Three dimensional stability of the axisymmetric Burgers vortex

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1-1. Motivation

· Several numerical simulations of turbulent flows have led to the general conclusion that vortex tubes play important roles in local structures of such flows.

cf. Townsend (1951), Robinson-Saffman (1984), Kida-Ohkitani (1992), Moffatt-Kida-Ohkitani (1994), · · ·

The Burgers vortex:

· Explicit stationary solution to 3D Navier-Stokes equations

· Simple model of vortex tubes

1-2. Flows with a background straining flow

$V = (V_1, V_2, V_3)^\top$: velocity field, P: pressure field

$$(NS)\quad \begin{cases} \partial_t V - \Delta V + (V,\nabla)V + \nabla P = 0 & t>0,\quad x\in\mathbb{R}^3, \\ \nabla\cdot V = 0 & t>0,\quad x\in\mathbb{R}^3, \end{cases}$$

To model ”vortex tubes” we assume that V takes the form

$$V = Mx + U, \qquad M = \begin{pmatrix} -\frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

$U = (U_1, U_2, U_3)$: unknown perturbation velocity

cf. Local in time solvability for general matrices M

Sawada (2004), Hieber-Sawada (2005)

1-3. Equations for vorticity fields: $\Omega = \nabla \times V$

$$\begin{cases} \partial_t \Omega - \Delta\Omega + (V,\nabla)\Omega - (\Omega,\nabla)V = 0 & t > 0, \quad x \in \mathbb{R}^3, \\ \nabla \cdot \Omega = 0 & t > 0, \quad x \in \mathbb{R}^3. \end{cases}$$

From $V = Mx + U$, we have

$$(V) \quad \begin{cases} \partial_t \Omega - L\Omega + (U,\nabla)\Omega - (\Omega,\nabla)U = 0 & t > 0, \quad x \in \mathbb{R}^3, \\ \nabla \cdot \Omega = 0 & t > 0, \quad x \in \mathbb{R}^3. \end{cases}$$

$$L\Omega = \Delta\Omega - (Mx,\nabla)\Omega + M\Omega$$

$$= \begin{pmatrix} \Delta_h \Omega_h + \frac{x_h}{2} \cdot \nabla_h \Omega_h - \frac{1}{2}\Omega_h + \partial_{x_3}^2 \Omega_h - x_3 \partial_{x_3} \Omega_h \\ \\ \Delta_h \Omega_3 + \frac{x_h}{2} \cdot \nabla_h \Omega_3 + \Omega_3 + \partial_{x_3}^2 \Omega_3 - x_3 \partial_{x_3} \Omega_3 \end{pmatrix}.$$

1-4. Biot-Savart law

Since $\Omega = \nabla \times V = \nabla \times (Mx + U) = \nabla \times U$, we formally recover U from Ω via the Biot-Savart law: $U = (-\Delta)^{-1}\nabla \times \Omega$

(i) 3D Biot-Savart law

$$U(x) = K_{3D} * \Omega\,(x) = -\frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{(x-y)\times\Omega(y)}{|x-y|^3}dy.$$

(ii) 2D Biot-Savart law

If $\Omega(x) = (0, 0, \Omega_3(x_h))^\top$ then $U(x) = (U_h(x_h), 0)^\top$ and

$$U_h(x_h) = K_{2D} * \Omega_3\,(x_h) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{(x_h-y_h)^\perp}{|x_h-y_h|^2}\Omega_3(y_h)dy_h,$$

where $x_h^\perp = (-x_2, x_1)^\top$.

1-5. Burgers vortex

$$(V)\quad \begin{cases} \partial_t \Omega - L\Omega + (U,\nabla)\Omega - (\Omega,\nabla)U &= 0, \\ U &= K_{3D} * \Omega, \\ \nabla\cdot\Omega &= 0. \end{cases}$$

We consider the vector field G defined by

$$G(x) = (0,\ 0,\ g(x_h))^{\top}, \qquad g(x_h) = \frac{1}{4\pi} e^{-|x_h|^2/4}.$$

Proposition 1 *For each* $\alpha \in \mathbb{R}$, αG *is a stationary sol. to* (V).

