International Conference on Mathematical Fluid Dynamics
March 10-13, 2010 (Mar 13)
63 bldg. Nishi-Waseda Campus
Waseda University, Tokyo, Japan

Lagrangian approach to wave interactions on vortices and weakly nonlinear stability of an elliptical flow

Yasuhide Fukumoto
Faculty of Mathematics,
Kyushu University, Fukuoka, Japan

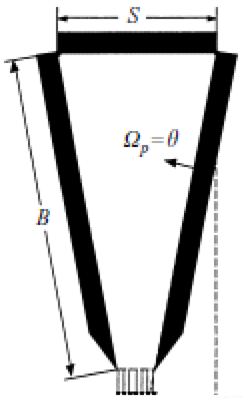
with

Makoto Hirota
Japan Atomic Energy Agency

Youichi Mie
Graduate School of Mathematics,
Kyushu University, Japan

Instability of an anti-parallel vortex pair

Leweke & Williamson: *J. Fluid Mech.* **360** (1998) 85



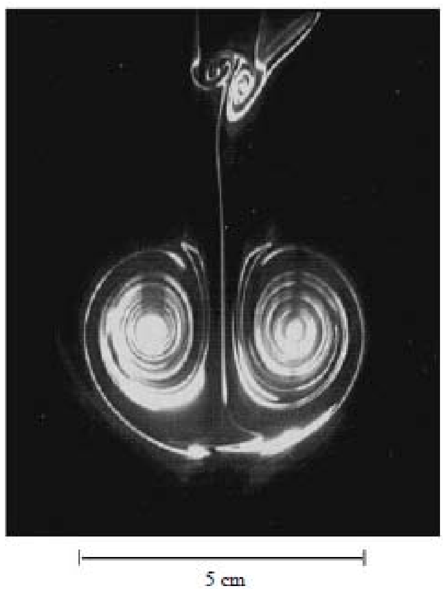


FIGURE 1. Fluorescent dye visualization of a symmetric counter-rotat
plane perpendicular to the vortex axes shortly after the end of

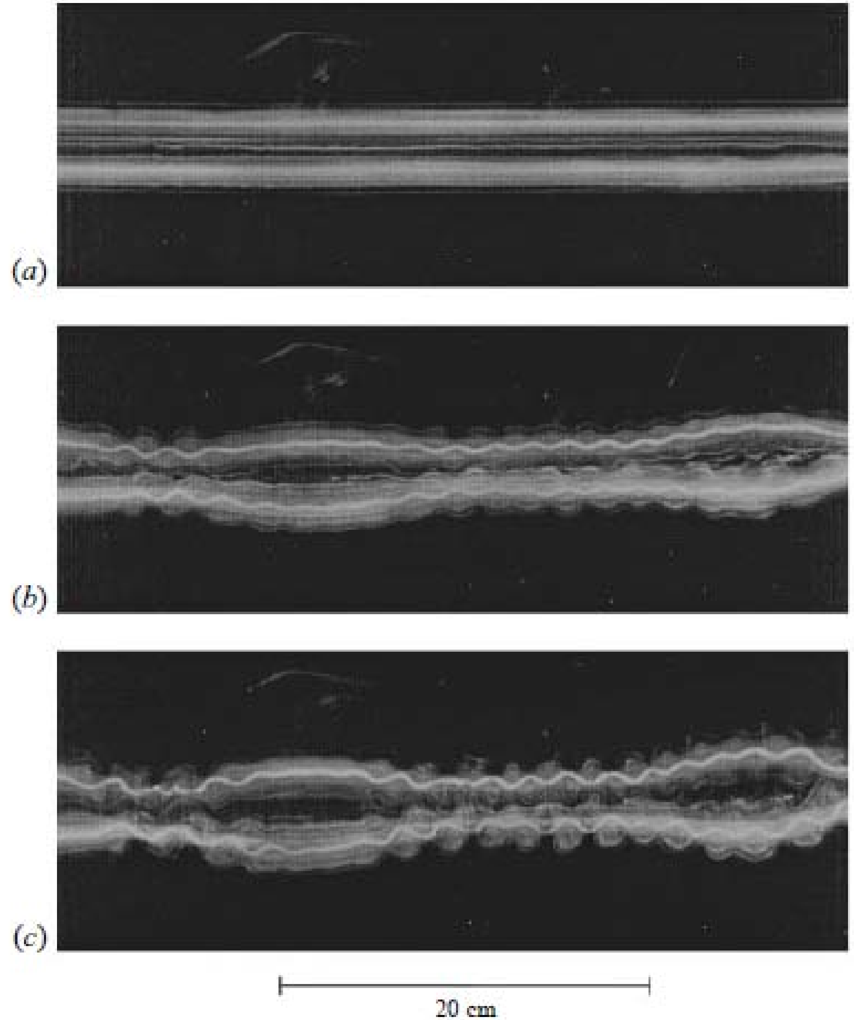


FIGURE 4. Visualization of vortex pair evolution under the combined action of long-wavelength (Crow) and short-wavelength instabilities. $Re = 2750$. The pair is moving towards the observer. (*a*) $t^* = 1.7$, (*b*) $t^* = 5.6$, (*c*) $t^* = 6.8$.

Close-up views of the short-wave instability

Leweke & Williamson: *J. Fluid Mech.* **360** (1998) 85

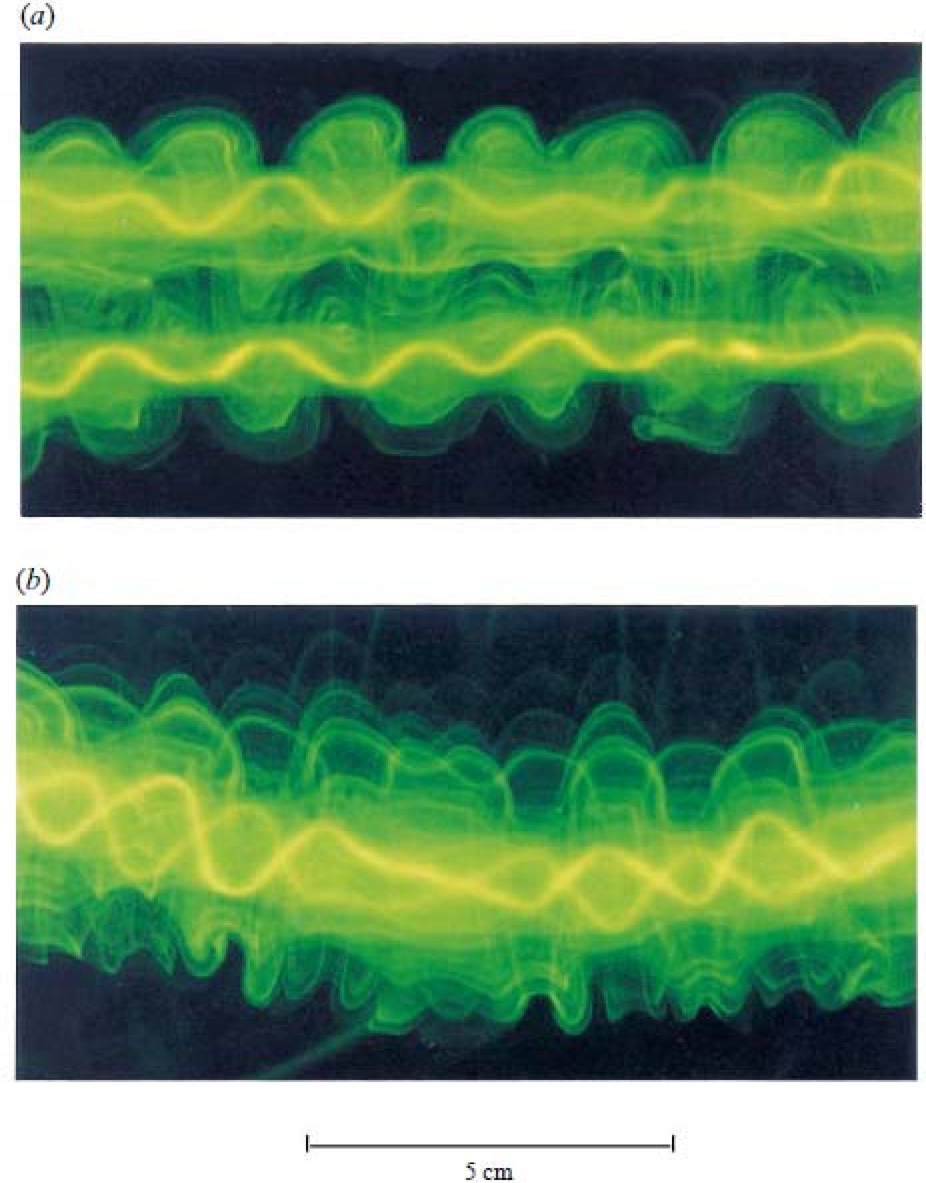


FIGURE 5. Simultaneous close-up views of the short-wavelength vortex pair perturbation in figure 4(*c*) from two perpendicular directions. $Re = 2750$, $t^* = 6.8$. (*a*) Front view (pair moving towards observer), (*b*) side view (pair moving down). The phase relation between the two vortices is clearly visible.

