Qualitative Behavior of Solutions for the Two-Phase Navier-Stokes Equations with Surface Tension

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Notations

$\Omega \subset \mathbb{R}^n$ open bounded domain with smooth boundary, $\Omega_i(t)$ subdomains occupied by immiscible incompressible fluid $i = 1, 2$, $\Gamma(t)$ interface separating the two phases. No boundary contact, i.e $\Gamma(t) \cap \partial\Omega = \emptyset$, no phase transitions, isothermal conditions and no external forces.

$u = u(t, x)$ velocity field, $\pi = \pi(t, x)$ pressure field
$S(t, x)$ stress tensor
$E(t, x) := \frac{1}{2}(\nabla u(t, x) + \nabla u(t, x)^{\mathsf{T}})$ rate of strain tensor
$\rho_i > 0$ densities, $\mu_i > 0$ viscosities in the phases
$\nu_\Gamma(t, x)$ the normal at $x \in \Gamma(t)$ directed into $\Omega_2(t)$
$V_\Gamma(t, x) = (u(t, x)|\nu_\Gamma(t, x))$ normal velocity of $\Gamma(t)$
$H_\Gamma(t, x) = -\mathrm{div}_\Gamma \nu_\Gamma(t, x)$ curvature of $\Gamma(t)$

$$[\![\phi]\!] = \lim_{h \to 0+} [\phi(t, x + h\nu_\Gamma(t, x)) - \phi(t, x - h\nu_\Gamma(t, x))]$$

is the jump of the quantity ϕ accross $\Gamma(t)$.

In the bulk phases:

$$\begin{aligned} \partial_t(\rho u) + \nabla \cdot (\rho u \otimes u - S) &= 0, \qquad x \in \Omega \backslash \Gamma(t),\ t > 0, \\ \nabla \cdot u &= 0, \qquad x \in \Omega \backslash \Gamma(t),\ t > 0, \\ \mu(\nabla u + \nabla u^{\mathsf{T}}) - \pi I &= S, \qquad x \in \Omega \backslash \Gamma(t),\ t > 0. \end{aligned}$$

At the interface:

$$\begin{aligned} [\![u]\!] &= 0, \quad x \in \Gamma(t),\ t > 0, \\ (u|\nu_\Gamma) &= V_\Gamma, \quad x \in \Gamma(t),\ t > 0, \\ -[\![S]\!]\,\nu_\Gamma &= \sigma H_\Gamma \nu_\Gamma, \quad x \in \Gamma(t),\ t > 0. \end{aligned}$$

Initial conditions:

$$u(0,x) = u_0(x) \quad x \in \Omega, \quad \Gamma(0) = \Gamma_0.$$

No-slip boundary conditions at $\partial\Omega$, i.e. $u = 0$ on $\partial\Omega$.

Literature - a selection

One-phase problems with surface tension:
Beale 1980, 1984, Solonnikov 1987-, Tani & Tannaka 1996, Shimizu & Shibata 2005-.

Two phase problems with surface tension:
Denisova 1988-, Tani 1996, Tannaka 1995, Shimizu & Shibata 2009, Prüss & Simonett 2009.

Of particular relevance for convergence of solutions are the papers of Beale and Solonnikov, both in the *one-phase case*.

Here we are interested in the **qualitative and asymptotic behaviour** of solutions of the two-phase problem.

Local Well-Posedness I

Theorem

Fix $p > n + 2$, *let* $\partial\Omega \in C^3$, *and suppose*

$$\Gamma_0 \in W_p^{3-2/p}, \quad u_0 \in W_p^{2-2/p}(\Omega\backslash\Gamma_0)^n.$$

Assume the compatibility conditions

$$\begin{aligned} \mathrm{div}\, u_0 = 0 \ \ \textit{in}\ \Omega\backslash\Gamma_0, \quad u = 0 \ \textit{on}\ \partial\Omega, \\ [\![\mathcal{P}_{\Gamma_0}\mu E_0\nu_{\Gamma_0}]\!] = 0,\ [\![u_0]\!] = 0 \ \textit{on}\ \Gamma_0, \end{aligned}$$

where $E_0 = \frac{1}{2}(\nabla u_0 + \nabla u_0^\mathsf{T})$, *and* $\mathcal{P}_{\Gamma_0} = I - \nu_{\Gamma_0} \times \nu_{\Gamma_0}$.