Rem. (i) This exact solution was found by Burgers (1948).

(ii) The parameter α is called the circulation number which represents the intensity of the Burgers vortex.

2.1 Stability of the Burgers vortex

Main goal: Show the asymptotic stability of the Burgers vortex αG for all circulation numbers.

Lemma 1 *Assume that* $\Omega = (\Omega_h, \Omega_3)^\top \in (L^1_{loc}(\mathbb{R}; L^1(\mathbb{R}^2)))^3$ *satisfies* $\nabla \cdot \Omega = 0$ *in the sense of distributions. Then there exists* α *such that* $\int_{\mathbb{R}^2} \Omega_3(x_h, x_3) dx_h = \alpha$ *for a.e.* $x_3 \in \mathbb{R}$. *Moreover, the value* $\int_{\mathbb{R}^2} \Omega_3(x_h, x_3, t) dx_h$ *is conserved under* (V).

Reminding this lemma, we study the equation for $\omega = \Omega - \alpha G$ with the condition $\int_{\mathbb{R}^2} \omega_3(x_h, x_3, 0) dx_h = 0$:

$$(V') \begin{cases} \partial_t \omega - (L - \alpha\Lambda)\omega + B(\omega, \omega) = 0, & t > 0, \quad x \in \mathbb{R}^3, \\ \nabla \cdot \omega = 0, & t > 0, \quad x \in \mathbb{R}^3, \\ \omega|_{t=0} = \omega_0, & x \in \mathbb{R}^3. \end{cases}$$

$$L\Omega = \Delta\Omega - (Mx, \nabla)\Omega + M\Omega,$$

$$\Lambda\omega = (U^G, \nabla)\omega - (\omega, \nabla)U^G + (K_{3D} * \omega, \nabla)G - (G, \nabla)K_{3D} * \omega,$$

$$U^G = K_{3D} * G,$$

$$B(\omega, \omega) = (K_{3D} * \omega, \nabla)\omega - (\omega, \nabla)K_{3D} * \omega.$$

2.2 Functional settings

· functional setting that allows for perturbations in the class to which the Burgers vortex belongs:

The Burgers vortex is essentially a two-dimensional flow. We thus assume that the perturbations are:

(i) localized in the horizontal direction,

(ii) bounded in the vertical direction.

(i) 2D space:

$$\rho_m(r) = \begin{cases} (1 + \frac{r}{4m})^m, & 0 \le m < \infty, \\ e^{r/4}, & m = \infty. \end{cases}$$

$$L^2(m) = \{f \in L^2(\mathbb{R}^2) \mid \int_{\mathbb{R}^2} |f(x_h)|^2 \rho_m(|x_h|^2) dx_h < \infty\}$$

$$L^2_0(m) = \{f \in L^2(m) \mid \int_{\mathbb{R}^2} f(x_h) dx_h = 0\}, \qquad \text{for } m > 1.$$

(ii) 3D space:

$$X(m) = BC(\mathbb{R};\ L^2(m)), \qquad \|f\|_{X(m)} = \sup_{x_3 \in \mathbb{R}} \|f(\cdot, x_3)\|_{L^2(m)},$$

$$X_0(m) = BC(\mathbb{R};\ L^2_0(m)).$$

$$\mathbb{X}(m) = (X(m))^2 \times X_0(m), \qquad m > 1.$$

2.3 Known results for stability of Burgers vortices $\{\alpha G\}_{\alpha \in \mathbb{R}}$

(i) Two dimensional stability: $\omega(x) = (0,\ 0,\ \omega_3(x_h))^\top$

It is already known that the Burgers vortex is globally stable in $L^1(\mathbb{R}^2)$ for all α.

Giga-Kambe (1988); $\|\omega_{0,3}\|_{L^1(\mathbb{R}^2)}, |\alpha| \ll 1.$
Carpio (1994), Giga-Giga (1999); $|\alpha| \ll 1.$
Gallay-Wayne (2005); $\forall \alpha \in \mathbb{R}.$

(ii) Three dimensional stability: $\omega = (\omega_1, \omega_2, \omega_3)^\top$

Schmid-Rossi (2004): linear stability by numerical calculations.