Weakly nonlinear stability of an elliptically strained vortex tube: Eulerian treatment

Sipp: *Phys. Fluids* **12** (2000) 1715
Waleffe: *PhD Thesis* (1989)

Lamb-Oseen vortex in a straining field

$$\mathbf{u}_0(r) = [0, r\Omega(r), 0], \tag{2}$$

$$\mathbf{u}_1(r,\theta) = [f/r \sin 2\theta, 1/2\, df/dr \cos 2\theta, 0], \tag{3}$$

$\mathbf{u}_0(r)$ represents the velocity field associated with the axisymmetrical vortex, $\Omega(r)$ designating the angular rotation. For a Lamb–Oseen vortex

$$\Omega(r) = \frac{1-\exp(-r^2)}{r^2}. \tag{4}$$

Eulerian treatment

Sipp: *Phys. Fluids* **12** (2000) 1715

$$\mathbf{u} = \underbrace{\mathbf{u}_0 + \epsilon \mathbf{u}_1 + \cdots}_{\text{Steady 2D strained vortex}}$$

$$+ \underbrace{\alpha \mathbf{u}_{01} + \alpha^2 \mathbf{u}_{02} + \alpha^3 \mathbf{u}_{03} + \epsilon\alpha \mathbf{u}_{11} + \alpha^4 \mathbf{u}_{04} + \epsilon\alpha^2 \mathbf{u}_{12} + \cdots}_{\text{Unsteady 3D perturbation}},$$

A combination of two helical waves

$$\mathbf{u}_{01}(r,\theta,z,t) = A e^{-i\theta} e^{ikz} \mathbf{u}_A(r) + B e^{+i\theta} e^{ikz} \mathbf{u}_B(r) + \text{c.c.},$$

Amplitude equations

$$\alpha^2 = \epsilon$$

$$\frac{dA}{dt} = +i\epsilon a B + i\epsilon A(b|A|^2 + c|B|^2 + dC),$$

$$\frac{dB}{dt} = -i\epsilon a A + i\epsilon B(c|A|^2 + b|B|^2 + dC),$$

$$\frac{dC}{dt} = +i\epsilon(A\bar{B} - \bar{A}B),$$

Solvability condition at $O(\alpha^2\epsilon)$

$\Rightarrow$ $\frac{\mathrm{d}C}{\mathrm{d}t}$ mean flow $\alpha^2 \mathbf{u}_C = \alpha^2 C \mathbf{t}(r)$

Indirect

Mean field exists without ellitical deformation and at any wavenumber k!

Eulerian treatment

Sipp: *Phys. Fluids* **12** (2000) 1715

Subspace

$$B = \bar{A}$$

$$\frac{dA}{d\tau} = +i\bar{A} + iA(D|A|^2 - D_0),$$

in which

$$D = D_{MF} - D_{NL}, \quad E_0 = |A_0|^2$$

$$D_0 = D_{MF}|A_0|^2.$$

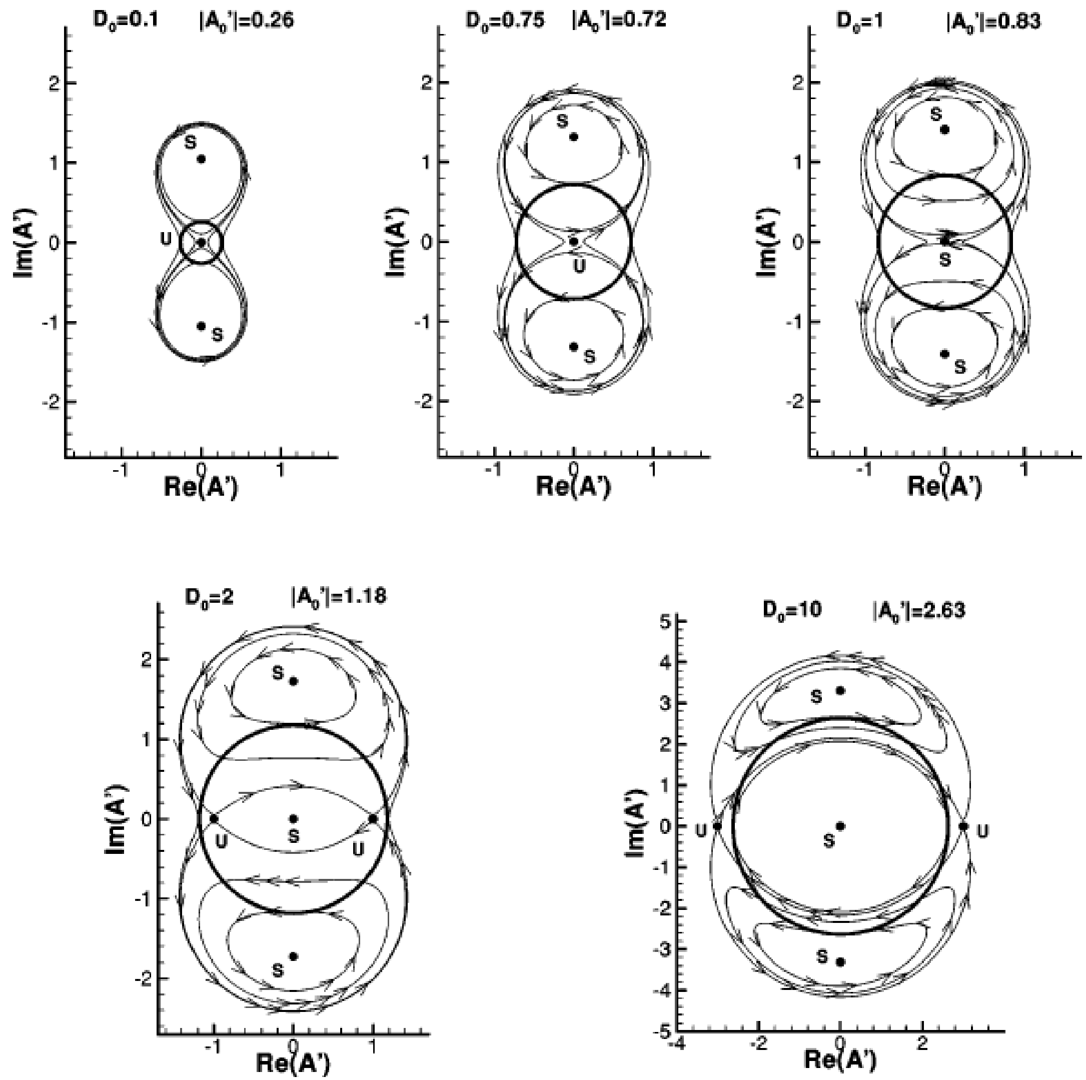


FIG. 6. Trajectories in the phase space projected on a plane $C''=cte$ in the cases $D_0=0.1,\ 0.75,\ 1,2,\ 10$. The circle in each figure represents the initial allowable conditions A'_0. Case $k=2.261$.

Hamiltonian normal form

Knobloch, Mahalov & Marsden
Physica D **73** (1994) 49

Energy of excited wave
at $O(\alpha^2)$ $\quad |A|^2 + \dot{C}' = E_0.$

$E_0 = 0$ → $D_0 = 0$

Mie & Y. F. (2010)

Contents

1. Introduction

2. Influence of a pure shear on spectra of Kelvin waves

Linear stability

Moore & Saffman ('75), Tsai & Widnall ('76)

Eloy & Le Dizés (‘01),

Y. F. ('03) *Solvable* model : an exact representation of spectra

3. Energy of Kelvin waves

Y. F. ('03) *Eulerain approach*

Hirota & Y. F. ('08a, ‘08b) ***Lagrangian approach***

”Kinematically accessible variations”

for both discrete and *continuous* spectra

⇩

4. Drift current of Kelvin waves

Y. F. & Hirota ('08)

5. Weakly nonlinear evolution to Kelvin waves

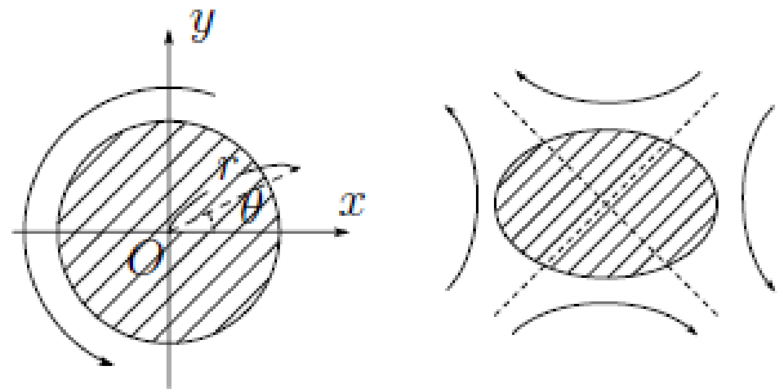
Mie & Y. F. (‘10)

Elliptically strained vortex

$$U = \varepsilon U_1(r,\theta) + \cdots \,, \quad V = V_0(r) + \varepsilon V_1(r,\theta) + \cdots \,,$$
$$\Phi = \Phi_0(\theta) + \varepsilon \Phi_1(r,\theta) + \cdots \,.$$