Local Well-Posedness II

Theorem

Then there exists $t_0 = t_0(u_0, \Gamma_0) > 0$ *and a unique classical solution* (u, π, Γ) *of the problem on* $(0, t_0)$*. The set*

$$\bigcup_{t \in (0,t_0)} \{t\} \times \Gamma(t)$$

is a real analytic manifold, and with

$$\mho := \{(t, x) \in (0, t_0) \times \Omega,\ x \notin \Gamma(t)\},$$

the function $(u, \pi) : \mho \to \mathbb{R}^{n+1}$ *is real analytic.*

Strategy for the proof

Approximate Γ_0 by a smooth hypersurface Σ.

a) **Transformation to a Fixed Domain (Hanzawa)**
Let $d(x)$ denote the signed distance of $x \in \mathbb{R}^n$ to Σ, and $\Pi(x)$ the projection of $x \in \mathbb{R}^n$ to Σ. Then

$$\Lambda : \Sigma \times (-a, a) \to \mathbb{R}^n$$
$$\Lambda(p, r) := p + r\nu_\Sigma(p), \quad \Lambda^{-1}(x) = (\Pi(x), d(x))$$

is a diffeomorphism from $\Sigma \times (-a, a)$ onto $\mathcal{R}(\Lambda) = \{x \in \mathbb{R}^n : |d(x)| < a\}$, provided

$$0 < a < \min\{r, 1/\kappa_j(x) : j = 1, \dots, n-1,\ x \in \Sigma\},$$

where $\kappa_j(x)$ mean the principal curvatures of Σ at $x \in \Sigma$ and

$$\bar{B}_r(p \pm r\nu_\Sigma(p)) \cap \Sigma = \{p\}, \quad p \in \Sigma.$$

Use this to parameterize $\Gamma(t)$ over Σ:

$$\Gamma(t) : p \mapsto p + h(t, p)\nu_\Sigma(p), \quad p \in \Sigma,\ t \geq 0.$$

Extend this diffeomorphism to all of Ω:

$$\Theta(t,x) = x + \chi(d(x))h(t,\Pi(x))\nu_\Sigma(\Pi(x)).$$

Here χ denotes a suitable cut-off function. This way $\Omega \backslash \Gamma(t)$ is transformed to the fixed domain $\Omega \backslash \Sigma$. Set

$$\begin{aligned} \bar{u}(t,x) &= u(t,\Theta(t,x)), \\ \bar{\pi}(t,x) &= \pi(t,\Theta(t,x)), \quad t>0,\ x \in \Omega\backslash\Sigma. \end{aligned}$$

This yields the problem (drop the bars!)

$$\begin{aligned}
&\rho\partial_t u - \mu\mathcal{A}(h)u + \mathcal{G}(h)\pi = \mathcal{R}(u,h) \quad \text{in } \Omega\backslash\Sigma, \\
&(\mathcal{G}(h)|u) = 0 \quad \text{in } \Omega\backslash\Sigma, \\
&u = 0 \quad \text{on } \partial\Omega, \\
&[\![-\mu(\mathcal{G}(h)u + [\mathcal{G}(h)u]^\mathsf{T}) + \pi]\!]\nu_\Gamma(h) = \sigma H_\Gamma(h)\nu_\Gamma(h) \text{ on } \Sigma, \\
&[\![u]\!] = 0 \quad \text{on } \Sigma, \\
&\partial_t h - (u|\nu_\Sigma) = -(u|\alpha(h)), \quad \text{on } \Sigma, \\
&u(0) = u_0, \quad h(0) = h_0.
\end{aligned} \tag{1}$$

Here $\mathcal{A}(h)$ and $\mathcal{G}(h)$ denote the transformed Laplacian, resp. gradient. With the curvature tensor L_Σ and the surface gradient ∇_Σ we have

$$\begin{aligned} \nu_\Gamma(h) &= \beta(h)(\nu_\Sigma - \alpha(h)), \qquad & \alpha(h) &= M(h)\nabla_\Sigma h, \\ M(h) &= (I - hL_\Sigma)^{-1}, \qquad & \beta(h) &= (1 + |\alpha(h)|^2)^{-1/2}, \end{aligned}$$

and

$$V = (\partial_t \Theta | \nu_\Gamma) = \partial_t h (\nu_\Gamma | \nu_\Sigma) = \beta(h) \partial_t h.$$

The curvature $H_\Gamma(h)$ becomes

$$\begin{aligned} H_\Gamma(h) = \beta(h)\{\mathrm{tr}[M(h)(L_\Sigma + \nabla_\Sigma \alpha(h))] \\ - \beta^2(h)(M(h)\alpha(h)|[\nabla_\Sigma \alpha(h)]\alpha(h))\}, \end{aligned}$$

a differential expression involving second order derivatives of h only linearly. Linearization of $H_\Gamma(h)$ at $h = 0$: $H'_\Gamma(0) = \mathrm{tr}\, L_\Sigma^2 + \Delta_\Sigma$, where Δ_Σ denotes the Laplace-Beltrami operator on Σ.

