Stochastic Navier-Stokes-Coriolis Equations

Wilhelm Stannat
Fachbereich Mathematik
TU Darmstadt

stannat@mathematik.tu-darmstadt.de

Tokyo, March 13, 2010

Contents

- Navier-Stokes-Coriolis equations
- Stochastic Navier-Stokes-Coriolis equations
  (joint with M. Hieber (TU Darmstadt))
- Invariant measures
- Results in 2D
  (partly joint with A. Es Sarhir (TU Berlin))

Navier-Stokes equation with Coriolis term

unbounded layer $\mathbb{R}^2 \times (0, b), \ b > 0$

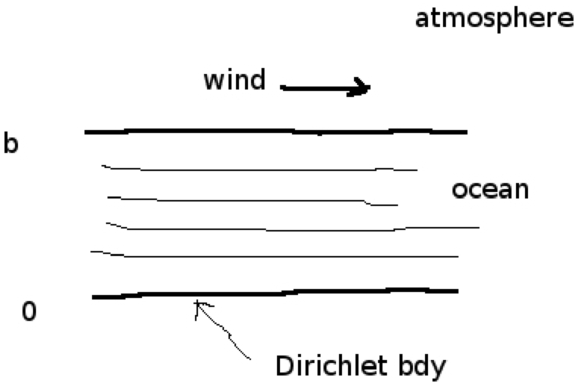
$$\begin{cases} \partial_t u - \nu \Delta u + 2\omega(e_3 \times u) + (u \cdot \nabla) u + \nabla p = 0 \\ div\ u = 0 \\ u(t, x_1, x_2, 0) = 0 \\ u(t, x_1, x_2, b) = u_b\, e_1 \\ u(0, \cdot) = u_0 \end{cases} \tag{1}$$

where

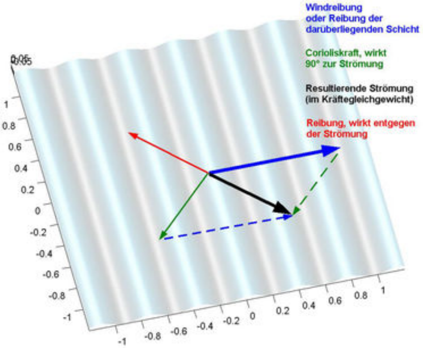
- u - velocity
- p - pressure
- $\nu > 0$ viscosity
- $\omega \in \mathbb{R}$ angular velocity
- $u_b e_1, \ u_b \in \mathbb{R}$ surface velocity

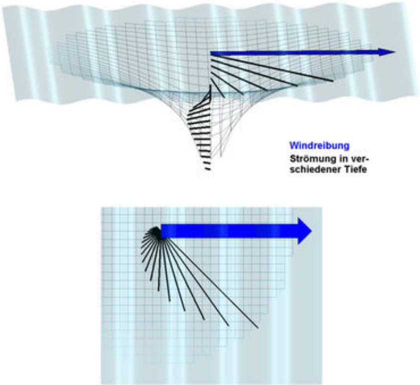
describes motion of rotating fluids under the influence of the Coriolis force

Background



Ekman spiral





(**from** Wikipedia)

Ekman spiral - analytic expression

explicit stationary solution of (1)

$$u_b^E(x_1, x_2, x_3) = \tilde{u}_b \begin{pmatrix} 1 - e^{-x_{3/\delta}} \cos\left(x_{3/\delta}\right) \\ e^{-x_{3/\delta}} \sin\left(x_{3/\delta}\right) \\ 0 \end{pmatrix}$$

$$p_b^E(x_1, x_2, x_3) = -\omega \tilde{u}_b x_2$$

with $\delta = \frac{b}{k\pi}, k \in \mathbb{Z}$, and

$$\tilde{u}_b = \begin{cases} u_b(1 - e^{-\frac{b}{\delta}})^{-1} & \text{if } k \text{ is even} \\ u_b(1 + e^{-\frac{b}{\delta}})^{-1} & \text{if } k \text{ is odd.} \end{cases}$$

Background: mathematical motivation

- (1) has a limiting equation for $|\omega| \to \infty$
- for arbitrary initial data u_0 and $|\omega| \gg 1$
  - ex. of global strong solutions (Babin, Mahalov, Nicolaenko 1999)
  - ex. and uniqueness of mild solutions (e.g.: Gallay, et al.: Global existence and Long-Time Asymptotics for Rotating Fluids in a 3D-Layer, 2009 on arXiv)