$O(\varepsilon^0)$ **Rankine vortex**

$$V_0 = \begin{cases} r & (r \le 1) \\ 1/r & (r > 1)\,. \end{cases}$$



$O(\varepsilon^1)$ **Pure shear**

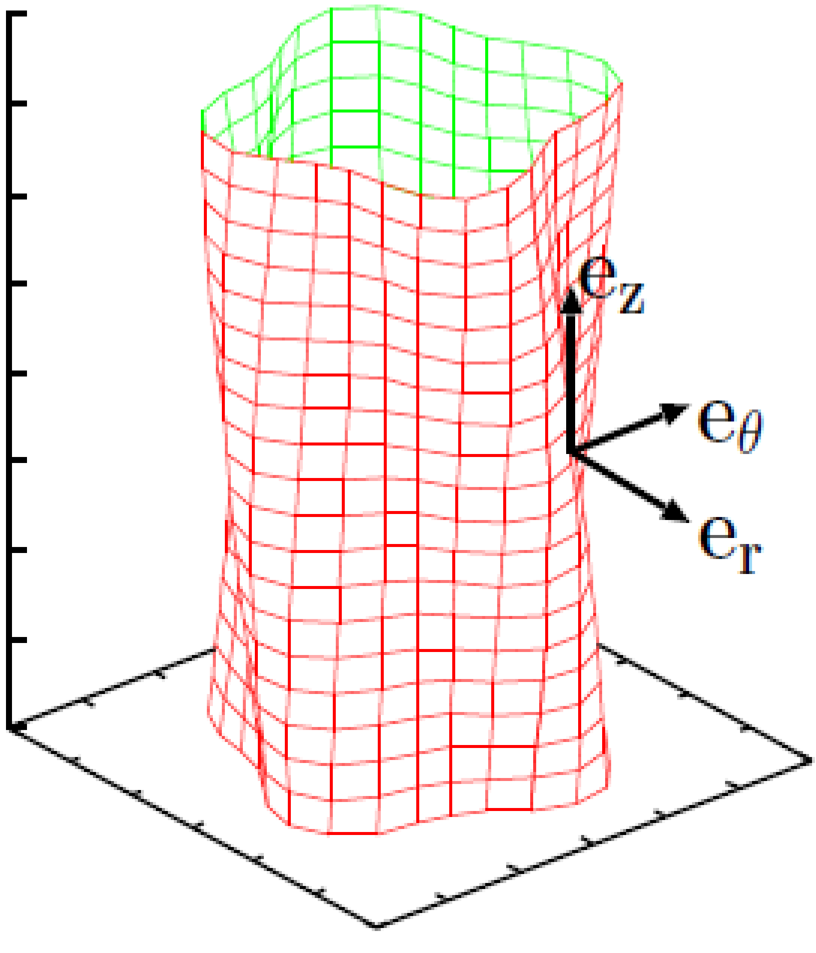
$$U_1 = -r \sin 2\theta\,, \quad V_1 = -r\cos 2\theta \quad \big(r < R(\theta,\varepsilon)\big)\,.$$

The boundary shape: $R(\theta,\varepsilon) \approx 1 + \frac{1}{2}\varepsilon \cos 2\theta$

<u>**Question**</u>: “Influence of **pure shear** upon *Kelvin waves* ?"

Example of a Kelvin wave *m=4*

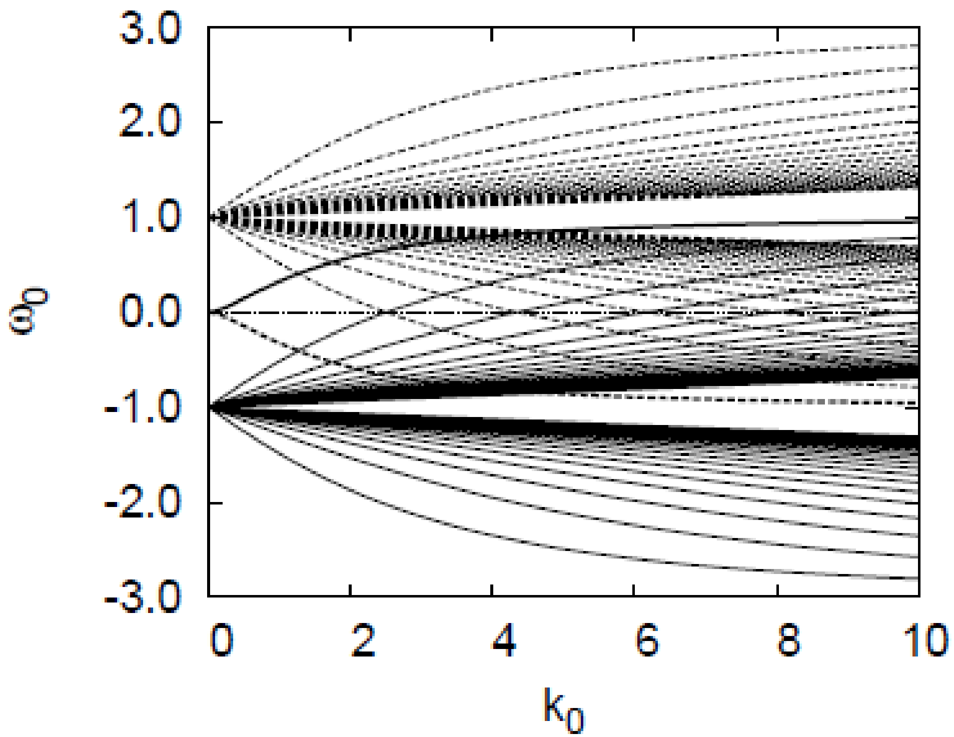
$$\tilde{\boldsymbol{u}} \propto \mathrm{e}^{\mathrm{i}(k_0 z + m\theta - \omega_0 t)}$$



Dispersion relation of Kelvin waves $m = \pm 1$

$$\eta_m J_{|m|}(\eta_m) K_{|m|-1}(k_0) - k_0 J_{|m|-1}(\eta_m) K_{|m|}(k_0) - \frac{2m(\eta_m/k_0)}{\omega_0 - m - \frac{2m}{|m|}} J_{|m|}(\eta_m) K_{|m|}(k_0) = 0$$

($J_{|m|}$ and $K_{|m|}$ are the (modified) Bessel functions)



m=-1 (solid lines) and *m=1* (dashed lines)

Equations for disturbance of $O(\varepsilon)$

$$\boldsymbol{u}_1 e^{\mathrm{i}(kz-\omega t)}; \qquad \boldsymbol{u}_1 = \{u_1, v_1, w_1, \pi_1, \phi_1\}$$

$$-\mathrm{i}\omega_0 u_1 + \frac{\partial u_1}{\partial \theta} - 2v_1 + \frac{\partial \pi_1}{\partial r} = i\omega_1 u_0 + \left(r\frac{\partial u_0}{\partial r} + u_0\right)\sin 2\theta + \frac{\partial u_0}{\partial \theta}\cos 2\theta\,,$$

$$\vdots$$

$$\frac{\partial u_1}{\partial r} + \frac{u_1}{r} + \frac{1}{r}\frac{\partial v_1}{\partial \theta} + ik_0 w_1 = -ik_1 w_0 \qquad (r<1)\,.$$

$$\frac{\partial^2 \phi_1}{\partial r^2} + \frac{1}{r}\frac{\partial \phi_1}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi_1}{\partial \theta^2} - k_0^2\phi_1 = 2k_1 k_0 \phi_0 \qquad (r>1)\,.$$

Disturbance field for the m, $m+2$ waves

Pose to $O(\varepsilon^0)$

$$\boldsymbol{u}_0 = \boldsymbol{u}_0^{(1)} e^{\mathrm{i}m\theta} + \boldsymbol{u}_0^{(2)} e^{\mathrm{i}(m+2)\theta}\,.$$

parametric resonance

$e^{2\mathrm{i}\theta}, e^{-2\mathrm{i}\theta}$ $(= \cos 2\theta, \sin 2\theta)$

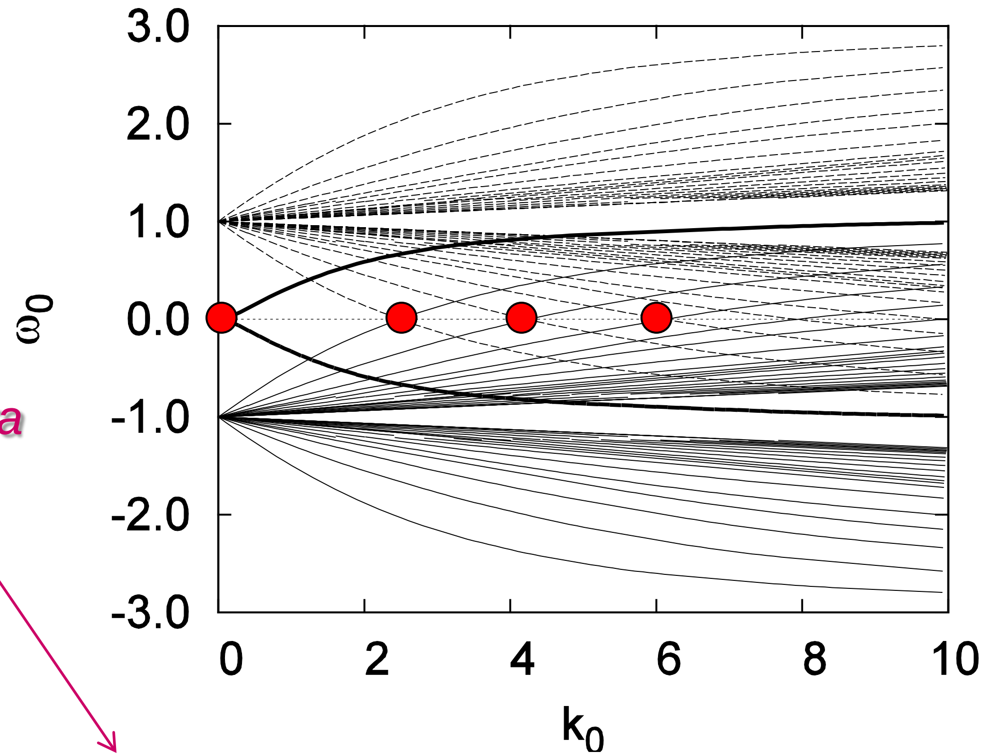
Then at $O(\varepsilon^1)$

$$\Rightarrow \boldsymbol{u}_1 = \boldsymbol{u}_1^{(1)} e^{\mathrm{i}m\theta} + \boldsymbol{u}_1^{(2)} e^{\mathrm{i}(m+2)\theta} + \boldsymbol{u}_1^{(3)} e^{\mathrm{i}(m-2)\theta} + \boldsymbol{u}_1^{(4)} e^{\mathrm{i}(m+4)\theta}$$

