A remark on the uniqueness of positive solutions to semilinear elliptic equations with double power nonlinearities

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1 Introduction

$$\begin{cases} \triangle u + f(u) = 0 & \text{in } \mathbb{R}^n, \quad n \geq 2, \\ u > 0 & \text{in } \mathbb{R}^n, \\ \lim_{|x| \to \infty} u(x) = 0. \end{cases}$$

$$\triangle = \sum_{j=1}^{n} \frac{\partial}{\partial x_j},$$

$$u : \mathbb{R}^n \ni x =^t (x_1, \ldots, x_n) \longrightarrow u(x) \in \mathbb{R}, \quad C^2.$$

Double Power:

$$f(u) = -\omega u + u^p - u^{2p-1},$$

where $p > 1, \quad 0 < \omega < \dfrac{p}{(p+1)^2} =: \omega_p \quad .$

Remark 1. (Motivation)
The Non Linear Schroedinger equation

$$\text{(NLS)} \quad i\partial_t \psi + \triangle \psi = -|\psi|^2\psi + |\psi|^4\psi,$$

$$\psi : \mathbb{R} \times \mathbb{R}^n \ni (t, x) \longrightarrow \psi(t, x) \in \mathbb{C}.$$

denotes **Boson gas interaction**.
Let standing wave solution $\psi(t, x) = e^{i\omega t}u(x)$ with $u > 0$, solves (NLS). Then u solves our elliptic equation with $p = 3$.

2 Known Results

Theorem 1. *(Rad. Sym. of the Sol.)*
For any $p > 1$ *and* $\omega > 0$*, any possible solution to our elliptic problem is radially symmetric with respect to some point.*
(Gidas, Ni and Nirenberg '79)

Theorem 2. *(Exist. of Sol.)*
For any $p > 1$ *and* $0 < \omega < \omega_p$*, there exists at least one solution. (Berestycki and Lions '83)*

$$
\begin{cases}
u_{rr} + \dfrac{n-1}{r} u_r - \omega u + u^p - u^{2p-1} = 0 \text{ in } (0,\infty), \\
u > 0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\; \text{in } [0,\infty), \\
u_r(0) = 0, \\
\lim\limits_{r\to\infty} u(r) = 0.
\end{cases}
$$

$$
u : [0,\infty) \ni r \longrightarrow u(r) \in \mathbb{R}, \quad C^2.
$$

Theorem 3. *(Nature of the rad. sol.)*

$$u' < 0 \text{ in } (0, \infty),$$

$$\lim_{r \to \infty} u'(r) = 0,$$

$$\lim_{r \to \infty} \frac{u'(r)}{u(r)} = -\sqrt{\omega},$$

u *has exp. decay at infty.*,

(Peletier and Serrin '83)

Remark 2. (Non Exist. of Sol.)
For any $p > 1$ and $\omega \geq \omega_p$, there can not exist a solution. (Peletier and Serrin '83)

Proof. Remark that $\omega \geq \omega_p$ is equivalent to $F(u) := \int_0^u f(s) \leq 0$ for any $u > 0$.
Assume that there exists a sol. u.
For the Energy

$$E(r) := \frac{u'(r)^2}{2} + F(u(r)),$$

We have

$$E'(r) = u''(r)u'(r) + f(u)u'(r) = -\frac{n-1}{r}u'(r)^2,$$

$$\lim_{r \to \infty} E(r) = 0.$$

These facts assert $E(0) = F(u(0)) > 0$. $\square$

Theorem 4. *(Uniqueess)*
The solution is unique for ω *which is 'sufficiently close' to* ω_p *with any* $p > 1.$ *(Mizumachi)*

Proof. Using perturbation theory. □

Remark 3. Uniqueness of rad. sol. in bouded domain is proved by Ouyang and Shi '98.

3 Main Result

Theorem 5. *The solution is unique for*

$$a_p := \frac{p(7p-5)}{4(p+1)(2p-1)^2} \leq \omega < \omega_p$$

with any $p > 1$.

4 Outline of Proof

Lemma 1. *The condition $a_p \leq \omega$ is equivalent to*

$$\alpha \leq \beta,$$

where $\alpha = \left[\frac{p}{2(2p-1)}\right]^{\frac{1}{p-1}}$ *is the unique inflection of our nonlinearity* f *and*

$\beta = \left[\frac{p}{p+1}\left(1 - \sqrt{1 - \frac{(p+1)^2}{p}\omega}\right)\right]^{\frac{1}{p-1}}$ *is the first zero of the integral* F.

Lemma 2. *(Peletier and Serrin '83)*
Let f satisfy the following condition;

$$G(u) := \frac{f(u)}{u-\beta} \text{ is nonincreasing in } (\beta, c),$$

where $c = \left[\frac{1+\sqrt{1-4\omega}}{2}\right]^{\frac{1}{p-1}}$ *is the second zero of the nonlinearity* f.
Then our ODE has exactly one positive solution.

Proof. (of the Theorem.)
It is enough to show that if $\alpha \leq \beta$, then

$G'(u)(u-\beta)^2 = f'(u)(u-\beta) - f(u) := k(u) \leq 0$

We calculate the derivative of $k(u)$

$$k'(u) = f''(u)(u-\beta),$$

and note that

$$f''(u) > 0 \qquad \text{in } (0, \alpha);$$
$$f''(u) < 0 \qquad \text{in } (\alpha, \infty).$$

So if $\alpha \leq \beta$, then $k'(u) < 0$ in (β, c),
i.e. k is decreasing in the interval.
Therefore

$$k(u) < k(\beta) = -f(\beta) < 0 \quad \text{for any } u \in (\beta, c).$$

This completes the proof. $\square$

5 Remark

If $\alpha > \beta$, we need to check that $k(\alpha) \leq 0$, i.e.

$$\frac{\alpha f'(\alpha) - f(\alpha)}{f'(\alpha)} \leq \beta. \tag{1}$$

This condition provides an implicit relation between ω and p. Besides,

Remark 4. The condition (1) does not cover all $\omega \in (0, \omega_p)$. That is for ω close to zero,

$$\frac{\alpha f'(\alpha) - f(\alpha)}{f'(\alpha)} > \beta.$$

Proof. The left hand side is estimated from below as

$$\frac{\alpha f'(\alpha) - f(\alpha)}{f'(\alpha)} = \frac{(p-1)\alpha^p(1-2\alpha^{p-1})}{-\omega + p\alpha^{p-1} - (2p-1)\alpha^{2(p-1)}}$$

$$> \frac{(p-1)\alpha^p(1-2\alpha^{p-1})}{p\alpha^{p-1} - (2p-1)\alpha^{2(p-1)}} > 0,$$

for all $\omega \in (0, \omega_p)$, whereas the right hand side β decreases to zero as ω decreases to zero. $\square$