Known results: existence, stability

Function spaces

for $v = u - u_b^E$

$$\mathcal{D} := \left\{ v \in C_0^\infty \left(\mathbb{R}^2 \times (0,b)\right)^3 : \ \operatorname{div} v = 0,\ v(x_1,x_2,0) = 0 = v(x_1,x_2,b) \right\}$$

$$H := \overline{\mathcal{D}}^{L^2\left(\mathbb{R}^2\times(0,b)\right)^3} \qquad \text{and} \qquad V := \overline{\mathcal{D}}^{H^{1,2}\left(\mathbb{R}^2\times(0,b)\right)^3}$$

Known results, ctd.

M. Hess (TU Darmstadt) 2009

- existence of global weak solutions

$$u = v + u_b^E\,, \quad v \in L^\infty(0,T;H) \cap L^2(0,T;V)$$

- if

$$\lambda_0 := \frac{\pi^2}{b^2}\left(\nu - \sqrt{2}\tilde{u}_b\left(\delta - (b+\delta)e^{-\frac{b}{\delta}}\right)\right) \geq 0$$

then $\exists$ (at least one) global weak solution $u = v + u_b^E$ such that $\lim_{t\to\infty}\|v(t)\|_H = 0$, in addition

$$\|v(t)\|_H \leq e^{-\lambda t}\|v_0\|_H \text{ for any } \lambda < \lambda_0$$

Next step

- stability of u_b^E for the perturbed NSC-equation
- fluctuations around u_b^E for mean zero perturbations

Stochastic Navier-Stokes-Coriolis equations

restricting to (x_1, x_2)-periodic solutions u

$\Omega_b := \mathbb{T}^2 \times (0, b)$, $\mathbb{T}^2$ 2D-Torus

Function spaces

for $v = u - u_b^E$

$$\mathcal{D} := \left\{ v \in C^\infty (\Omega_b)^3 : \text{ div } v = 0, v(x_1, x_2, 0) = 0 = v(x_1, x_2, b) \right\}$$

$$H := \overline{\mathcal{D}}^{L^2(\Omega_b)^3} \qquad \text{and} \qquad V := \overline{\mathcal{D}}^{H^{1,2}(\Omega_b)^3}$$

$$\begin{cases} du_t = [\nu \Delta u_t - \omega(e_3 \times u_t) - (u_t \cdot \nabla) u_t + \nabla p_t] \, dt + dW_t \\ div \ u_t = 0 \\ u_t(x_1, x_2, 0) = 0 \\ u_t(x_1, x_2, b) = u_b \, e_1 \end{cases} \tag{2}$$

$(W_t)_{t \geq 0}$ Wiener process on H

The Wiener process $(W_t)_{t\geq 0}$

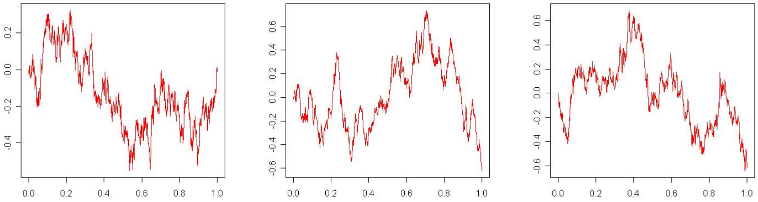
simplest example:

$$W_t = \sum_k \alpha_k \beta_k(t) e_k$$

where

- $\{e_k\}$ - complete orthonormal system of H
- $\{\alpha_k\} \subset \ell^2(\mathbb{R})$
- $\{\beta_k\}$ independent 1-dim Brownian motions on $(\Omega, \mathcal{F}, P)$

Brownian motion β_k



cont. stoch. process, such that for $0 \leq t_0 < t_1 < \ldots < t_n$ the increments

$$\beta_k(t_i) - \beta_k(t_{i-1}) \qquad i = 1, \ldots, n\,,$$

- independent
- $N(0, t_i - t_{i-1})$-distributed

fundamental for the Ito-calculus: a typical path $t \mapsto \beta_k(t)(\omega)$ is