Growth rate of helical waves

(m=±1)

Closed form representation of spectra



For stationary mode $(\omega_0 = 0)$ the growth rate is

$$\sigma_{1\max} = \frac{3}{2}\left\{2\left(\frac{k_0K_0}{K_1}\right)^2 + \frac{3k_0K_0}{K_1} + k_0^2 + 1\right\} \Big/ \left\{6\left(\frac{k_0K_0}{K_1}\right)^2 + \frac{8k_0K_0}{K_1} + 2k_0^2 + 3\right\}$$

where $K_0 := K_0(k_0), \quad K_1 := K_1(k_0)$

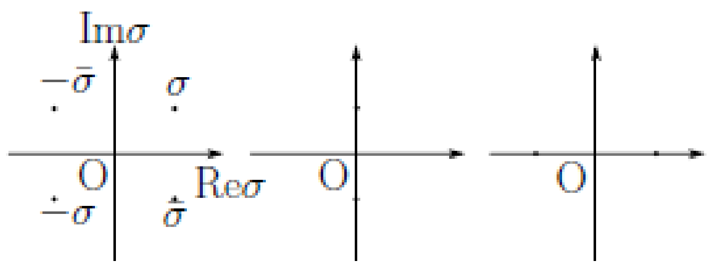
k_0	σ_{1max}	Δk_1
0	0.5	∞
2.504982369	0.5707533917	2.145502816
4.349076726	0.5694562098	3.518286549
6.174012330	0.5681222780	4.883945142
7.993536550	0.5671646287	6.247280752
9.810807288	0.5664714116	7.609553122

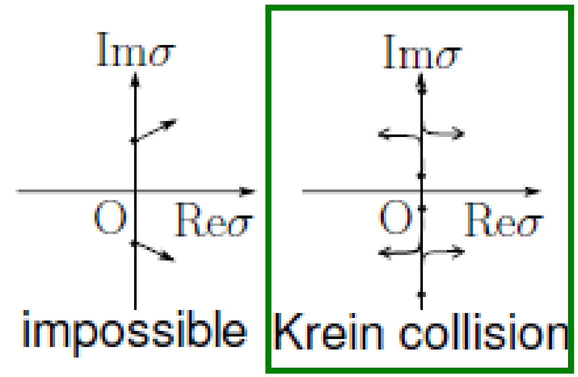
Instability occurs at ***every*** intersection points of dispersion curves of (*m*, *m*+2) waves !?

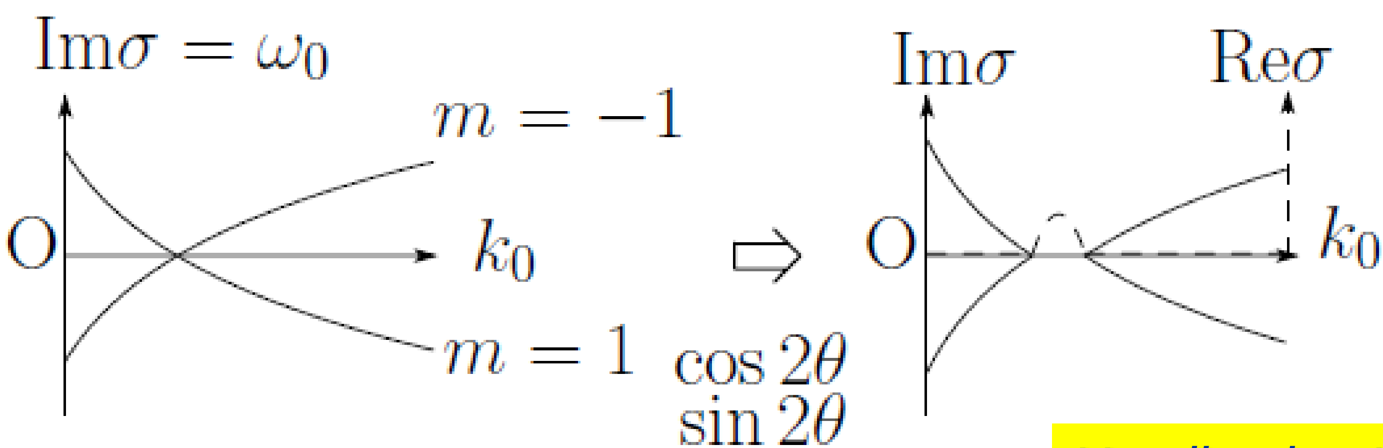
Krein's theory of Hamiltonian spectra

Spectra of a *finte*-dimensional Hamilton system

$$z(t) \propto e^{\sigma t}$$







Hamiltonian Hopf bifurcation

Wave energy: Difficulty in Eulerian treatment

base flow $\qquad$ disturbance

$$\boldsymbol{u} = \boldsymbol{U} + \tilde{\boldsymbol{u}}; \qquad \tilde{\boldsymbol{u}} = \alpha \tilde{\boldsymbol{u}}_{01} + \frac{1}{2}\alpha^2 \tilde{\boldsymbol{u}}_{02}$$

Excess energy:

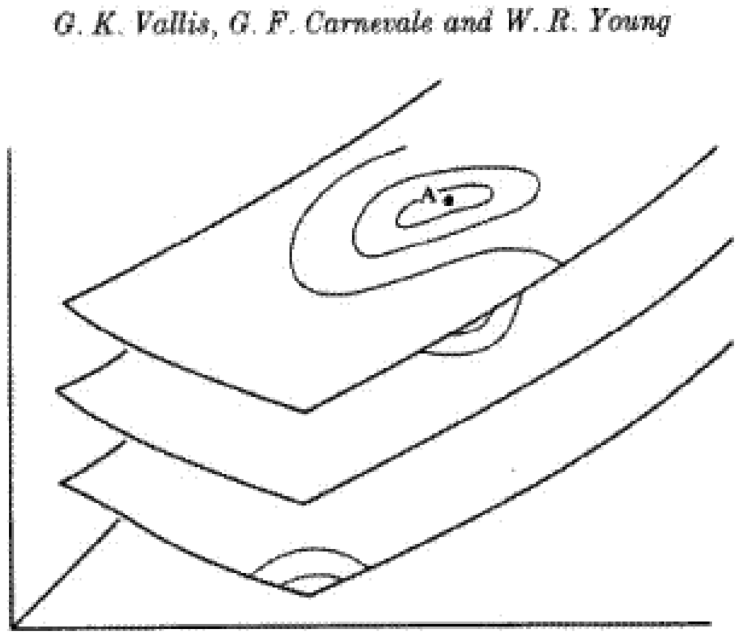
$$\frac{1}{2}\int \boldsymbol{u}^2 \mathrm{d}V - \frac{1}{2}\int \boldsymbol{U}^2 \mathrm{d}V$$

$$= \alpha \delta H + \frac{1}{2}\alpha^2 \delta^2 H;$$

$$\delta H = \int \boldsymbol{U} \cdot \tilde{\boldsymbol{u}}_{01} \mathrm{d}V, \quad \delta^2 H = \int \left(\tilde{\boldsymbol{u}}_{01}^2 + \boldsymbol{U} \cdot \tilde{\boldsymbol{u}}_{02}\right) \mathrm{d}V$$

* $\delta H \neq \text{const.} \quad \delta^2 H \neq \text{const.}$
* Complicated calculation would be required for $\tilde{\boldsymbol{u}}_{02}$

Steady Euler flows



isovortical sheets

Kinematically accessible variation
(= preservation of circulation)

$$\omega := \frac{1}{2}\epsilon_{ijk}\omega_k(\boldsymbol{x},t)\mathrm{d}x_i \wedge \mathrm{d}x_j$$

$$\boldsymbol{x} \to \tilde{\boldsymbol{x}} \quad \Rightarrow \quad \omega = \tilde{\omega};$$

$$\frac{1}{2}\epsilon_{ijk}\omega_k(\boldsymbol{x},t)\mathrm{d}x_i \wedge \mathrm{d}x_j$$
$$= \frac{1}{2}\epsilon_{pqr}\tilde{\omega}_r(\tilde{\boldsymbol{x}},t)\mathrm{d}\tilde{x}_p \wedge \mathrm{d}\tilde{x}_q$$
$$(\tilde{\omega}_r = \omega_r + \delta\omega_r)$$

Theorem (Kelvin, Arnold '65) A steady Euler flow is a coditional extremum of energy *H* on *an isovortical sheet* (= w.r.t. *kinematically accessible variations*).

