Stochastic Power Law Fluids : the Existence and the Uniqueness of the Weak Solution*

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0. Motivation

• Object:
Dynamics of viscous, incompressible fluids subject to a **random** external forcing.

• The most studied model so far:
A stochastic PDE called the **stochastic Navier-Stokes eq.(SNS)**:

• The **stochastic power law fluids** $(\mathrm{SPLF})_p$, $p > 0$ are generalization of (SNS) such that:

$$(\mathrm{SPLF})_p = \begin{cases} \text{(SNS)} & \text{for } p = 2, \\ \text{stochastic } \textbf{non-Newtonian} \text{ fluid} & \text{for } p \neq 2. \end{cases}$$

("shear thinning" for $p < 2$, "shear thickening" for $p > 2$).

Newtonian vs. non-Newtonian fluids

- $u = (u_j)_{j=1}^d$: the velocity field of the fluid.

- The force exerted to the fluid per volume is given by:

$$-\nabla \Pi + \mathrm{div}\, \tau \in \mathbb{R}^d,$$

where

$$\begin{array}{lcll} \Pi & = & \Pi(u) \in \mathbb{R} & (\textbf{pressure}) \\ \tau & = & \tau(u) \in \mathbb{R}^d \otimes \mathbb{R}^d & (\textbf{extra stress}) \\ \mathrm{div}\, \tau & = & \left(\sum_{j=1}^d \partial_j \tau_{ij}\right)_{i=1}^d \in \mathbb{R}^d & (\textbf{friction}) \end{array}$$

- $\tau(u)$ is a function of the (symmetrized) **velocity gradient**:

$$e(u) = \left(\frac{\partial_i u_j + \partial_j u_i}{2}\right) \in \mathbb{R}^d \otimes \mathbb{R}^d.$$

• **Newtonian** fluids (e.g., air, water,...) are characterized by:

$$\textbf{Stokes' law: } \tau(u) = 2\nu e(u) \quad (\nu = \text{viscosity} > 0).$$

This, together with div $u = 0$ implies that:

$$\text{div } \tau(u) = \nu \Delta u \ \ (\Rightarrow \textbf{Navier-Stokes eq.}).$$

• More generally, the viscosity can be variable in $|e(u)|$:

$$\tau(u) = 2 \underbrace{F(|e(u)|)}_{\text{viscosity} > 0} e(u).$$

Many **non-Newtonian** fluids ($F \not\equiv$const.) are applied in science and engineering. Typical ones are:

• **Shear thinnig** fluid: (F is $\searrow$ in $|e(u)|$)

automobile engine oil, pipelines for crude oil,...

• **Shear thickening** fluid: (F is $\nearrow$ in $|e(u)|$)

bullet proof vests, automobile 4WD systems,...

• **PDE results for non-Newtonian fluids**:

[Málek, Nečas, Rokyta, Růžička, 1996]

$\exists$ sol. and $\exists 1$ sol. in some cases.

Plan for the rest of the talk:

1. The stochastic power law fluid (SPLF)

2. The existence thm for a weak solution

3. The proof of the existence thm

4. Future works

1. The stochastic power law fluid (SPLF)

• Container of the fluid:

▶ $\mathbb{T}^d = (\mathbb{R}/\mathbb{Z})^d \cong [0,1]^d$

Given the velocity field $u = (u_j)_{j=1}^d$, the extra stress is given by:

▶ $\tau(u) = 2\underbrace{(1+|e(u)|^2)^{\frac{p-2}{2}}}_{\text{viscosity}} e(u) : \mathbb{T}^d \to \mathbb{R}^d \otimes \mathbb{R}^d,$

with $p \in (1,\infty)$.

Note

$$p \begin{cases} < 2 & \text{"shear thinning"} \\ = 2 & \text{"Newtonian"} \\ > 2 & \text{"shear thickening"} \end{cases}$$

• <u>SPDE for Stochastic Power Law Fluids</u>:

▶ $u = (u_i(t,x))_{i=1}^d$: velocity of the fluid,

▶ $\Pi = \Pi(t,x)$: pressure,

▶ $W = (W_i(t,x))_{i=1}^d$: BM in $L^2(\mathbb{T}^d \to \mathbb{R}^d)$, trace class cov., div $W = 0$.

▶ $(\text{SPLF})_p$:

2) div $u = 0$ (imcompressible);

2') $$\underbrace{\partial_t u + (u \cdot \nabla) u}_{\text{``acceleration''}} = -\nabla \Pi + \underbrace{\operatorname{div} \tau(u)}_{\text{``friction''}} + \partial_t W.$$

$$(\text{SPLF})_p = \begin{cases} (\text{SNS}) & \text{for } p = 2, \\ \text{stochastic } \textbf{non-Newtonian} \text{ fluid} & \text{for } p \neq 2. \end{cases}$$

2. The existence thm for the weak solution.

- Test Functions:

  ▶ $\mathcal{V}$ = "div-free smooth vector fields"
  $= \{v : \mathbb{T}^d \to \mathbb{R}^d$; trigono. polynom., div $v = 0\}$.

- Spaces of the solutions:

  ▶ $V_{p,\alpha}$ = "Sobolev space. of div-free vector. fields"
  $= \| \cdot \|_{p,\alpha}$-completion of $\mathcal{V}$,

where $p \in [1, \infty)$, $\alpha \in \mathbb{R}$ and:

$$\|v\|_{p,\alpha}^p = \int_{\mathbb{T}^d} |(1 - \Delta)^{\alpha/2} v|^p.$$

• <u>The weak solution</u>:

▶ $\mu_0 \in \mathcal{P}(V_{2,0})$ = prob.'s on $V_{2,0}$.