- of unbounded variation
- but of finite quadratic variation

$$\langle \beta_k(\omega) \rangle_t := \lim_{n \to \infty} \sum_{t_i \in \tau_n} \left(\beta_k(t_{i+1})(\omega) - \beta_k(t_i)(\omega) \right)^2 = t$$

Stochastic evolution equation

orthogonal-projection $\Pi : L^2(\Omega_b)^3 \to H$, applied to (6), yields

$$du_t = [\nu A_S u_t - \omega \Pi(e_3 \times u_t) - \Pi(u_t \cdot \nabla) u_t]\, dt + dW_t \qquad (3)$$

on the space H

$$A_S u = \nu\, \Pi \Delta u \qquad \text{Stokes operator}$$

in the following: decompose

$$u_t = v_t + u_b^E \qquad \text{w.r.t. the Ekman spiral } u_b^E$$

then v is a solution of

$$dv_t = [(\nu A_S + B) v_t - \Pi(v_t \cdot \nabla) v_t]\, dt + dW_t$$

where

$$B^{u_b^E} v := -\omega\, \Pi(e_3 \times v) - \Pi(u_b^E \cdot \nabla v) - \Pi(v_3 \partial_3 u_b^E)\, , v \in D(A_S),$$

Perturbed stoch. Navier-Stokes eq

$$dv_t = [(\nu A_S + B)v_t - \Pi(v_t \cdot \nabla)v_t]\, dt + dW_t \tag{4}$$

Assumptions

A.1 $B : V \to H$, bounded, linear, s.th. $\exists\ \omega_0 > 0$, $\omega_1 \geq 0$ with

$$\langle(\nu A_S + B)u, u\rangle_H \leq -\omega_0 \|u\|_V^2 + \omega_1 \|u\|_H^2 \qquad \forall u \in D(A_S)\,.$$

A.2 $(W_t)_{t\geq 0}$ is a Wiener-process on H with covariance Q having finite trace

Rem A.1 implies that $(\nu A_S + B, D(A_S))$ generates an analytic C_0-semigroup $(T_t)_{t\geq 0}$ satisfying $\|T_t\|_{op} \leq e^{(\omega_1 - \tilde{\omega}_0)t}$, $t \geq 0$, $\tilde{\omega}_0 = \omega_0 \frac{\pi^2}{b^2}$

Weak martingale solutions (up to time T)

given initial distribution μ_0 on H

a probability measure P on $\Omega = C([0,T];H)$ such that the *canonical process* $v_t : \Omega \to H$, $\omega \mapsto \omega(t)$ satisfies

- $$P\left[\sup_{t\in[0,T]} \|v_t\|_H + \int_0^T \|v_s\|_V^2 \, ds < \infty\right] = 1$$

- $$\begin{aligned} M_t^\varphi &:= \langle v_t, \varphi\rangle_H \\ &\quad - \int_0^t \nu\langle v_s, A_S\varphi\rangle_H - \langle Bv_s, \varphi\rangle_H - \langle (v_s\nabla)\,\varphi, v_s\rangle_H \, ds \end{aligned}$$

  is a continuous square-integrable martingale with

  $$\langle M^\varphi\rangle_t = \|\sqrt{Q}\varphi\|_H^2 \cdot t \qquad \text{for all } \varphi \in \mathcal{D}$$

- $P \circ v_0^{-1} = \mu_0$

Main existence result

Theorem (S., Hieber 2009) Let $\xi : \Omega \to H$ be $\mathcal{F}_0$-measurable, in $\mathcal{L}^2(P)$, independent of $(W_t)_{t\geq 0}$.

Then $\exists$ weak martingale solution $((v_t), (\tilde{W}_t))$ of (4) satisfying $\tilde{P} \circ v_0^{-1} = P \circ \xi^{-1}$

Moreover

$$\tilde{E}\left(\sup_{0\leq t\leq T} \|v_t\|_H^2 + \int_0^T \|v_t\|_V^2 \, dt\right) < \infty$$

proof similar to the unperturbed case (e.g. Flandoli, Gatarek [PTRF 1995]) via Galerkin approximation, tightness, ...