Rewrite this problem in quasilinear form.

$$
\begin{aligned}
&\rho\partial_t u - \mu\Delta u + \nabla\pi = F(u,\pi,h) \quad \text{in } \Omega\backslash\Sigma,\\
&\operatorname{div} u = F_d(u,h) \quad \text{in } \Omega\backslash\Sigma,\\
&u = 0 \quad \text{on } \partial\Omega,\\
&[\![-\mu(\nabla u + \nabla u^{\mathsf{T}}) + \pi]\!]\nu_\Sigma - \sigma(\Delta_\Sigma h)\nu_\Sigma = G(u,[\![\pi]\!],h) \quad \text{on } \Sigma,\\
&[\![u]\!] = 0 \quad \text{on } \Sigma, \qquad (2)\\
&\partial_t h - (u|\nu_\Sigma) = G_h(u,h) \quad \text{on } \Sigma,\\
&u(0) = u_0, \quad h(0) = h_0
\end{aligned}
$$

The right hand sides in this problem consist of lower order terms and of terms of the same order appearing on the left, but carrying a factor $|\nabla_\Sigma h|$, which is small by construction.

b) **The Linear Problem**

Establish maximal L_p-regularity for the linear problem defined by the l.h.s. of (2). In particular, find the right spaces for the data, such that the *solution-to-data map* for the following inhomogeneous linear problem becomes an *isomorphism*.

$$\begin{aligned} &\rho\partial_t v - \mu\Delta v + \nabla q = f_v \quad \text{in } \Omega\backslash\Sigma,\\ &\operatorname{div} v = f_d \quad \text{in } \Omega\backslash\Sigma\\ &v = 0 \quad \text{on } \partial\Omega,\\ &[\![-\mu(\nabla v + \nabla v^{\mathsf{T}}) + q]\!]\nu_\Sigma - \sigma(\Delta_\Sigma h)\nu_\Sigma = g \quad \text{on } \Sigma,\\ &[\![v]\!] = 0 \quad \text{on } \Sigma,\\ &\partial_t h - (v|\nu_\Sigma) = g_h \quad \text{on } \Sigma,\\ &v(0) = v_0, \quad h(0) = h_0. \end{aligned} \tag{3}$$

This is proved by localization and perturbation, and a corresponding result for a flat interface, i.e. $\Omega = \mathbb{R}^n$, $\Sigma = \mathbb{R}^{n-1} \times \{0\}$; see Prüss & Simonett 2009.

The solutions of the transformed problem will belong to the following class:

$$v \in H_p^1(J; L_p(\Omega)^n) \cap L_p(J; H_p^2(\Omega\backslash\Sigma)^n), \quad q \in L_p(J; \dot{H}_p^1(\Omega\backslash\Sigma)),$$

$$[\![q]\!] \in W_p^{1/2-1/2p}(J; L_p(\Sigma)) \cap L_p(J; W_p^{1-1/p}(\Sigma)),$$

$$h \in W_p^{2-1/2p}(J; L_p(\Sigma)) \cap H_p^1(J; W_p^{2-1/p}(\Sigma)) \cap L_p(J; W_p^{3-1/p}(\Sigma)),$$

provided

$$v_0 \in W_p^{2-2/p}(\Omega\backslash\Sigma)^n, \quad h_0 \in W_p^{3-2/p}(\Sigma),$$

and the natural compatibility conditions hold.

The Linear Problem - Main Result

Theorem

Let $1 < p < \infty$, ρ_j, μ_j, σ *be positive constants,* $j = 1, 2$; *set* $J = [0, a]$. *Then the two-phase Stokes problem* (3) *admits a unique solution* (v, q, h) *with regularity*

$$v \in H^1(J; L_p(\Omega))^n \cap L_p(J; H_p^2(\Omega \backslash \Sigma))^n, \quad q \in L_p(J; \dot{H}_p^1(\Omega \backslash \Sigma)),$$

$$[\![q]\!] \in W_p^{1/2-1/2p}(J; L_p(\Sigma)) \cap L_p(J; W_p^{1-1/p}(\Sigma)),$$

$$h \in W_p^{2-1/2p}(J; L_p(\Sigma)) \cap H_p^1(J; W_p^{2-1/p}(\Sigma) \cap L_p(J; W_p^{3-1/p}(\Sigma)).$$

if and only if the data $(v_0, h_0, f, f_d, g, g_h)$ *satisfy the following regularity and compatibility conditions.*

