A generalized Cartan decomposition for the non-symmetric space $SL(2n+1,\mathbb{C})/Sp(n,\mathbb{C})$

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Why do we treat $SL(2n+1,\mathbb{C})/Sp(n,\mathbb{C})$?

$G/H := SL(2n+1,\mathbb{C})/Sp(n,\mathbb{C})$: spherical variety

- (Algebraic geometry) $B := \{$upper triangular matrices in $G\}$

  $\Rightarrow \exists x \in G/H$ s.t. $B \cdot x \subset G/H$: open dense
- (Representation theory)

  $K := SU(2n+1) \curvearrowright \mathcal{O}(G/H)$: multiplicity-free representation
- (Analysis)

  $\mathbb{D}(G/H) := \{G$-invariant differential operators on $G/H\}$

  $\Rightarrow \mathbb{D}(G/H)$ is a commutative algebra

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  $\Rightarrow \mathbb{D}(G/H)$ is a commutative algebra

- (Lie theory, differential geometry) G/H : non-symmetric

  $\Rightarrow$ There is NO structural theory

Our Problem

$$G/H := SL(2n+1,\mathbb{C})/Sp(n,\mathbb{C})$$: spherical variety

Problem

Finding a concrete description for

$$K\backslash G/H := SU(2n+1)\backslash SL(2n+1,\mathbb{C})/Sp(n,\mathbb{C})$$

	left action	Space	right action
1	K	G	H
2	K	G/H	
3		$K\backslash G$	H

Background — symmetric case

K	$\subset$	G	$\supset$	H
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$\exists A \simeq \mathbb{R}^l$, $l = \mathrm{rank}_{\mathbb{R}} G/H$ s.t. $\boxed{G = KAH}$ [Flensted–Jensen '78]

- $H = K \Rightarrow \boxed{G = KAK}$ $\cdots$ (classical) Cartan decomposition

Remarkable aspects

K	$\subset$	G	$\supset$	H
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- Algebraically $K \backslash G / H$ — reduction theory, normal forms

  e.g. $O(n) \backslash GL(n, \mathbb{R}) / O(n) \leftrightarrow$ diagonalization of symm. matrix

- Geometrically $K \curvearrowright G/H$ — $A/(A \cap H)$ meets every K-orbit

  $A/(A \cap H)$: flat totally geodesic

- Analytically — analysis on $L^1(G//K)$

  $$L^1(G)^{K \times H} \simeq L^1(K \backslash G / H)$$

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- $(U(n_1) \times U(n_2) \times U(n_3)) \backslash U(n) / (U(p) \times U(q))$ [Kobayashi '07]
- $\mathbb{C}^\times$-bundles $G_\mathbb{C}/[K_\mathbb{C}, K_\mathbb{C}] \to G_\mathbb{C}/K_\mathbb{C}$

  G/K : non-tube type Hermitian symmetric spaces [S- '09]
- $Spin(8) \backslash Spin(8, \mathbb{C}) / G_2(\mathbb{C}) \to SO(8) \backslash SO(8, \mathbb{C}) / G_2(\mathbb{C})$ [S-]
- Today : $SU(2n+1) \backslash SL(2n+1, \mathbb{C}) / Sp(n, \mathbb{C})$

Main Theorem

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$\exists A \subset G$: $2n$-dim. subset s.t. $\boxed{G = KAH}$

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Theorem

$\exists A \subset G$: $2n$-dim. subset s.t. $\boxed{G = KAH}$

- $\mathbb{T} \simeq SO(2) \Rightarrow A \simeq \mathbb{R}^2 \cdot \underbrace{\mathbb{T} \cdot \cdots \cdot \mathbb{T}}_{n-1} \cdot \mathbb{R}^{n-1}$ Explicit
- A is no longer a group

Strategy of Proof

H

$\cap$

G

$\hookleftarrow$

K

Table: Herringbone stitch for $K\backslash G/H$

Strategy of Proof

$$
\begin{array}{l}
L = 1 \times SL(2n, \mathbb{C}) \supset \; U = 1 \times SU(2n) \supset U_1 = E_2 \times SU(2n-1) \\
\qquad\qquad\qquad\qquad\quad \cup \\
\qquad\qquad\qquad\qquad\quad U_2 = 1 \times SU(2)^n
\end{array}
$$

$$
\begin{array}{ccc}
 & H & \\
 & \cap & \\
 & G & \\
\subset & & \\
K & &
\end{array}
$$