Geometric formulation: economical derivation

$$G = \mathrm{SDiff}(\mathcal{D}) : \text{ volume preserving diffeomorphism of } \mathcal{G}$$

$$\mathcal{G} : \text{Lie algebral of } G, \quad \mathcal{G}^* : \text{dual space of } \mathcal{G}.$$

Lie-Poisson bracket $\{F_1, F_2\} := \left\langle \left[\frac{\delta F_1}{\delta v}, \frac{\delta F_2}{\delta v} \right], v \right\rangle ; \quad v \in \mathcal{G}^*$ $F_1, F_2 : \mathcal{G}^* \to \mathbb{R}$

Hamiltonian functional $H : \mathcal{G}^* \to \mathbb{R}$

Hamiltonian equation $\frac{\partial F}{\partial t} = \{F, H\}$ for $F : \mathcal{G}^* \to \mathbb{R}$

$$\left\langle \frac{\delta F}{\delta v}, \frac{\partial v}{\partial t} \right\rangle = \left\langle \left[\frac{\delta F}{\delta v}, \frac{\delta H}{\delta v} \right], v \right\rangle = -\left\langle \mathrm{ad}\left(\frac{\delta H}{\delta v} \right) \frac{\delta F}{\delta v}, v \right\rangle$$

$$\frac{\partial v}{\partial t} = -\mathrm{ad}^*\left(\frac{\delta H}{\delta v} \right) v$$

If $\frac{\delta H}{\delta v} = u$ ⟹

$$\frac{\partial v}{\partial t} = -\mathrm{ad}^*(u)\, v$$

Euler-Poincaré eq

$$\langle u, \mathrm{ad}(\xi)^* v \rangle := \langle \mathrm{ad}(\xi) u, v \rangle$$

Adjoint representation $\mathrm{ad}(\xi)u = [\xi, u] := (\boldsymbol{u} \cdot \nabla)\boldsymbol{\xi} - (\boldsymbol{\xi} \cdot \nabla)\boldsymbol{u}$ for $\xi, u \in \mathcal{G}(\mathcal{D})$

Euler flows

$$v \sim v + \mathrm{d}f \in \mathcal{G}^*(\mathcal{D}), \quad u = \boldsymbol{u} \cdot \frac{\partial}{\partial \boldsymbol{x}} \in \mathcal{G}(\mathcal{D})$$

$$\langle u, v \rangle = \int_{\mathcal{D}} v(u) \mathrm{d}V = \int_{\mathcal{D}} v_j u^j \mathrm{d}V$$

Adjoint representation

Lie derivative

$$\mathrm{ad}(\xi)u = [\xi, u] := (\boldsymbol{u} \cdot \nabla)\boldsymbol{\xi} - (\boldsymbol{\xi} \cdot \nabla)\boldsymbol{u} = \nabla \times (\boldsymbol{\xi} \times \boldsymbol{u}) = -\mathcal{L}_{\xi} \quad \text{for } \xi, u \in \mathcal{G}(\mathcal{D})$$

$$\langle u, \mathrm{ad}(\xi)^* v \rangle := \langle \mathrm{ad}(\xi) u, v \rangle$$

$$\begin{aligned} \mathrm{ad}^*(\xi)v &= \xi^j \left(\frac{\partial v_i}{\partial x^j} - \frac{\partial v_j}{\partial x^i} \right) \mathrm{d}x^i + \frac{\partial f}{\partial x^i} \mathrm{d}x^i \\ &= \left[-\boldsymbol{\xi} \times (\nabla \times \boldsymbol{v}) + \nabla f \right]_i \mathrm{d}x^i \end{aligned}$$

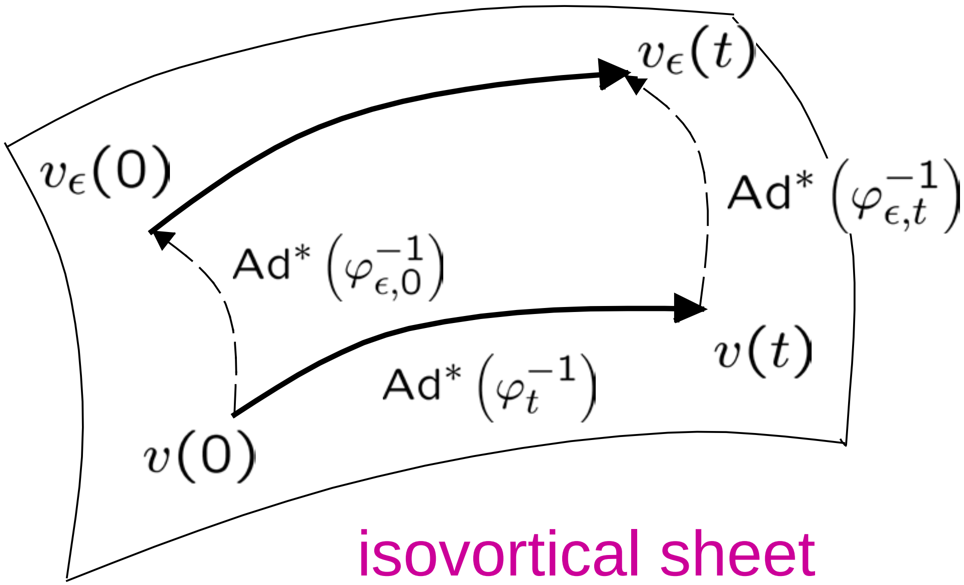
$$\frac{\partial v}{\partial t} = -\mathrm{ad}^* \left(\frac{\delta H}{\delta v} \right) v \quad \Longrightarrow \quad \frac{\partial \boldsymbol{v}}{\partial t} = \boldsymbol{\xi} \times (\nabla \times \boldsymbol{v}) - \nabla f; \quad \boldsymbol{\xi} = \frac{\delta H}{\delta v}$$

Euler Poincaré equation

Velocity field in Lagrangian displacement

$$\varphi_t \in \mathrm{SDiff}(\mathcal{D})$$
$$\varphi_{\epsilon,t} \in \mathrm{SDiff}(\mathcal{D})$$

$$\begin{aligned} v_\epsilon(t) &= \mathrm{Ad}^*\left(\left(\varphi_{\epsilon,t} \circ \varphi_t\right)^{-1}\right) v(0) \\ &= \mathrm{Ad}^*\left(\varphi_{\epsilon,t}^{-1}\right) v(t) \end{aligned}$$



$$\exists \xi_\epsilon(t) \in \mathcal{G} \;\; s.t. \;\; \varphi_{\epsilon,t} = \exp \xi_\epsilon(t)$$

$$u_\epsilon(t_0) = \frac{\partial}{\partial t}\bigg|_{t_0} \left(\varphi_{\epsilon,t} \circ \varphi_t \circ \varphi_{t_0}^{-1} \circ \varphi_{\epsilon,t_0}^{-1}\right) = u + \sum_{n=0}^{\infty} \frac{1}{(n+1)!} [\mathrm{ad}(\xi_\epsilon)]^n \left(\frac{\partial \xi}{\partial t} - \mathrm{ad}(v)\xi\right)$$

$$\xi_\epsilon = \epsilon \xi_1 + \frac{\epsilon^2}{2}\xi_2 + \cdots ,$$
$$u_\epsilon = u + \epsilon u_1 + \frac{\epsilon^2}{2} u_2 + \cdots$$

$$u_1 = \frac{\partial \xi_1}{\partial t} - \mathrm{ad}(u)\xi_1$$
$$u_2 = \frac{\partial \xi_2}{\partial t} - \mathrm{ad}(u)\xi_2 + \mathrm{ad}(\xi_1)\left(\frac{\partial \xi_1}{\partial t} - \mathrm{ad}(u)\xi_1\right)$$

Equation of Lagrangian displacement

$$v_\epsilon(t) = \mathrm{Ad}^*\left(\varphi_{\epsilon,t}^{-1}\right) v(t) = \sum_{n=0}^{\infty} \frac{1}{n!}\left[-\mathrm{ad}^*(\xi_\epsilon)\right]^n v$$

$$v_\epsilon = v + \epsilon v_1 + \frac{\epsilon^2}{2} v_2 + \cdots$$

$$v_1 = -\mathrm{ad}^*(\xi_1) v,$$
$$v_2 = -\mathrm{ad}^*(\xi_2) v + \mathrm{ad}^*(\xi_1)\mathrm{ad}^*(\xi_1) v$$

$$\boldsymbol{v}_1 = \mathcal{P}\left[\boldsymbol{\xi}_1 \times \boldsymbol{\omega}\right],$$
$$\boldsymbol{v}_2 = \mathcal{P}\left[\boldsymbol{\xi}_1 \times \left(\nabla \times (\boldsymbol{\xi}_1 \times \boldsymbol{\omega})\right) + \boldsymbol{\xi}_2 \times \boldsymbol{\omega}\right]$$