Gallay-Wayne (2006): $\lim_{t\to\infty} \|\omega(t)\|_{\mathbb{X}(m)} = 0$ if $|\alpha| + \|\omega_0\|_{\mathbb{X}(m)} \ll 1.$

2-5. Main result: Stability for all circulation numbers

Theorem 1 *Let $m > 2$ and $\alpha \in \mathbb{R}$. Then there exists $\delta = \delta(\alpha, m) > 0$ such that for any $\omega_0 \in \mathbb{X}(m)$ with $\nabla \cdot \omega_0 = 0$ and $\|\omega_0\|_{\mathbb{X}(m)} \le \delta$, Eq. (V') has a unique time global solution ω satisfying*

$$\|\omega(t)\|_{\mathbb{X}(m)} \le Ce^{-\frac{t}{2}}\|\omega_0\|_{\mathbb{X}(m)}, \qquad t > 0.$$

The number δ satisfies $\lim_{|\alpha|\to\infty} \delta(\alpha, m) = 0$.

Rem. (i) This result gives a time global solution to (NS) of the form

$$V(t) = Mx + \alpha K_{3D} * G + K_{3D} * \omega(t).$$

(ii) We have a uniform decay rate in time for perturbations. The key point is to show that $L - \alpha\Lambda$ has a uniform spectral gap for $\alpha \in \mathbb{R}$.

3-1. Analysis of the linearized problem

$$L\omega = \Delta\omega - (Mx, \nabla)\omega + M\omega,$$

$$\Lambda\omega = (U^G, \nabla)\omega - (\omega, \nabla)U^G + (K_{3D} * \omega, \nabla)G - (G, \nabla)K_{3D} * \omega.$$

Lemma 2 *Let $m > 2$ and $\alpha \in \mathbb{R}$. Then for any $\omega_0 \in \mathbb{X}(m)$ with $\nabla \cdot \omega_0 = 0$, we have*

$$||e^{t(L-\alpha\Lambda)}\omega_0||_{\mathbb{X}(m)} \leq Ce^{-\frac{t}{2}}||\omega_0||_{\mathbb{X}(m)}, \qquad t > 0.$$

Here C depends only on α and m.

3-2. Basic strategy in the study of $e^{t(L-\alpha\Lambda)}$

Step 1: Show the exponential time decay of $\partial_{x_3}^k e^{t(L-\alpha\Lambda)}$ for $k \gg 1$.

Key: Use the relations

$$[L,\ \partial_{x_3}] = L\partial_{x_3} - \partial_{x_3}L = -\partial_{x_3}, \tag{1}$$

$$[\Lambda,\ \partial_{x_3}] = 0. \tag{2}$$

(i) The property (1) comes from the term $-x_3\partial_{x_3}$ in L.

(ii) The property (2) follows from the fact that the Burgers vortex is a 2D vorticity field.

$$\partial_{x_3}^{\mathbf{k}} e^{t(L-\alpha\Lambda)} = e^{-kt} e^{t(L-\alpha\Lambda)} \partial_{x_3}^{\mathbf{k}},$$

$$\|\partial_{x_3}^{\mathbf{k}} e^{t(L-\alpha\Lambda)} \omega_0\|_{\mathbb{X}(m)} \leq C_\alpha e^{-\mathbf{k}t+C(\alpha)t} \|\partial_{x_3}^{\mathbf{k}} \omega_0\|_{\mathbb{X}(m)}.$$

Step 2: Get the estimate of $e^{t(L-\alpha\Lambda)}$ by the interpolation.

