

A generalized Cartan decomposition for the non-symmetric space $SL(2n + 1, \mathbb{C})/Sp(n, \mathbb{C})$

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International Workshop on Mathematical Fluid Dynamics

at Waseda University

March 12, 2010

Why do we treat $SL(2n + 1, \mathbb{C})/Sp(n, \mathbb{C})$?

$G/H := SL(2n + 1, \mathbb{C})/Sp(n, \mathbb{C})$: **spherical** variety

- (Algebraic geometry) $B := \{\text{upper triangular matrices in } G\}$

$\Rightarrow \exists x \in G/H$ s.t. $B \cdot x \subset G/H$: open dense

- (Representation theory)

$K := SU(2n + 1) \curvearrowright \mathcal{O}(G/H)$: multiplicity-free representation

- (Analysis)

$\mathbb{D}(G/H) := \{G\text{-invariant differential operators on } G/H\}$

$\Rightarrow \mathbb{D}(G/H)$ is a commutative algebra

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- (Lie theory, differential geometry) G/H : **non-symmetric**

\Rightarrow There is **NO** structural theory

Our Problem

$G/H := SL(2n + 1, \mathbb{C})/Sp(n, \mathbb{C})$: spherical variety

Problem

Finding a concrete description for

$$K \backslash G/H := SU(2n + 1) \backslash SL(2n + 1, \mathbb{C})/Sp(n, \mathbb{C})$$

	left action	Space	right action
1	K	G	H
2	K	G/H	
3		$K \backslash G$	H

Background — symmetric case

K	\subset	G	\supset	H
maximal compact subgp.		non-compact reductive Lie gp.		reductive closed subgp.

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$H \subset G$: **symmetric** subgp.

i.e. $\exists \tau \in \text{Aut}(G) : \tau^2 = \text{id}$ s.t. $(G^\tau)_0 \subset H \subset G^\tau$

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$\exists A \simeq \mathbb{R}^l$, $l = \text{rank}_{\mathbb{R}} G/H$ s.t. $G = KAH$ [Flensted–Jensen '78]

- $H = K \Rightarrow G = KAK \cdots$ (classical) Cartan decomposition

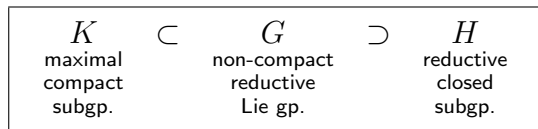
Remarkable aspects

K maximal compact subgp.	\subset	G non-compact reductive Lie gp.	\supset	H reductive closed subgp.
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$H \subset G$: **symmetric** subgp. $\Rightarrow \exists A \simeq \mathbb{R}^l$ s.t. $G = KAH$

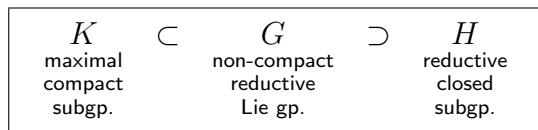
- Algebraically $K \backslash G / H$ — reduction theory, normal forms
e.g. $O(n) \backslash GL(n, \mathbb{R}) / O(n) \leftrightarrow$ diagonalization of symm. matrix
- Geometrically $K \curvearrowright G / H$ — $A / (A \cap H)$ meets every K -orbit
 $A / (A \cap H)$: flat totally geodesic
- Analytically — analysis on $L^1(G // K)$
 $L^1(G)^{K \times H} \simeq L^1(K \backslash G / H)$

Non-symmetric case



$H \subset G$: non-symmetric \Rightarrow there is **NO** structural theory

Non-symmetric case



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- $(U(n_1) \times U(n_2) \times U(n_3)) \backslash U(n) / (U(p) \times U(q))$ [Kobayashi '07]
- \mathbb{C}^\times -bundles $G_{\mathbb{C}} / [K_{\mathbb{C}}, K_{\mathbb{C}}] \rightarrow G_{\mathbb{C}} / K_{\mathbb{C}}$
 G/K : non-tube type Hermitian symmetric spaces [S- '09]
- $Spin(8) \backslash Spin(8, \mathbb{C}) / G_2(\mathbb{C}) \rightarrow SO(8) \backslash SO(8, \mathbb{C}) / G_2(\mathbb{C})$ [S-]
- **Today** : $SU(2n+1) \backslash SL(2n+1, \mathbb{C}) / Sp(n, \mathbb{C})$