Corollary $B = B^{u_b^E}$, then $u_t := v_t + u_b^E$ is a weak martingale solution of (3).

stationary solutions

a stochastic process $(v_t)_{t \geq 0}$ is called **stationary** if

$$\tilde{P} \circ v_{t+\cdot}^{-1} = \tilde{P} \circ v_{\cdot}^{-1} \qquad t \geq 0$$

in this case

$$\mu = \tilde{P} \circ v_t^{-1} \qquad \text{is time-invariant}$$

μ describes the long-time fluctuations/statistics of $(v_t)_{t \geq 0}$

stationary martingale solutions of (4)

consider again the perturbed stochastic Navier-Stokes equation (4)

assume

$$\langle(\nu A_S + B)u, u\rangle_H \leq -\tilde{\omega}_0\|u\|_V^2 \quad \forall u \in D(A_S)$$

with $\tilde{\omega}_0 := \omega_0 - \frac{1}{\nu}\frac{b^2}{\pi^2}\omega_1 > 0$

Theorem (S., Hieber, 2009) $\exists$ a stat. martingale solution of (4)
Moreover, the invariant distribution $\mu = \tilde{P} \circ v_t^{-1}$ satisfies the moment estimate

$$\int \left(1 + \|v\|_V^2\right) e^{\varepsilon\|v\|_H^2}\mu(dv) < \infty \qquad \text{for } \varepsilon < \varepsilon_0 := \frac{\tilde{\omega}_0}{\|Q\|_{op}}$$

open dependence on ω

at least: exponential moments ind. of ω

Navier-Stokes-Coriolis eq on $\mathbb{T}^2$

appropriate projection of (1) onto (x_1, x_2)-plane

$$\begin{cases} \partial_t u - \nu \Delta u + \ell e_3 \times u - (u \cdot \nabla) u + \nabla p = 0 \\ \operatorname{div} u = 0 \\ u(0, \cdot) = u_0 \end{cases} \tag{5}$$

where

- $\ell = \omega + \beta x_2$
- $$\ell e_3 \times u = \omega \begin{bmatrix} -u_2 \\ u_1 \end{bmatrix} + \beta x_2 \begin{bmatrix} -u_2 \\ u_1 \end{bmatrix}$$

ex. & uniq. well-known

Stoch. Navier-Stokes-Coriolis eq on $\mathbb{T}^2$

$$\begin{cases} du_t = [\nu\Delta u_t - \ell e_3 \times u_t - (u_t \cdot \nabla)u_t + \nabla p_t]\, dt + dW_t \\ div\; u_t = 0 \end{cases} \tag{6}$$

$(W_t)_{t\geq 0}$ Wiener process on $H := \overline{\mathcal{D}}^{L^2(\mathbb{T}^2)^2}$

where

$$\mathcal{D} := \left\{u \in C^\infty\left(\mathbb{T}^2\right)^2 : \text{ div } u = 0, \int_{T^2} u\, dx = 0\right\}$$

Stoch. Navier-Stokes-Coriolis eq on $\mathbb{T}^2$, ctd.

applying the Helmholtz/Leray-projection yields

$$du_t = [\nu A_S u_t + Bu_t - \Pi (u_t \cdot \nabla) u_t] \, dt + dW_t \tag{7}$$

where

$$Bu = \Pi \left(\omega \begin{bmatrix} -u_2 \\ u_1 \end{bmatrix} + \beta x_2 \begin{bmatrix} -u_2 \\ u_1 \end{bmatrix} \right) = \beta \Pi \left(x_2 \begin{bmatrix} -u_2 \\ u_1 \end{bmatrix} \right)$$

Main features of B

- $\langle Bu, u \rangle = 0$ (cons. of energy)
- $\langle Bu, A_S u \rangle = 0$ (cons. of enstrophy)

Rem ex. of (stationary) weak martingale solution (should be) similar to 3D (not done)

Direct construction of invariant measures

characterization via the ass. Kolmogorov operator

$$LF(u) := \frac{1}{2}\,\mathrm{tr}\left(QD^2F(u)\right) + \langle \nu A_S u + Bu - \Pi(u\cdot\nabla)u, DF(u)\rangle$$

$$F \in \mathcal{F}C_b^2 := \{F(u) = \varphi(\langle e_1, u\rangle, \ldots, \langle e_n, u\rangle) : n \geq 1,$$
$$e_1, \ldots, e_n \in D(A_S), \varphi \in C_b^2(\mathbb{R}^n)\}$$