Key: Roughly speaking, we establish the estimate of $e^{t(L-\alpha\Lambda)}\omega_0$ in terms of ω_0 and $\partial_{x_3} e^{t(L-\alpha\Lambda)}\omega_0$, such as

$$||e^{t(L-\alpha\Lambda)}\omega_0||_{\mathbb{X}(m)} \leq C(\alpha, m)e^{-\frac{t}{2}}||\omega_0||_{\mathbb{X}(m)} + C(\alpha, m)\int_0^t e^{-\frac{t-s}{2}}||\partial_{x_3} e^{s(L-\alpha\Lambda)}\omega_0||_{\mathbb{X}(m)} ds.$$

$$||\partial_{x_3} e^{s(L-\alpha\Lambda)}\omega_0||_{\mathbb{X}(m)} \leq C||e^{s(L-\alpha\Lambda)}\omega_0||_{\mathbb{X}(m)}^{1-\frac{1}{k}}||\partial_{x_3}^k e^{s(L-\alpha\Lambda)}\omega_0||_{\mathbb{X}(m)}^{\frac{1}{k}}.$$

Remark. (i) Step 1 is easy to establish, although it is crucial in our proof.

(ii) The proof of Step 2 is more complicated and technical. The key idea is to introduce the decomposition

$$L - \alpha\Lambda = L_{2D,\alpha} + \partial_{x_3}^2 - x_3\partial_{x_3} - \alpha H.$$

(I) $L_{2D,\alpha}$ is a purely (but vectorial) 2D operator.

(II) H is regarded as a remainder in the sense that we have the estimates such as

$$||H\omega||_{\mathbb{X}(m)} \leq C||\omega||_{\mathbb{X}(m)}^{\gamma}||\partial_{x_3}\omega||_{\mathbb{X}(m)}^{1-\gamma}, \quad \gamma \in (0,1).$$

3-4. Decomposition of $L - \alpha\Lambda$

Set $\mathcal{L}_h = \Delta_h + \frac{x_h}{2} \cdot \nabla_h + 1.$

$$L_{2D,\alpha}\omega = \begin{pmatrix} L_{2D,\alpha,h}\omega_h \\ \\ L_{2D,\alpha,3}\omega_3 \end{pmatrix} = \begin{pmatrix} \mathcal{L}_h\omega_h - \frac{3}{2}\omega_h - \alpha(U_h^G, \nabla_h)\omega_h + \alpha(\omega_h, \nabla_h)U_h^G \\ \\ \mathcal{L}_h\omega_3 - \alpha(U_h^G, \nabla_h)\omega_3 - \alpha(K_{2D} * \omega_3, \nabla_h)g \end{pmatrix},$$

$$H\omega = (K_{3D} * \omega, \nabla)G - (K_{2D} * \omega_3, \nabla)G - (G, \nabla)K_{3D} * \omega.$$

$$\Longrightarrow \qquad L - \alpha\Lambda = L_{2D,\alpha} + \partial_{x_3}^2 - x_3\partial_{x_3} - \alpha H.$$

Remark. If $\omega(x) = \omega(x_h)$ then $H\omega = 0$.

The important step is to study the eigenvalue problem of $L_{2D,\alpha}$.

$$L_{2D,\alpha}\omega = \begin{pmatrix} L_{2D,\alpha,h}\omega_h \\ \\ L_{2D,\alpha,3}\omega_3 \end{pmatrix} = \begin{pmatrix} \mathcal{L}_h\omega_h - \frac{3}{2}\omega_h - \alpha(U_h^G, \nabla_h)\omega_h + \alpha(\omega_h, \nabla_h)U_h^G \\ \\ \mathcal{L}_h\omega_3 - \alpha(U_h^G, \nabla_h)\omega_3 - \alpha(K_{2D} * \omega_3, \nabla_h)g \end{pmatrix}.$$

Lemma 3 *Let $m > 2$. Let λ_h and λ_3 be the eigenvalues of $L_{2D,\alpha,h}$ and $L_{2D,\alpha,3}$ in $(L^2(m))^2$ and $L_0^2(m)$, respectively.*
Then $\mathrm{Re}\,(\lambda_h) \leq -\frac{3}{2}$ *and* $\mathrm{Re}\,(\lambda_3) \leq -\frac{1}{2}$.

Remark. The estimate of the eigenvalues for $L_{2D,\alpha,3}$ is obtained by Gallay-Wayne (2005).