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\end{array}$$

$$\begin{array}{ccccccc}
 & & {\color{red}H} & = & H & & \\
 & & & \rotatebox{45}{$\subset$} & & \rotatebox{45}{$\subset$} & \\
 & & & L & & & U_2 \\
 & & \rotatebox{45}{$\supset$} & & \rotatebox{-45}{$\supset$} & \rotatebox{45}{$\supset$} & \\
 & & {\color{red}G} & & U & & \\
 & \rotatebox{-45}{$\subset$} & & \rotatebox{-45}{$\subset$} & & & \\
{\color{red}K} & \supset & U_1 & & & &
\end{array}$$

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Strategy of Proof

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$$\begin{array}{ccccccccc}
 & & H & & = & & H & & \\
 & & & \rotatebox{-45}{$\cap$} & & & & \rotatebox{-45}{$\cap$} & \\
 & & & & L & & & & U_2 \\
 & & & \rotatebox{-135}{$\cap$} & & \rotatebox{-45}{$\cup$} & & \rotatebox{-135}{$\cap$} & \\
 & & G & & & & U & & \\
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		H		$=$		H		
			$\subset$				$\subset$	
				L				U_2
			$\supset$		$\supset$		$\supset$	
		G				U		
	$\subset$				$\subset$			
K		$\supset$		U_1				

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$$L = 1 \times SL(2n,\mathbb{C}) \supset \color{red}{U = 1 \times SU(2n) \supset U_1 = E_2 \times SU(2n-1)}$$
$$\cup$$
$$\color{red}{U_2 = 1 \times SU(2)^n}$$

$$\begin{array}{ccccccc}
 & & H & = & H & & \\
 & & & \rotatebox{-45}{$\subset$} & & \rotatebox{-45}{$\subset$} & \\
 & & & L & & & \color{red}{U_2} \\
 & & \rotatebox{45}{$\supset$} & & \rotatebox{-45}{$\supset$} & \color{red}{\rotatebox{45}{$\supset$}} & \\
 & & G & & & \color{red}{U} & \\
 & \rotatebox{45}{$\subset$} & & & \color{red}{\rotatebox{45}{$\subset$}} & & \\
K & \supset & \color{red}{U_1} & & & &
\end{array}$$

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Proof

Step 1 ▸ Link [S- '09] $\Rightarrow \exists A_1 \simeq \mathbb{R}^2$ s.t. $G = KA_1L$

Step 2 ▸ Link [F-J '78] $\Rightarrow \exists A_2 \simeq \mathbb{R}^{n-1}$ s.t. $L = UA_2H$

Step 3 ▸ Link $U_1A_1 = A_1U_1$, $U_2A_2 = A_2U_2$

and $\exists A_3 \simeq \underbrace{\mathbb{T} \cdot \cdots \cdot \mathbb{T}}_{n-1}$ s.t. $U = U_1A_3U_2$

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$$\begin{aligned} G &= KA_1L \quad \leftarrow \quad \text{Step 1} \\ &= KA_1(UA_2H) \quad \leftarrow \quad \text{Step 2} \\ &= KA_1(U_1A_3U_2)A_2H \quad \leftarrow \quad \text{Step 3} \\ &= KU_1(A_1A_3A_2)U_2H \\ &= K(A_1A_3A_2)H. \end{aligned}$$

Explicit description

- $\mathfrak{a}_1 = \mathbb{R}(E_{1,2} + E_{2,1})$

  $\mathfrak{b}_1 = \mathbb{R}\{(2n-1)(E_{1,1} + E_{2,2}) - 2(E_{3,3} + \cdots + E_{2j+1,2j+1})\}$
- $\mathfrak{a}_2 = \left\{ \sum_{j=1}^{n} t_j (E_{2j,2j} + E_{2j+1,2j+1}) : t_1, \ldots, t_n \in \mathbb{R}, \sum_{j=1}^{n} t_j = 0 \right\}$
- $B_j = \exp \mathbb{R}(E_{2j+2,2j} - E_{2j,2j+2}) \ (j = 1, 2, \ldots, n-1)$

$$\Rightarrow \begin{cases} A_1 = (\exp \mathfrak{a}_1)(\exp \mathfrak{b}_1) \simeq \mathbb{R}^2 \\ A_2 = \exp \mathfrak{a}_2 \simeq \mathbb{R}^{n-1} \\ A_3 = B_1 B_2 \cdots B_{n-1} \simeq \underbrace{\mathbb{T} \cdot \cdots \cdot \mathbb{T}}_{n-1} \\ A = A_1 A_3 A_2 \end{cases}$$