of the partial vector field** X.

Hamilton-Pontryagin Principle on Lie Groups

- The **Hamilton-Pontryagin principle** states the stationary condition of the action functional for curves $(g(t), v(t), p(t))$, $t \in [a, b]$ in the Pontryagin bundle $TG \oplus T^*G$, which is given by

$$\delta \int_a^b \{L(g(t), v(t)) + \langle p(t), \dot{g}(t) - v(t)\rangle\}\, dt = 0,$$

  with the endpoints of $g(t)$ fixed.

- This variational principle induces **implicit Euler–Lagrange equations** on $TG \oplus T^*G$:

$$p = \frac{\partial L}{\partial v}, \quad \dot{g} = v, \quad \dot{p} = \frac{\partial L}{\partial g}.$$

- This Hamilton-Pontryagin principle is quite consistent with the **canonical Dirac structure !**

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Let us see how reduction procedure will be going !

Reduced Hamilton-Pontryagin Principle

- The **reduced Hamilton-Pontryagin principle** is given by

$$\delta \int_{t_1}^{t_2} \{ l(\eta(t)) + \langle \mu(t), \xi(t) - \eta(t) \rangle \} \; dt = 0,$$

with variations of $\xi(t) = \dot{g}g^{-1}$ in the form

$$\delta\xi(t) = \dot{\zeta}(t) - [\xi(t), \zeta(t)],$$

where $\zeta(t) = \delta g g^{-1} \in \mathfrak{g}$ satisfies $\zeta(t_i) = 0$.

- Then, the action functional vanishes for any $\delta\eta \in \mathfrak{g}$, $\zeta \in \mathfrak{g}$ and $\delta\mu \in \mathfrak{g}^*$ if and only if **implicit Euer-Poincaré equations** holds:

$$\mu = \frac{\delta l}{\delta \eta}, \quad \xi = \eta, \quad \dot{\mu} = -\mathrm{ad}_{\xi}^{*} \mu.$$

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$$\mu = \frac{\delta l}{\delta \eta}, \quad \xi = \eta, \quad \dot{\mu} = -\mathrm{ad}^*_{\xi} \mu.$$

- Let us call this reduction procedure **Lie-Dirac reduction**.

The H-P Principle with Advected Parameters

- Using this basic construction, we can easily develop the **Lie-Dirac reduction for a parameter dependent Lagrangian** $L_{a_0} : TG \times V^* \to \mathbb{R}$ by

$$\delta \int_{t_0}^{t_1} \{ l(\eta(t), a(t)) + \langle \mu(t), \xi(t) - \eta(t) \rangle \} \; dt = 0,$$

with variations of $\xi = \dot{g} g^{-1}$ in the form

$$\delta \xi = \dot{\eta} - [\xi, \eta]$$

and

$$\delta a = -\eta a.$$

- This reduction theory induces the **implicit Euer-Poincaré equations with advected variables** as

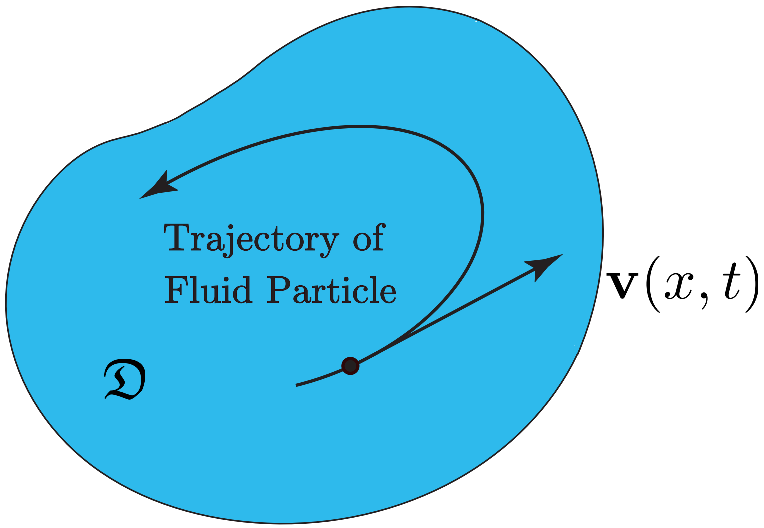
$$\mu = \frac{\delta l}{\delta \eta}, \quad \xi = \eta, \quad \dot{\mu} = -\mathrm{ad}_{\xi}^{*} \mu + \frac{\partial l}{\partial a} \diamond a.$$