3-5. Eigenvalues of $L_{2D,\alpha,h}$ in $(L^2(\infty))^2 = (L^2(\frac{dx_h}{g(x_h)}))^2$

$$\mathcal{L}_h\omega_h - \frac{3}{2}\omega_h - \alpha(U_h^G, \nabla_h)\omega_h + \alpha(\omega_h, \nabla_h)U_h^G = \lambda\omega_h, \qquad \omega_h \in (L^2(\infty))^2.$$

$\Rightarrow$ Re $\lambda \le -\frac{3}{2}$

Key properties:

(i) $\mathcal{L}_h = \Delta_h + \frac{x_h}{2} \cdot \nabla_h + 1$ is self-adjoint in $L^2(\infty)$, and

$$\langle -\mathcal{L}_h\phi,\ \phi\rangle \ \ge 0, \qquad \phi \in L^2(\infty) \cap D(\mathcal{L}_h),$$
$$\langle -\mathcal{L}_h\phi,\ \phi\rangle \ \ge \frac{1}{2}\|\phi\|^2_{L^2(\infty)} \qquad \phi \in L^2_0(\infty) \cap D(\mathcal{L}_h).$$

(ii) Skew-symmetry of $(U_h^G, \nabla_h)\cdot$:

$$\text{Re}\ \langle (U_h^G, \nabla_h)\phi,\ \phi\rangle = 0.$$

Rem. The property (ii) follows from the fact that $U_h^G(x_h) \perp (x_1, x_2)^\top$.

$$\mathcal{L}_h\omega_h - \frac{3}{2}\omega_h - \alpha(U_h^G, \nabla_h)\omega_h + \alpha(\omega_h, \nabla_h)U_h^G = \lambda\omega_h, \qquad \omega_h \in (L^2(\infty))^2. \tag{3}$$

By taking the inner product with ω_h in (3) we have

$$\begin{aligned}
&\mathrm{Re}\lambda\, \|\omega_h\|^2 \\
= \;& \mathrm{Re}\langle(\mathcal{L}_h - \frac{3}{2}I)\omega_h, \omega_h\rangle - \alpha\mathrm{Re}\langle(U_h^G, \nabla_h)\omega_h, \omega_h\rangle + \alpha\mathrm{Re}\langle(\omega_h, \nabla_h)U_h^G, \omega_h\rangle \\
\leq \;& -\frac{3}{2}\|\omega_h\|^2 + \alpha\mathrm{Re}\langle(\omega_h, \nabla_h)U_h^G, \omega_h\rangle \\
= \;& -\frac{3}{2}\|\omega_h\|^2 + 2\alpha\mathrm{Re}\int_{\mathbb{R}^2} x_h \cdot \omega_h\; x_h^\perp \cdot \bar{\omega}_h\; f(|x_h|^2)\,\frac{dx_h}{g(x_h)}.
\end{aligned}$$

where $f(r) = \frac{1}{2\pi}\frac{d}{dr}(1 - e^{-\frac{r}{4}})/r$.

The function $x_h \cdot \omega_h$ satisfies

$$\lambda x_h \cdot \omega_h = (\mathcal{L}_h - 2I)x_h \cdot \omega_h - \alpha(U_h^G, \nabla)x_h \cdot \omega_h - 2\nabla_h \cdot \omega_h,$$

thus we have

$$\mathrm{Re}\lambda \, \|x_h \cdot \omega_h\|^2 \leq -2\|x_h \cdot \omega_h\|^2 - 2\mathrm{Re}\langle \nabla_h \cdot \omega_h, x_h \cdot \omega_h \rangle.$$

The function $\nabla_h \cdot \omega_h$ satisfies

$$\lambda \nabla_h \cdot \omega_h = (\mathcal{L}_h - I)\nabla_h \cdot \omega_h - \alpha(U_h^G, \nabla_h)\nabla_h \cdot \omega_h,$$

which leads to

$$\mathrm{Re}\lambda \, \|\nabla_h \cdot \omega_h\|^2 \leq -\frac{3}{2}\|\nabla_h \cdot \omega_h\|^2.$$

4. Summary

(i) The axisymmetric Burgers vortex is locally stable with respect to 3D perturbations for all circulation numbers.