Fact μ invariant for (7) implies μ infinitesimally invariant for L, i.e.,

$$L(\mathcal{F}C_b^2) \subset L^1(\mu) \qquad \text{and} \qquad \int LF\,d\mu = 0 \quad \forall F \in \mathcal{F}C_b^2$$

Invariant measures - finite trace

Theorem (S. 2009)

- $\mathrm{tr}_H(Q) < \infty \Longrightarrow \exists\ \mu$ inf. invariant for L with

$$\int \left(1 + \|u\|_V^2\right) e^{\varepsilon \|u\|_H^2}\, \mu(du) < \infty$$

if $\varepsilon < \frac{\nu}{\|Q\|_{L(H)}}$

- $\mathrm{tr}_V(Q) < \infty \Longrightarrow \mu$ satisfies in addition

$$\int \left(1 + \|u\|_{H^2}^2\right) e^{\varepsilon \|u\|_V^2}\, \mu(du) < \infty \tag{8}$$

if $\varepsilon < \frac{\nu}{\|Q\|_{L(V)}}$

L^p-uniqueness - finite trace

Theorem (S. 2009)
Let μ be inf. invariant, satisfying (8), then $(L, \mathcal{F}C_b^2)$ is L^p-unique for all $p \in [1, \infty[$

basic ingredient: pointwise gradient estimate for finite-dim. Galerkin approx. L^N of L

$$\|DT_t^N F(u)\|_H \le \exp\left(\frac{\varepsilon\nu}{4(\nu^2 - \varepsilon\|\sqrt{Q}\|^2_{L(H^2,V)})} \|u\|_V^2 + C(\varepsilon, Q, \nu)\, t \right) \|F\|_{Lip(H)}$$

uniformly in N

Rem $B = 0$ result obtained by Barbu, Da Prato (2004)

white noise $Q = id$

recall invariants for $\Pi(u \cdot \nabla)u)$ and B

- energy $\langle \Pi(u \cdot \nabla)u, u\rangle = 0 = \langle Bu, u\rangle$
- enstrophy $\langle \Pi(u \cdot \nabla)u, A_S u\rangle = 0 = \langle Bu, A_S u\rangle$

implies

$$L_{\sigma,\nu}F(z) = \frac{\sigma^2}{2}\text{tr } (D^2F)(u) + \langle \nu A_S u + Bu - \Pi(u \cdot \nabla)u, DF(u)\rangle$$

has (inf.) invariant measure $\mu_{2,\sigma,\nu} = N(0, \frac{\sigma^2}{\nu}(-A_S)^{-1})$

(independent of ω)

A priori estimate in the case $C = 0$

Theorem (S. 2007, IDAQP) Let $\alpha \in (-1, 0)$. Then there exists $C(\alpha)$ such that:

- If $\nu^3 > \sigma^2 C(\alpha)$ for some $\alpha \in (-1, 0)$ then the closure $\overline{L}_{\sigma,\nu}$ generates a Markovian C_0-semigroup on $L^1(\mu_{2,\sigma,\nu})$.
- The associated resolvent $\overline{R}_{\sigma,\nu,\lambda} = \left(\lambda - \overline{L}_{\sigma,\nu}\right)^{-1}$, $\lambda > 0$, satisfies

$$\int |D\overline{R}_{\sigma,\nu,\lambda}F(u)|^2_{H^{2+\alpha}}\,\mu(du) \le \frac{\nu^2}{\lambda\,(\nu^3 - \sigma^2 C(\alpha))} \int |DF(u)|^2_{H^{1+\alpha}}\,\mu(du)$$

  in part. $\overline{R}_{\sigma,\nu,\lambda}$ operates as a bounded operator

$$\overline{R}_{\sigma,\nu,\lambda} : H^{1,2}_{H^{1+\alpha}}(\mu_{2,\sigma,\nu}) \to H^{1,2}_{H^{2+\alpha}}(\mu_{2,\sigma,\nu})$$

  for $\alpha \in (-1, 0)$.