Postulate

$$u_\epsilon(t) = \left.\frac{\delta H}{\delta v}\right|_\epsilon (t)$$

$$u_1 = \frac{\delta^2 H}{\delta v^2} v_1,$$
$$u_2 = \frac{\delta^2 H}{\delta v^2} v_2 + \frac{\delta^3 H}{\delta v^3}(v_1, v_1)$$

$$\frac{\partial \boldsymbol{\xi}_1}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{\xi}_1 - (\boldsymbol{\xi}_1\cdot\nabla)\boldsymbol{u} = \boldsymbol{v}_1$$
$$\frac{\partial \boldsymbol{\xi}_2}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{\xi}_2 - (\boldsymbol{\xi}_2\cdot\nabla)\boldsymbol{u} + (\boldsymbol{u}_1\cdot\nabla)\boldsymbol{\xi}_1 - (\boldsymbol{\xi}_1\cdot\nabla)\boldsymbol{u}_1 = \boldsymbol{v}_2$$

Wave energy

$$H(v_\epsilon) = H(v) + \epsilon H_1 + \frac{\epsilon^2}{2} H_2 + \cdots$$

$$v_1 = -\mathrm{ad}^*(\xi_1)v,$$

$$v_2 = -\mathrm{ad}^*(\xi_2)v + \mathrm{ad}^*(\xi_1)\mathrm{ad}^*(\xi_1)v$$

$$H_1 = \left\langle \frac{\delta H}{\delta v}, v_1 \right\rangle = \left\langle \frac{\delta H}{\delta v}, -\mathrm{ad}^*(\xi_1)v \right\rangle = -\left\langle \mathrm{ad}(\xi_1)\frac{\delta H}{\delta v}, v \right\rangle$$

$$= \left\langle \xi_1, \mathrm{ad}^*\left(\frac{\delta H}{\delta v}\right) v \right\rangle = -\left\langle \xi_1, \frac{\partial v}{\partial t} \right\rangle = 0 \quad \text{if } v \text{ is steady.}$$

$$H_2 = \left\langle \frac{\delta H}{\delta v}, v_2 \right\rangle + \left\langle \frac{\delta^2 H}{\delta v^2} v_1, v_1 \right\rangle = -\left\langle \xi_2, \frac{\partial v}{\partial t} \right\rangle - \left\langle \xi_1, \frac{\partial v_1}{\partial t} \right\rangle$$

For steady flow

$$\begin{aligned} H_2 &= -\left\langle \xi_1, \frac{\partial v_1}{\partial t} \right\rangle = \left\langle \xi_1, \mathrm{ad}^*\left(\frac{\partial \xi_1}{\partial t}\right) v \right\rangle = \left\langle \mathrm{ad}\left(\frac{\partial \xi_1}{\partial t}\right) \xi_1, v \right\rangle \\ &= \int \boldsymbol{\omega} \cdot \left(\frac{\partial \boldsymbol{\xi}_1}{\partial t} \times \boldsymbol{\xi}_1\right) \mathrm{d}V \end{aligned}$$

Energy of Kelvin waves

Lagrangian dispalcement $\boldsymbol{\xi}_1 = \mathrm{Re}\left[C_0\hat{\boldsymbol{\xi}}(r;\omega_0,m,k_0)e^{\mathrm{i}(m\theta+k_0-\omega_0 t)}\right];$

$$\hat{\xi}_r^{(m)} = \frac{\omega_0-m}{\sqrt{4-(\omega_0-m)^2}}\left\{\frac{m}{r}(\omega_0-m-2)J_m(\eta_m r)-(\omega_0-m)\eta_m J_{m+1}(\eta_m r)\right\},$$

$$\hat{\xi}_\theta^{(m)} = \mathrm{i}\frac{\omega_0-m}{\sqrt{4-(\omega_0-m)^2}}\left\{-\frac{m}{r}(\omega_0-m-2)J_m(\eta_m r)-2\eta_m J_{m+1}(\eta_m r)\right\},$$

$$\hat{\xi}_z^{(m)} = -\mathrm{i}k_0\sqrt{4-(\omega_0-m)^2}J_m(\eta_m r), \qquad \text{where} \quad \eta_m := k_0\sqrt{4/(\omega_0-m)^2-1}.$$

The wave energy per unit length in *z* is $E_0 = \omega_0\mu_0;$

$$\mu_0 = 2\pi\left|C_0\right|^2\frac{\omega_0-m}{2}\int_0^1|\hat{\boldsymbol{\xi}}|^2\mathrm{d}r$$

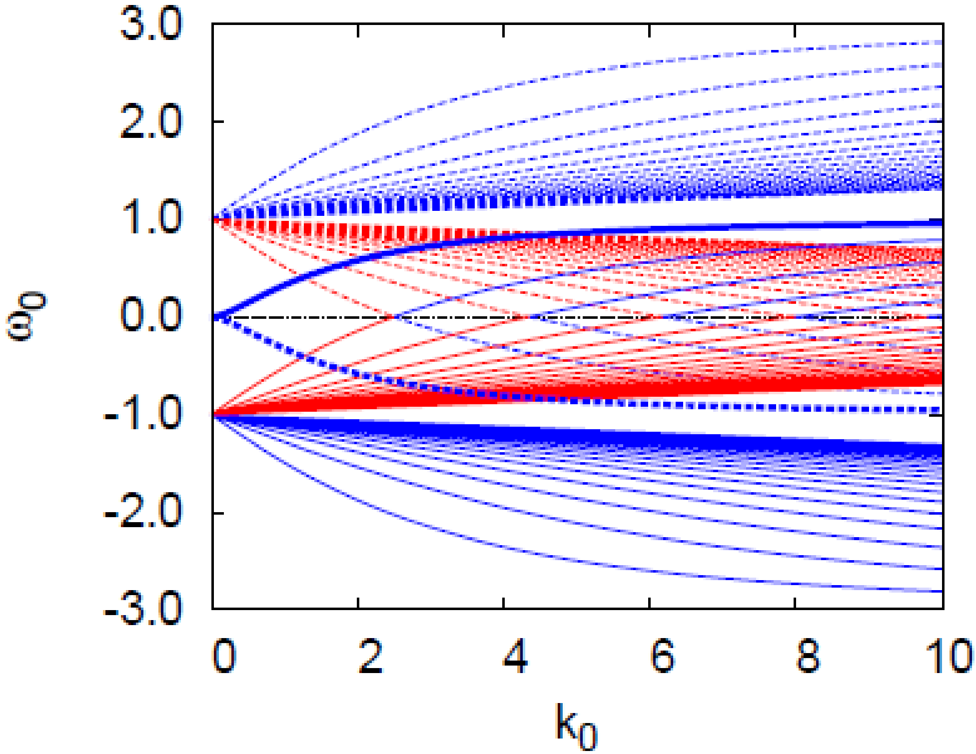
$$= \pi\left|C_0\right|^2\frac{\partial D}{\partial\omega_0}(\omega_0;m,k);$$

$$D(\omega_0,m,k) := (\omega_0-m)^3 J_m(\eta_m)\left[(\omega_0-m)\eta_m J_{m-1}(\eta_m)-m(\omega_0-m+2)J_m(\eta_m)\right]$$

$\mu_0 = E_0/\omega_0:$ wave action, $D=0:$ dispersion relation

Energy signature of helical waves (m=±1)

- **Blue**: positive wave-energy
- **Red**: negative wave-energy



m=-1 (solid lines) and *m=1* (dashed lines)

Drift current

For $\eta \in \mathcal{G}$,

$$J_\epsilon = < \eta, v > + \epsilon < \eta, v_1 > + \frac{\epsilon^2}{2} < \eta, v_2 > + \cdots$$

$$J_1 = \langle \eta, v_1 \rangle = \langle \eta, -\mathrm{ad}^*(\xi_1) v \rangle = \langle \xi_1, \mathrm{ad}^*(\eta) v \rangle$$

$$J_2 = \langle \eta, v_2 \rangle = \langle \xi_2, \mathrm{ad}^*(\eta) v \rangle + \langle \xi_1, \mathrm{ad}^*(\eta) v_1 \rangle$$

If the basic flow has a symmetry $\mathrm{ad}^*(\eta) v = 0$

$$J_1 = 0, \qquad J_2 = \langle \xi_1, \mathrm{ad}^*(\eta) v_1 \rangle$$

$$\begin{aligned} J_2 &= \langle \xi_1, \mathrm{ad}^*(\eta) v_1 \rangle = \langle \mathrm{ad}(\eta) \xi_1, -\mathrm{ad}^*(\xi_1) v \rangle = \langle -\mathcal{L}_{\boldsymbol{\eta}} \boldsymbol{\xi}_1, \boldsymbol{\xi}_1 \times \boldsymbol{\omega} \rangle \\ &= \int \boldsymbol{\omega} \cdot (\boldsymbol{\xi}_1 \times \mathcal{L}_{\boldsymbol{\eta}} \boldsymbol{\xi}_1) \, \mathrm{d}V \end{aligned}$$