Main Theorem

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Theorem

$\exists A \subset G : 2n\text{-dim. subset s.t. } \boxed{G = KAH}$

- $\mathbb{T} \simeq SO(2) \Rightarrow A \simeq \mathbb{R}^2 \cdot \underbrace{\mathbb{T} \cdots \mathbb{T}}_{n-1} \cdot \mathbb{R}^{n-1}$ [Explicit](#)
- A is **no longer** a group

Strategy of Proof

$$\begin{array}{c} H \\ \cap \\ G \\ \subset \\ K \end{array}$$

Table: Herringbone stitch for $K \backslash G / H$

Strategy of Proof

$$L = 1 \times SL(2n, \mathbb{C}) \supset U = 1 \times SU(2n) \supset U_1 = E_2 \times SU(2n - 1) \\ \cup \\ U_2 = 1 \times SU(2)^n$$

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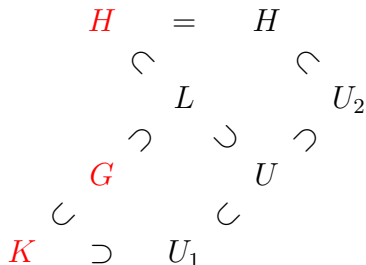


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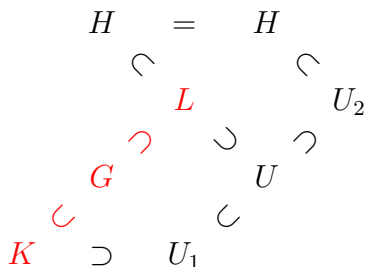


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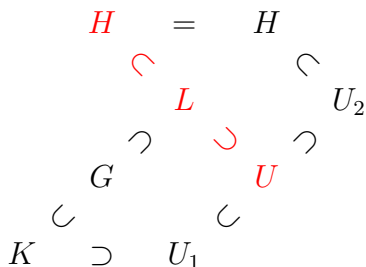


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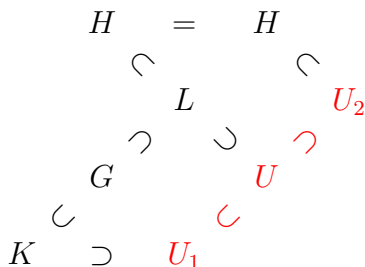


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Proof

Step 1 [▶ Link](#) [S- '09] $\Rightarrow \exists A_1 \simeq \mathbb{R}^2$ s.t. $G = KA_1L$

Step 2 [▶ Link](#) [F-J '78] $\Rightarrow \exists A_2 \simeq \mathbb{R}^{n-1}$ s.t. $L = UA_2H$

Step 3 [▶ Link](#) $U_1A_1 = A_1U_1, U_2A_2 = A_2U_2$

and $\exists A_3 \simeq \underbrace{\mathbb{T} \cdots \cdots \mathbb{T}}_{n-1}$ s.t. $U = U_1A_3U_2$

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$$\begin{aligned} G &= KA_1L \leftarrow \text{Step 1} \\ &= KA_1(UA_2H) \leftarrow \text{Step 2} \\ &= KA_1(U_1A_3U_2)A_2H \leftarrow \text{Step 3} \\ &= KU_1(A_1A_3A_2)U_2H \\ &= K(A_1A_3A_2)H. \end{aligned}$$

Explicit description

- $\mathfrak{a}_1 = \mathbb{R}(E_{1,2} + E_{2,1})$

$$\mathfrak{b}_1 = \mathbb{R}\{(2n - 1)(E_{1,1} + E_{2,2}) - 2(E_{3,3} + \cdots + E_{2j+1,2j+1})\}$$

- $\mathfrak{a}_2 = \left\{ \sum_{j=1}^n t_j (E_{2j,2j} + E_{2j+1,2j+1}) : t_1, \dots, t_n \in \mathbb{R}, \sum_{j=1}^n t_j = 0 \right\}$

- $B_j = \exp \mathbb{R}(E_{2j+2,2j} - E_{2j,2j+2}) \quad (j = 1, 2, \dots, n - 1)$

$$\Rightarrow \begin{cases} A_1 = (\exp \mathfrak{a}_1)(\exp \mathfrak{b}_1) \simeq \mathbb{R}^2 \\ A_2 = \exp \mathfrak{a}_2 \simeq \mathbb{R}^{n-1} \\ A_3 = B_1 B_2 \cdots B_{n-1} \simeq \underbrace{\mathbb{T} \cdots \mathbb{T}}_{n-1} \\ A = A_1 A_3 A_2 \end{cases}$$