(ii) The rate of convergence is estimated uniformly in the circulation numbers.

(iii) Key idea is to use:

· Stabilizing effect of $-x_3\partial_{x_3}$ (for x_3 derivatives)

· Decomposition of the linearized operator into the **horizontal part** and the **vertical part**.

· Special cancelation (**symmetry**) in the **horizontal part** of the linearized operator in a weighted L^2 space.

We denote by $r_{ess}(T)$ the essential spectral bound of a bounded linear operator T.

Proposition 2 *Let $m \in (1, \infty)$ and $\alpha \in \mathbb{R}$. Then in $(L^2(m))^2$ we have for each $t > 0$,*

$$r_{ess}(e^{tL_{2D,\alpha,h}}) = r_{ess}(e^{t(\mathcal{L}_h - \frac{3}{2}I)}) = e^{-(\frac{m}{2}+1)t}.$$

By Proposition 2 it suffices to study the behavior of discrete eigenvalues of $L_{2D,\alpha,h}$. So we consider the eigenvalue problem in $(L^2(m))^2$:

$$(L_{2D,\alpha,h}\omega_h =)\mathcal{L}_h\omega_h - \frac{3}{2}\omega_h - \alpha(U_h^G, \nabla_h)\omega_h + \alpha(\omega_h, \nabla_h)U_h^G = \lambda\omega_h. \quad (4)$$

Proposition 3 *Let $m \in [0, \infty)$ and $\alpha \in \mathbb{R}$. If $\omega_h \in (L^2(m))^2$ satisfies (3) with* $\mathrm{Re}\,\lambda > -\frac{m}{2} - 1$, *then $\omega_h \in (L^2(\infty))^2$.*

$R_\alpha(t)$: semigroup of $L_{2D,\alpha} + \partial_{x_3}^2 - x_3\partial_{x_3}$

$$(R_\alpha(t)f)(x) = \frac{1}{\sqrt{2\pi(1-e^{-2t})}} \int_{\mathbb{R}} e^{-\frac{|x_3 e^{-t} - y_3|^2}{2(1-e^{-2t})}} (e^{tL_{2D,\alpha}}(f(\cdot, y_3))(x_h) dy_3.$$

Since $L - \alpha\Lambda = L_{2D,\alpha} + \partial_{x_3}^2 - x_3\partial_{x_3} - \alpha H$, the solution $\omega(t) = e^{t(L-\alpha\Lambda)}\omega_0$ satisfies

$$\omega(t) = R_\alpha(t)\omega_0 - \alpha \int_0^t R_\alpha(t-s) H\omega(s) ds.$$

We need:

(i) Estimate of $R_\alpha(t)$ (obtained from the estimate of $e^{tL_{2D,\alpha}}$)

(ii) Estimate of H (obtained from the Biot-Savart law)

3-4. Proof of Step 2: II. Estimate of $R_\alpha(t)$

$R_\alpha(t)$: semigroup of $L_{2D,\alpha} + \partial_{x_3}^2 - x_3\partial_{x_3}$ in $\mathbb{X}(m) = (X(m))^2 \times X_0(m)$

$L_{2D,\alpha} = (L_{2D,\alpha,h},\ L_{2D,\alpha,3})^\top$

$R_{\alpha,h}(t)$: semigroup of $L_{2D,\alpha,h} + \partial_{x_3}^2 - x_3\partial_{x_3}$ in $(X(m))^2 = (BC(\mathbb{R}; L^2(m)))^2$

$R_{\alpha,3}(t)$: semigroup of $L_{2D,\alpha,3} + \partial_{x_3}^2 - x_3\partial_{x_3}$ in $X_0(m) = BC(\mathbb{R}; L_0^2(m))$