Hamiltonian Noether's theorem

Suppose that $\exists \eta \in \mathcal{G}$ $\quad s.t. \{ < \eta, v >, H \} = 0$

then $< \eta, v > = \mathrm{const.}$

Drift current in a cylindrical vortex

F. & Hirota '08

$$\boxed{J_2 = \int \boldsymbol{\omega}\cdot(\boldsymbol{\xi}_1 \times \mathcal{L}_{\boldsymbol{\eta}}\boldsymbol{\xi}_1)\,\mathrm{d}V}$$

$$\boldsymbol{\eta} = r\boldsymbol{e}_\theta; \quad J_{2\theta} = \int \boldsymbol{\omega}\cdot\left(\boldsymbol{\xi}_1 \times \frac{\partial \boldsymbol{\xi}_1}{\partial \theta}\right)\mathrm{d}V$$

$$\boldsymbol{\eta} = \boldsymbol{e}_z; \quad J_{2z} = \int \boldsymbol{\omega}\cdot\left(\boldsymbol{\xi}_1 \times \frac{\partial \boldsymbol{\xi}_1}{\partial z}\right)\mathrm{d}V$$

Substitute the ***Kelvin wave*** $\boldsymbol{\xi}_1 = \mathrm{Re}\left[C_0\hat{\boldsymbol{\xi}}\mathrm{e}^{\mathrm{i}(m\theta + k_0 z - \omega_0 t)}\right]$

$$\boxed{J_{2z} := \int \overline{v_{2z}}\mathrm{d}A = k_0|C_0^2|\frac{\mathrm{i}}{2}\int \boldsymbol{\omega}\cdot\left(\hat{\boldsymbol{\xi}}^* \times \hat{\boldsymbol{\xi}}\right)\mathrm{d}A = k_0\mu_0}$$

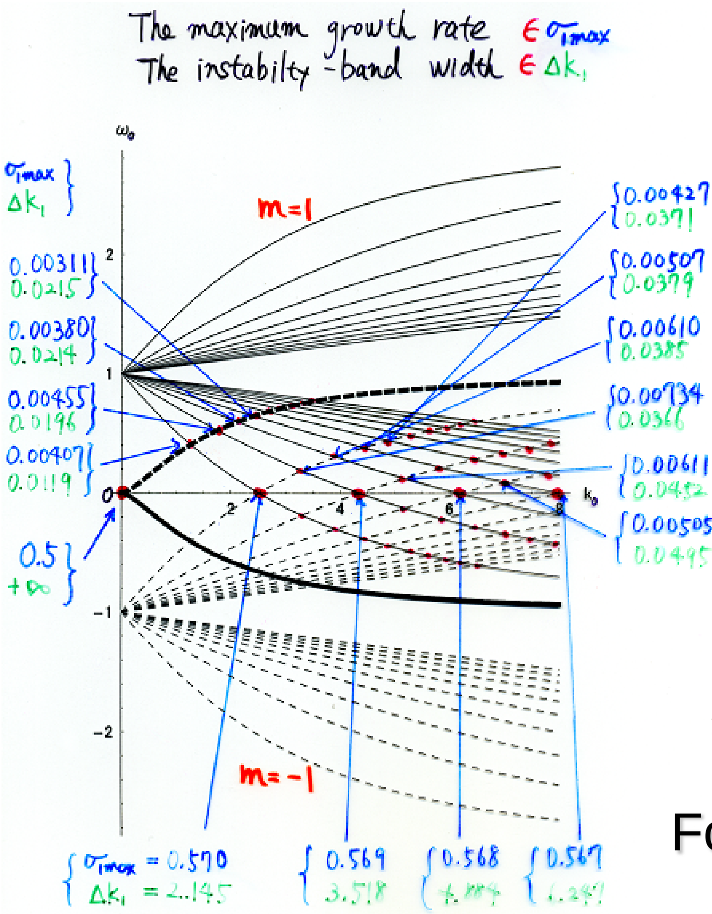
$$k_0 = 0 \Rightarrow J_{2z} = 0$$

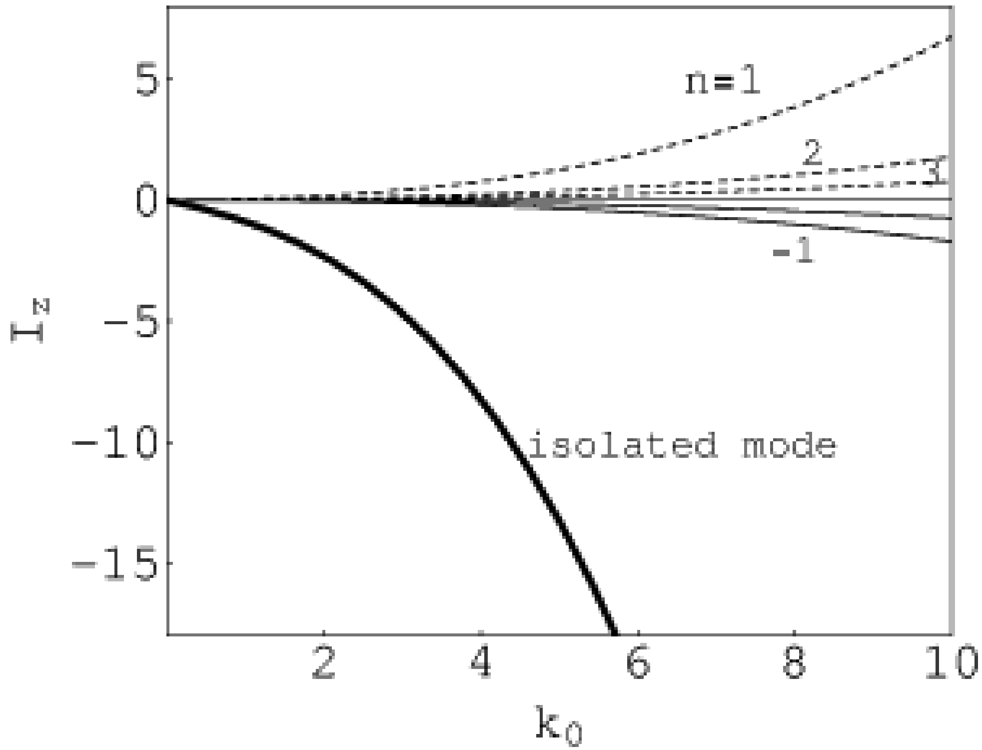
genuinly *3D* effect !!

$$\boxed{H_2 = \omega_0\mu_0, \quad J_{2\theta} = m\mu_0, \quad J_{2z} = k_0\mu_0,}$$

pseudomomentum

Axial flow-flux of a helical wave *(m=1)*





For the *principal* mode （＝stationary)

$$k_0 = \qquad I_z^{(-1)} + I_z^{(1)} = 0$$

- 2.505
- 4.349

Confined geometry: a cylinder with elliptic cross-section

Experimental Study of the Multipolar Vortex Instabilty

Eloy, Le Gal & Le Dizès:
Phys. Rev. Lett. **85** (2000) 3400

Instabilty of flows with *elliptic streamlines*

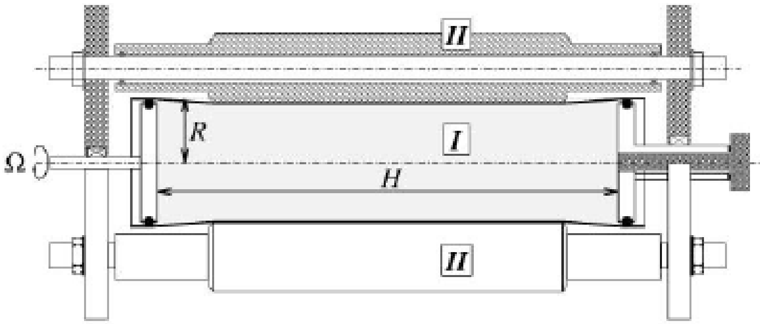


FIG. 1. Experimental setup: (I) plastic elastic cylinder filled with water; (II) rollers.