Consequences

- existence of diffusions generated by $\overline{L}_{\sigma,\nu}$ using Dirichlet form techniques
- uniqueness of ass. martingale problem

A priori estimate in the case $C = 0$

Theorem (S. 2007, IDAQP) Let $\alpha \in (-1,0)$. Then there exists $C(\alpha)$ such that:

- If $\nu^3 > \sigma^2 C(\alpha)$ for some $\alpha \in (-1,0)$ then the closure $\overline{L}_{\sigma,\nu}$ generates a Markovian C_0-semigroup on $L^1(\mu_{2,\sigma,\nu})$.
- The associated resolvent $\overline{R}_{\sigma,\nu,\lambda} = \left(\lambda - \overline{L}_{\sigma,\nu}\right)^{-1}$, $\lambda > 0$, satisfies

$$\int |D\overline{R}_{\sigma,\nu,\lambda}F(u)|^2_{H^{2+\alpha}}\,\mu(du) \le \frac{\nu^2}{\lambda\left(\nu^3 - \sigma^2 C(\alpha)\right)} \int |DF(u)|^2_{H^{1+\alpha}}\,\mu(du)\,.$$

$B \neq 0$ **conjecture** for ν^3/σ^2 sufficiently large

- the same integr. gradient estimates hold
- $(L_{\sigma,\nu}, \mathcal{F}C_b^2)$ is L^1-unique

NB

- invariant measure is independent of angular velocity ω
- can therefore study the limit $|\omega| \to \infty$ in $L^1(\mu_{2,\sigma,\nu})$

rough noise $Q = (-A_S)^{-\frac{1}{2}-\varepsilon}$

ex. of (stationary) weak martingale solution (should be) similar to the case without Coriolis forcing term (not done)

ex. proof for the stoch. Navier-Stokes without Coriolis forcing term by Flandoli, Gatarek [PTRF, 1995] suggests

$$\int \log\left(1 + \|u\|^2\right) \, \mu(du) < \infty$$

for an invariant distribution

can be improved to

$$\Longrightarrow \qquad \int \left(1 + \|u\|_\gamma^2\right) \|u\|_H^{2m} \, \mu(du) < \infty \qquad \forall m \geq 0 \, , \gamma < \frac{1}{4} + \varepsilon$$

cf. **Es-Sarhir, S., 2009**, (to appear in JFA) Improved moment estimates for inv. measures of semilinear diffusions in Hilbert spaces and applications

improved moment estimates

improved moment estimates for spde of the type

$$dX_t = (AX_t + B(X_t))\, dt + \sqrt{Q}\, dW_t \text{ on } H$$

A self-adjoint, of negative type, compact resolvent satisfying

$$\langle Ay + B(y+w), y\rangle \le -\alpha_1 \|y\|^2_{V_{\frac{1}{2}}} + \alpha_2 \|w\|^s_{V_{\gamma_0}} + \alpha_3 \|w\|^s_{V_{\gamma_0}} \|y\|^2_{V_{\gamma_1}} + \alpha_4$$

Rem $\alpha_3 = 0$: exponential moments (cf. Es Sarhir, S., JEE 2008)

$\alpha_3 > 0$ - main ingredient

Prop (Da Prato, Debussche, ...) Let $\delta \in]0, \frac{1}{2}[$ and $\gamma \in \mathbb{R}$. Then

$$\sup_{0 \le t \le T} \|W_{A-\lambda}(t)\|^2_{V_\gamma} \le C_\delta^2 \sum_{k=1}^{\infty} \frac{\lambda_k^{2\gamma}}{(\lambda + \lambda_k)^{2\delta}} M_k(\delta, T)^2$$

where

- $W_{A-\lambda}(t) = \int_0^t e^{(t-s)A} \sqrt{Q}\, dW_s$
- (λ_k) - eigenvalues of $-A$
- $$M_k(\delta, T) := \sup_{0 \le s < t \le T} \frac{|\beta_k(t) - \beta_k(s)|}{|t-s|^\delta}$$

  ind. r.v., having finite moments of any order. Here,

  $$\beta_k(t) := \langle W(t), e_k \rangle \qquad (e_k)_k \text{ eigenfunctions of } A$$

  ind. 1-dim Brownian motions

in part.: if for some $\varepsilon > 0$

$$\sum_{k=1}^{\infty} \lambda_k^{-2(\delta-\gamma-\varepsilon)} < \infty$$

then

$$\sup_{0 \le t \le T} \|W_{A-\lambda}(t)\|_{V_\gamma}^2 \le \lambda^{-2\varepsilon} M_{\delta,\gamma,\varepsilon}$$

where $M_{\delta,\gamma,\varepsilon}$ ind. of λ, finite moments