Incompressible Ideal Fluids

- The **group of diffeomorphisms** $G = \mathrm{Diff}(\mathfrak{D})$ of a bounded region $\mathfrak{D}$ in $\mathbb{R}^3$ is a configuration space.
- Set a **reference point** $X \in \mathfrak{D}$, and a **current point** is denoted by

$$x = \eta_t(X) \in \mathfrak{D},$$

  where $\eta_t : \mathfrak{D} \to \mathfrak{D}$ is the diffeomorphism.
- In other words, a **motion** of a fluid is a family of time-dependent elements of G, which is written by $x = \eta(X, t)$.



- The **Lagrangian** or **material velocity field** is defined by taking the time derivative of the motion keeping the particle lable X fixed:

$$\mathbf{V}(X,t) := \frac{\partial \eta(X,t)}{\partial t} := \left.\frac{\partial}{\partial t}\right|_X \eta_t(X)$$

while the **Eulerian** or **spatial velocity field** is defined by taking the time derivative of the motion keeping the particle lable x fixed:

$$\mathbf{v}(x,t) := \mathbf{V}(X,t) := \left.\frac{\partial}{\partial t}\right|_x \eta_t(X).$$

Thus, the Eulerian velocity $\mathbf{v}$ is a time dependent vector field on $\mathfrak{D}$, namely, $\mathbf{v}_t \in \mathfrak{X}(\mathfrak{D})$, where $\mathbf{v}_t(x) := \mathbf{v}(x,t)$.

- The map from the space of $(\eta, \dot{\eta})$ (**material velocity**) to the space of $\mathbf{v}$ (**spatial velocity**) is given by

$$\mathbf{v} = \dot{\eta} \circ \eta^{-1}, \quad \text{i.e.,} \quad \mathbf{v}_t = \mathbf{V}_t \circ \eta_t^{-1},$$

where $\mathbf{V}_t(X) := \mathbf{V}(X,t)$.

Implicit Euler-Poincaé Equations

- The reduced Lagrangian $l : \mathfrak{g} \times V^* \longrightarrow \mathbb{R}$ is given by

$$l(\mathbf{u}, \rho) = \int_{\mathfrak{D}} \frac{1}{2} \rho \mathbf{u} \cdot \mathbf{u} \, d^3x = \frac{1}{2} \langle \rho \mathbf{u}, \mathbf{u} \rangle ,$$

where $\mathbf{u} \in \mathfrak{X}(\mathfrak{D})$ denote some Eulerian velocity.

- Let us consider the **reduced Hamilton-Pontryagin variational principle** in Eulerian coordinates:

$$\delta \int_{t_1}^{t_2} \int_{\mathfrak{D}} l(\mathbf{u}, \rho) + p(1 - \rho) + \langle \mathbf{\Pi}, \mathbf{v} - \mathbf{u} \rangle \, d^3x \, dt = 0,$$

with variations of the form (**Lin constraints**)

$$\delta \mathbf{v} = \dot{\mathbf{w}} - [\mathbf{v}, \mathbf{w}] \quad \text{and} \quad \delta(\rho d^3x) = -\pounds_{\mathbf{w}}(\rho d^3x) = -\nabla \cdot (\rho \mathbf{w}) \, d^3x,$$

where $\mathbf{v} = \dot{\eta} \circ \eta^{-1}, \mathbf{w} = \delta\eta \circ \eta^{-1} \in \mathfrak{X}(\mathfrak{D})$ and $\rho \otimes d^3x \in V^* \subset \mathfrak{T}(\mathfrak{D}) \otimes \mathrm{Den}(\mathfrak{D})$ denotes the **mass density**.