▶ $(u, W) = ((u_t, W_t))_{t \geq 0}$: a process s.t.

$$u \in L_{p,\mathrm{loc}}(\mathbb{R}_+ \to V_{p,1}) \bigcap L_{\infty,\mathrm{loc}}(\mathbb{R}_+ \to V_{2,0}) \bigcap C(\mathbb{R}_+ \to V_{p' \wedge 2, -\beta}), \ \ \exists \beta > 0,$$

W : BM in $V_{2,0}$, trace class cov. Γ.

▶ (u, W) is a **weak sol.** to (SPLF)$_p$ with init. law μ_0, if:

3) $P(u_0 \in \cdot) = \mu_0$,

3') $\langle \varphi, u_t - u_0 \rangle = \int_0^t (\langle u_s, (u_s \cdot \nabla)\varphi \rangle - \langle e(\varphi), \tau(u_s) \rangle) \, ds + \langle \varphi, W_t \rangle, \ \ \forall \varphi \in \mathcal{V}.$

Remark: 2),2') $\stackrel{\text{IBP}}{\Longrightarrow}$ 3'), when Π disappears, since

$$\langle \varphi, \nabla \Pi \rangle = -\langle \operatorname{div} \varphi, \Pi \rangle = 0.$$

Theorem 1 Suppose:

- $p \in \exists I_d$ (e.g., $I_2 = (3/2, \infty)$, $I_3 = (9/5, \infty)$, $I_4 = (2, \infty)$,...)
- $\mu_0 \in \mathcal{P}(V_{2,1})$, $\int \|v\|_{2,1}^2 \mu_0(dv) < \infty$.
- $\Delta\Gamma = \Gamma\Delta$, $\{\Gamma, \Gamma\Delta\} \subset$ trace class.

Then, $\exists$ weak sol. (u, W) to $(\mathrm{SPLF})_p$ with init. law μ_0. Moreover,

$$E\left[\sup_{t \le T} \|u_t\|_2^2 + \int_0^T \|u_t\|_{p,1}^p dt\right] \le (1+T)C < \infty.$$

Remarks:

- $d = 2, 3$, $p = 2 \Rightarrow$ result for SNS cf. [Flandoli 2008] and ref.'s therein.

- $W \equiv 0 \Rightarrow$ PDE result [Málek et al. '96].

- Pathwise uniqueness: OK for $p \geq \frac{d+2}{2}$.

(looks VERY hard for $p < \frac{d+2}{2}$, e.g. 3D NS.)

Technical difference: (S)PLF $\leftrightarrow$ (S)NS

• (S)PLF is L_p (Banach sp.)-theory as opposed to L_2 (Hilbert sp.)-theory for (S)NS.

• Extra non-linearity in the friction term div $\tau(u)$: the proofs of some a priori bounds are much harder to get.

3. The proof of the existence thm.

Step 1 (Galerkin approximation)

• Set up finite dim. subspaces $\mathcal{V}^{(n)} \nearrow \mathcal{V}$.

• Solve an approximating eq. "(SPLF)$_{p,n}$" in $\mathcal{V}^{(n)}$.
$\Rightarrow \exists 1$ sol. $u^{(n)} \in \mathcal{V}^{(n)}$.

Step 2 (A priori bds)

• Establish some a priori bds for $u^{(n)}$ unif in n, e.g.,

$$\sup_{n \geq 1} E\left[\sup_{t \leq T} \|u_t^{(n)}\|_2^2 + \int_0^T \|u_t^{(n)}\|_{p,1}^p dt\right] \leq (1+T)C < \infty.$$

Technique:
Itô calculus, Martingale ineq.'s (e.g.,B-D-G),
Sobolev imbedding.

Step 3 (Tightness)

• Prove the tightness (i.e, relative compactness of the laws) of $u^{(n)}$, $n \geq 1$ in Sobolev sp.'s of the form:

$$X = L_{p_1,\alpha_1}([0,T] \to V_{p_2,\alpha_2})$$

so that

$$u^{(n)} \stackrel{n\to\infty}{\longrightarrow} \exists u \text{ in law along a subseq.}$$

To this end, we choose a subspace $X_1 \subset X$ s.t.

a) $X_1 \hookrightarrow X$ compactly (cpt imbedding thm's for Sobolev sp's).

b) $\exists \delta > 0,\ \sup_n E[\|u^{(n)}\|_{X_1}^{\delta}] \leq C_T$ (A priori bds used here).

a) $X_1 \hookrightarrow X$ <u>compactly</u> (cpt imbedding thm's for Sobolev sp's).

b) $\exists \delta > 0$, $\sup_n E[\|u^{(n)}\|_{X_1}^{\delta}] \le C_T$ (A priori bds used here).

Then, the desired tightness in X can be seen as follows:

$$\{u \in X_1 \, ; \, \|u\|_{X_1} \le R\} \overset{\text{a)}}{\subset\subset} X,$$

$$\sup_n P(\|u^{(n)}\|_{X_1} > R) \overset{\text{Chebyshev}}{\le} \frac{\sup_n E[\|u^{(n)}\|_{X_1}^{\delta}]}{R^{\delta}} \overset{\text{b)}}{\le} \frac{C_T}{R^{\delta}} \xrightarrow{R\to\infty} 0.$$

Step 4 (Verification of SPDE)

- By Step 3,

$$u^{(n)} \xrightarrow{n\to\infty} \exists u \text{ in law along a subseq.}$$

Then, $\exists$ BM W s.t. (u, W) is a w-sol. to (SPLF)$_p$.

4. Future works

• Invariant measure
(an example of "non-equilibrium steady state")

• Ergodicity
(a starting point to discuss the turbulence)

• (In the distant future ??)
Approach to Kolmogorov's K41 theory,
Onsager conjecture