Proposition 4 *Let $m > 2$ and $\alpha \in \mathbb{R}$. Then we have*

$$\partial_{x_3} R_\alpha(t) = e^{-t} R_\alpha(t)\partial_{x_3}, \tag{5}$$

$$\|R_{\alpha,h}(t) f_h\|_{(X(m))^2} \le C_{\alpha,\epsilon} e^{-(\frac{3}{2}-\epsilon)t} \|f_h\|_{(X(m))^2}, \qquad f_h \in (X(m))^2, \tag{6}$$

$$\|R_{\alpha,3}(t) f_3\|_{X(m)} \le C_{\alpha,\epsilon} e^{-\frac{1}{2}t} \|f_3\|_{X(m)}, \qquad f_3 \in X_0(m). \tag{7}$$

3-4. Proof of Step 2: III. Estimate of H

$$Hf = (K_{3D} * f, \nabla)G - (K_{2D} * f_3, \nabla)G - (G, \nabla)K_{3D} * f.$$

Then $Hf = (H_h f,\ H_3 f)^\top$, where

$$\begin{pmatrix} H_h f \\ \\ H_3 f \end{pmatrix} = \begin{pmatrix} -g\partial_{x_3}(K_{3D} * f)_h \\ \\ (K_{3D} * f, \nabla)g - (K_{2D} * f_3, \nabla)g - g\partial_{x_3}(K_{3D} * f)_3 \end{pmatrix},$$

Proposition 5 *Let* $m > 1$. *Then we have*

$$\partial_{x_3}H = H\partial_{x_3}, \tag{8}$$
$$\|H_h f\|_{(X(m))^2} \le C\|\partial_{x_3} f\|_{\mathbb{X}(m)}, \tag{9}$$
$$\|H_3 f\|_{X(m)} \le C(\|\partial_{x_3} f\|_{\mathbb{X}(m)} + \|f_h\|_{(X(m))^2}). \tag{10}$$

If f is a 2D vorticity field, then $f(x) = (0, 0, f_3(x_h))^\top$ and $Hf = 0$.

3-4. Proof of Step 2: V. Estimates of $R_\alpha(t)$.

$R_\alpha(t)$: semigroup of $L_{2D,\alpha} + \partial_{x_3}^2 - x_3\partial_{x_3}$

$$(R_{\alpha,h}(t) f_h)(x) = \frac{1}{\sqrt{2\pi(1-e^{-2t})}} \int_{\mathbb{R}} e^{-\frac{|x_3 e^{-t} - y_3|^2}{2(1-e^{-2t})}} (e^{tL_{2D,\alpha,h}}(f_h(\cdot, y_3))(x_h) dy_3,$$

$$(R_{\alpha,3}(t) f_3)(x) = \frac{1}{\sqrt{2\pi(1-e^{-2t})}} \int_{\mathbb{R}} e^{-\frac{|x_3 e^{-t} - y_3|^2}{2(1-e^{-2t})}} (e^{tL_{2D,\alpha,3}}(f_3(\cdot, y_3))(x_h) dy_3.$$

It is not difficult to see

$$\|R_{\alpha,h}(t) f_h\|_{(X(m))^2} \le \sup_{y_3\in\mathbb{R}} \|e^{tL_{2D,\alpha,h}}(f_h(\cdot, y_3))\|_{(L^2(m))^2},$$

$$\|R_{\alpha,3}(t) f_3\|_{X(m)} \le \sup_{y_3\in\mathbb{R}} \|e^{tL_{2D,\alpha,3}}(f_3(\cdot, y_3))\|_{L^2(m)}.$$

Our goal is to show

$$\|e^{tL_{2D,\alpha,h}}\phi_h\|_{(L^2(m))^2} \le C_\epsilon e^{-(\frac{3}{2}-\epsilon)t}\|\phi_h\|_{(L^2(m))^2}, \qquad \phi_h \in (L^2(m))^2, \quad (11)$$

$$\|e^{tL_{2D,\alpha,3}}\phi_3\|_{L^2(m)} \le C_\epsilon e^{-\min\{\frac{1}{2},\frac{m-1}{2}-\epsilon\}t}\|\phi_3\|_{L^2(m)}, \qquad \phi_3 \in L^2_0(m). \quad (12)$$

Rem. (12) is obtained by Gallay-Wayne (2005).