- In the above, we impose the constraint $\rho = 1$ associated with the incompressibility by using the **Lagrange multiplier** p.

- Notice that the Lagrange multipliers $p \in V$ and $\mathbf{\Pi} \otimes d^3x \in \mathfrak{X}(\mathfrak{D})^* = \Omega^1(\mathfrak{D}) \otimes \mathrm{Den}(\mathfrak{D})$ are eventually to be the **pressure** and the tensor product of the **momentum density**.
- The **Neumann boundary conditions** are given by

$$\delta\eta(x, t_i) = 0, \quad \text{for} \;\; x \in \mathfrak{D}; \qquad \delta\eta(x,t)\cdot\mathbf{n}(x) = 0, \quad \text{for all} \;\; t_1 \leq t \leq t_2.$$

- Thus, we can obtain **implicit Euler-Poincaré equations** for **incompressible ideal fluids** as

$$(1)\ \mathbf{\Pi} = \frac{\delta l}{\delta \mathbf{u}} = \rho\mathbf{u} \quad : \text{Momentum},$$

$$(2)\ \rho = 1 \qquad : \text{Incompressibility Constraint},$$

$$(3)\ \mathbf{v} = \mathbf{u} \qquad : \text{Second-order Vector Field},$$

$$(4)\ \frac{\partial \mathbf{\Pi}}{\partial t} + \mathrm{ad}^*_{\mathbf{v}}\mathbf{\Pi} = \rho\nabla\left(\frac{1}{2}\,|\mathbf{u}|^2 - p\right) \quad : \text{Equation of Motion}.$$

- Of course, by some rearrangements, we can obtain **Euler equations** for motion of ideal incompressible fluids:

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\nabla\, p, \qquad \nabla \mathbf{v} = 0.$$

- Of course, by some rearrangements, we can obtain **Euler equations** for motion of ideal incompressible fluids:

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- **Kelvin's Circulation Theorem:** The implicit Euler-Poincaré equation preserves the **circulation integral** $I(t)$ associated to a Lagrangian loop C_t as

$$I(t) = \oint_{C(\mathbf{u})} \mathbf{u} \cdot d\mathbf{x} = \oint_{C(\mathbf{u})} \frac{1}{\rho} \cdot \frac{\delta l}{\delta \mathbf{u}} = \oint_{C(\mathbf{u})} \frac{\mathbf{\Pi}}{\rho},$$

since

$$\begin{aligned}\frac{d}{dt}\left(\oint_{C(\mathbf{u})} \frac{1}{\rho} \cdot \frac{\delta l}{\delta \mathbf{u}}\right) &= \oint_{C(\mathbf{u})} \nabla \left(\frac{1}{2}\left|\mathbf{u}\right|^2 - p\right) \cdot d\mathbf{x} \\ &= \oint_{C(\mathbf{u})} \mathbf{d}\left(\frac{1}{2}\left|\mathbf{u}\right|^2 - p\right) \\ &= 0.\end{aligned}$$

Concluding Remarks

- **Lagrange-Dirac Systems:** We have considered the Hamilton-Pontryagin principle on Lie groups. We have shown how implicit Euler-Lagrange equations can be constructed associated with the canonical Dirac structure.
- **Lie-Dirac Reduction with Advected Parameters:** For the case in which a Lagrangian is dependent on advected parameters, we have shown the implicit Euler-Poincaré equations with advected parameters can be developed from the reduced Hamilton-Pontryagin principle.
- **Incompressible Ideal Fluids:** We have demonstrated how the Lie-Dirac reduction with advected parameters can be applied to incompressible ideal fluids and we have also demonstrated how the ideal fluid dynamics can be formulated in the context of implicit Euler-Poincaré equations with advected parameters.