Relation of the covariant and Lie derivatives and its application to Hydrodynamics

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- The group $G = SDiff(M)$ of all diffeomorphisms of M preserving the volume form μ is a Lie group.
- The Lie algebra $\mathfrak{g} = SVect(M)$ for this group is formed by divergence-free vector fields on M(tangent to the boundary if $\partial M \neq \emptyset$).

The Euler hydrodynamic equation on manifolds

Definition

The Euler equation of an ideal incompressible fluid on M is the following evolution equation on the velocity field v of the fluid on the manifold:

$$\begin{cases} \frac{\partial v}{\partial t} = -(v, \nabla)v - \nabla p, \\ \\ div_\mu v = 0, \end{cases} \tag{1}$$

where the second equation means that the field v preserves the volume form μ. Here p is a time-dependent function on M. The expression $(v, \nabla)v$ denotes the covariant derivative $\nabla_v v$ of the field v along itself on M.

The goal of this talk

The goal of this talk is to obtain the generalized Euler equation on the dual space $\mathfrak{g}^*$ of the Lie algebra of divergence-free vector fields on M using the relation of the covariant and Lie derivates.

Relation of the covariant and Lie derivatives

- Every vector field on a Riemannian manifold defines a differential 1-form: the pointwise inner product with vectors of the field. For a vector field v we denote by v^b the 1-form whose value on a tangent vector at a point x is the inner product of the tangent vector with the vector $v(x)$.

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- Every vector field also defines a flow, which transports differential forms. For instance, one might transport the 1-form corresponding to some vector field by means of the flow of this field and get a new differential 1-form.
- Infinitesimally this transport is described by the Lie derivative of the 1-form(corresponding to the field) along the field itself, and the result is again a 1-form. This natural derivative of a 1-form is related to the covariant derivative of the corresponding vector field along itself by the following formula.

Relation of the covariant and Lie derivatives

Theorem

The Lie derivative of the one-form corresponding to a vector field on a Riemannian manifold differs from the one-form corresponding to the covariant derivative of the field along itself by a complete differential:

$$L_v(v^b) = (\nabla_v v)^b + \frac{1}{2} d\langle v, v\rangle. \tag{2}$$

Here $\langle v, v\rangle$ is the function on the manifold equal at each point x to the Riemannian square of the vector $v(x)$.

The dual space to the Lie algebra of divergence-free fields

Theorem

For an n-dimensional compact manifold M with boundary ∂M, the dual space $\mathfrak{g}^*$ of the Lie algeba $\mathfrak{g} = SVect(M)$ of divergence-free vector fields on M (tangent to ∂M) is naturally isomorphic to the quotient space $\Omega^1/d\Omega^0$ of all differential 1-forms on M, modulo all exact 1-forms (i.e., modulo differentials of all functions) on M in the following sense:

The dual space to the Lie algebra of divergence-free fields

- If α is the differential of a function ($\alpha = df$) and $v \in \mathfrak{g}$, then $\iint\limits_M \omega_v \wedge \alpha = 0$.

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- If $\iint_M \omega_v \wedge \alpha = 0$ for all $\alpha = df$, then $v \in \mathfrak{g}$(i.e., v is a divergence-free field on M tangent to ∂M).

The generalized Euler equation on the dual space $\mathfrak{g}^* = \Omega^1/d\Omega^0$ of the Lie algebra of divergence-free vector fields on M

Theorem

The Euler equation

$$\frac{\partial v}{\partial t} = -\nabla_v v - \nabla p \tag{3}$$

on the Lie algebra $\mathfrak{g} = SVect(M)$ of divergence-free vector fields is mapped by the inertia operator $A : \mathfrak{g} \to \mathfrak{g}^*$ to the Euler equation

$$\frac{\partial [u]}{\partial t} = -L_v[u] \tag{4}$$

on the dual space $\mathfrak{g}^* = \Omega^1/d\Omega^0$ of this algebra. Here the field v and the 1-form u are related by means of the Riemannian metric: $u = v^b$, and $[u] \in \Omega^1/d\Omega^0$ is the coset of the form u.

Proof

The inertia operator $A : SVect(M) \to \Omega^1/d\Omega^0$ sends a divergence-free field v to the 1-form $u = v^b$ considered up to the differential of a function. By the above theorem, it also sends the covariant derivate $\nabla_v v$ to the Lie derivate $L_v u$ modulo the differential of another function. Hence the Euler equation for the 1-form u assumes the form

$$\frac{\partial u}{\partial t} = -L_v u - df,$$

with the function $f = p - \frac{1}{2}\langle v, v\rangle$. It is equivalent to equation (4) for the coset $[u]$.

Thank you for your attentions!