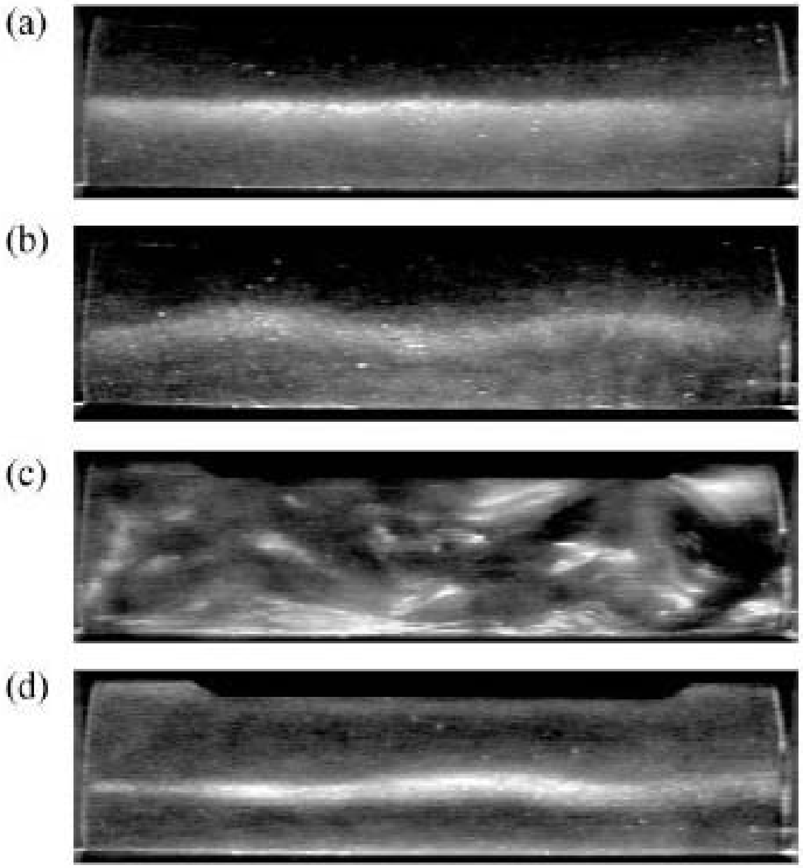


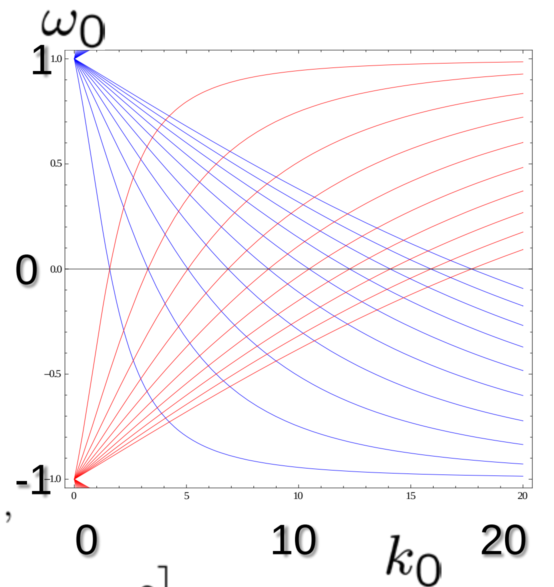
FIG. 4. Four successive images of the flow for $n = 2$, Re = 5000, $H/R = 7.96$, and (a) $\Omega t = 294$, solid body rotation; (b) $\Omega t = 715$, appearance of mode $(-1, 1, 1)$; (c) $\Omega t = 943$, vortex breakup; (d) $\Omega t = 1113$, relaminarization.

Weakly nonlinear amplitude equations

Mie & Fukumoto (2010)

$$\boldsymbol{u}_{01} = \boxed{A_-(t)\boldsymbol{u}_-(r)e^{-\mathrm{i}\theta}e^{\mathrm{i}k_0 z}} + \boxed{A_+(t)\boldsymbol{u}_+(r)e^{\mathrm{i}\theta}e^{\mathrm{i}k_0 z}} + \text{c.c.}$$

$$\frac{\mathrm{d}A_\pm}{\mathrm{d}t} = \pm\mathrm{i}\left[\epsilon a A_\mp + \alpha^2 A_\pm (b|A_\pm|^2 + c|A_\mp|^2)\right]$$



$$a = \frac{3(3k_0^2+1)}{8(2k_0^2+1)}, \quad \eta = \sqrt{3}k_0,$$

$$b = \frac{2k_0^4}{3(2k_0^2+1)}\left[\frac{4}{J_0(\eta)^2}\int_0^1 rJ_0(\eta r)^2 J_1(\eta r)^2 \mathrm{d}r - (11k_0^4 + 13k_0^2 + 5)J_0(\eta)^2\right],$$

$$c = \frac{-k_0^2}{12(2k_0^2+1)}\left[\frac{64k_0^2}{J_0(\eta)^2}\int_0^1 rJ_0(\eta r)^2 J_1(\eta r)^2 \mathrm{d}r + (20k_0^6 + 97k_0^4 + 14k_0^2 - 27)J_0(\eta)^2\right],$$

wavenumber	k_0	1.579	3.286	5.061	6.856	8.659	10.47
coefficients	a(>0)	0.5312	0.5542	0.5589	0.5605	0.5613	0.5617
	b(>0)	0.3976	8.286	40.45	118.4	266.1	509.5
	c(<0)	-5.222	-53.39	-212.8	-562.1	-1185	-2170

Restricted dynamics

$A = A_+ = \overline{A_-}$

$\epsilon = \alpha^2$
$\tau = \epsilon t$

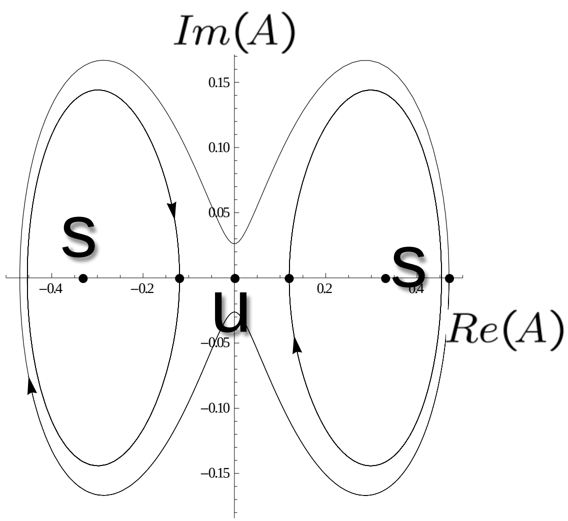
$$\frac{dA}{d\tau} = i\left[\overline{A} + (b+c)|A|^2 A\right]$$

$b+c \longrightarrow -k_0^3 \log k_0/\sqrt{3}\pi$

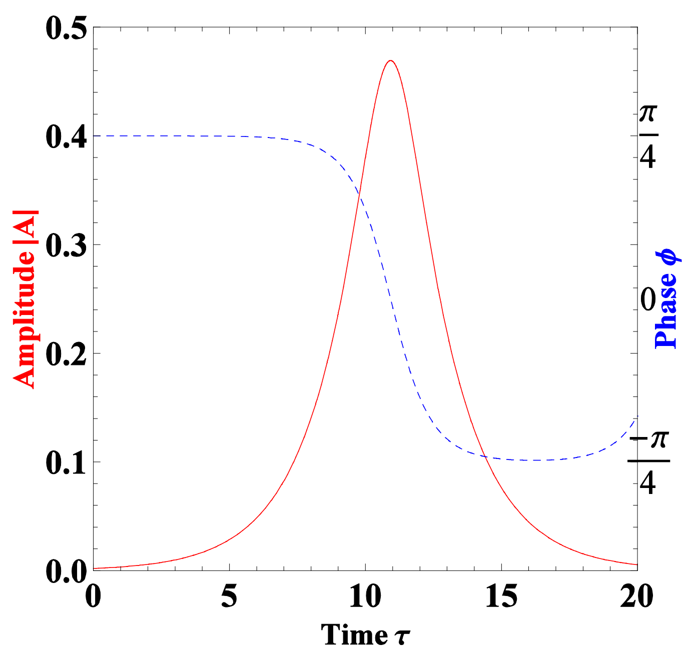
k_0	1.579	3.286	5.061	6.856	8.659	10.47
a (>0)	0.5312	0.5542	0.5589	0.5601	0.5613	0.5617
b+c(<0)	-4.824	-45.10	-172.3	-443.7	-919.3	-1661

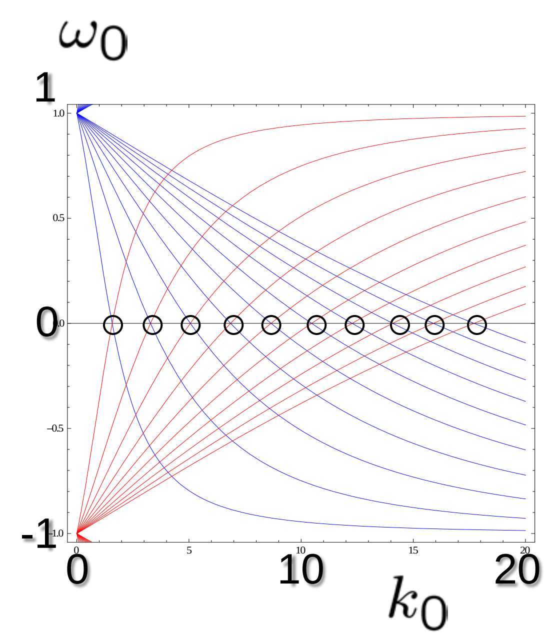
Equilibrium Amplitude

$$|A| = \sqrt{a/|b+c|} \approx \frac{3}{4}\left(\frac{\sqrt{3}\pi}{k_0^3 \log k_0}\right)^{1/2}$$



$k_0 = 1.579$





Summary

Linear stability of an *elliptic vortex*, a straight vortex tube subject to a pure shear, to three-dimensional disturbances is calculated.
This is a parametric resonance instability between two Kelvin waves caused by a perturbation breaking *S*^{1}-symmetry of the circular core.

1. **Lagrangian approach**: ***Energy*** of the Kelvin waves is calculated by restricting disturbances to ***kinematically accessible field***

   linear perturbation is sufficient to calculate energy, quadratic in amplitude!

   ⇨ Modification of mean field at 2nd order : $\overline{v_{2\theta}}$

2. ***Axial current***: For the *Rankine vortex*, 2nd-order drift current $\overline{\boldsymbol{v}_2}$ includes not only azimuthal but also *axial* component $\overline{v_{2z}}$.

energy $\xleftarrow{\times\omega_0}$ ***wave action*** $\xrightarrow{\times k_0}$ ***pseudomomentum***

3. **Weakly nonlinear amplitude equation:** *Hamiltonian normal form*

   Its coefficients are all determined *explicitly.*

4. ***Short-wave asymptotics***: The equlibrium amplitude is obtained.

⇨ Secondary instability (*three-wave resonance*)