Theorem

(a) $f_v \in L_p(J \times \Omega)^n$, $v_0 \in W_p^{2-2/p}(\Omega\backslash\Sigma)^n, v_0 = 0$ *on* $\partial\Omega$;
(b) $f_d \in H_p^1(J; H_p^{-1}(\Omega\backslash\Sigma)) \cap L_p(J; H_p^1(\Omega\backslash\Sigma)), \quad \mathrm{div}\, v_0 = f_d(0)$;
(c) $g \in W_p^{1/2-1/2p}(J; L_p(\Sigma)) \cap L_p(J; W_p^{1-1/p}(\Sigma))$;
(d) $[\![v_0]\!] = 0, \quad \mathcal{P}_\Sigma[\![\mu(\nabla v_0 + \nabla v_0^{\mathsf{T}})]\!] = g_v(0)$;
(e) $h_0 \in W_p^{3-2/p}(\Sigma)$;
(f) $g_h \in W_p^{1-1/2p}(J; L_p(\Sigma)) \cap L_p(J; W_p^{2-1/p}(\Sigma))$.

The solution map $(v_0, h_0, f, f_d, g, g_h) \mapsto (v, q, h)$ *is continuous between the corresponding spaces.*

c) **The Nonlinear Problem**

Based on *maximal regularity*, use the *implicit function theorem* to obtain local well-posedness of the nonlinear problem.

Use a variant of ANGENENT'S *parameter trick* to obtain real analyticity via *maximal regularity* and the *implicit function theorem*; cf. ESCHER, PRÜSS & SIMONETT 2003.

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The Induced Semiflow

Recall that the closed C^2-hypersurfaces contained in Ω form a C^2-manifold, which we denote by $\mathcal{MH}^2$. Charts are obtained via parametrization over a fixed hypersurface. As an ambient space for the phase-manifold $\mathcal{PM}$ of the two-phase Navier-Stokes problem with surface tension we consider the product space $C(\bar{\Omega})^n \times \mathcal{MH}^2$. We define $\mathcal{PM}$ as follows.

$$\mathcal{PM} := \{(u,\Gamma) \in C(\bar{\Omega})^n \times \mathcal{MH}^2 : u \in W_p^{2-2/p}(\Omega\backslash\Gamma)^n, \Gamma \in W_p^{3-2/p}, \operatorname{div} u = 0 \text{ in } \Omega\backslash\Gamma, \ u = 0 \text{ on } \partial\Omega, \mathcal{P}_\Gamma [\![\mu E]\!] \nu_\Gamma = 0 \text{ on } \Gamma\}. \quad (4)$$

The charts for this manifold are obtained by the charts induced by $\mathcal{MH}^2$, followed by a HANZAWA transformation.

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The charts for this manifold are obtained by the charts induced by $\mathcal{MH}^2$, followed by a Hanzawa transformation.

Observe that the compatibility conditions

$$\begin{aligned} &\operatorname{div} u = 0 \ \text{ in } \Omega\backslash\Gamma, \quad u = 0 \text{ on } \partial\Omega, \\ &\mathcal{P}_\Gamma \mu[\![(\nabla u + \nabla u^{\mathsf{T}})]\!]\nu_\Gamma = 0, \ [\![u]\!] = 0 \quad \text{on } \Gamma, \end{aligned}$$

as well as regularity are preserved by the solutions.
This yields the following result

Theorem

Let $p > n + 2$. Then the two-phase Navier-Stokes problem with surface tension generates a local semiflow on the phase-manifold $\mathcal{PM}$. Each solution (u, Γ) exists on a maximal time interval $[0, t_*)$.

The energy functional and equilibria

Define the *energy functional* by means of

$$\Phi(u,\Gamma) := \frac{1}{2}\|\rho^{1/2}u\|^2_{L_2(\Omega)} + \sigma \,\mathrm{mes}\,\Gamma(t).$$

Then

$$\partial_t\Phi(u,\Gamma) + 2\|\mu^{1/2}E\|^2_{L_2(\Omega)} = 0,$$

hence the energy functional is a Ljapunov functional, even a strict one.

Proposition

Let $\rho_i, \mu_i, \sigma > 0$ be constants. Then
(a) The **energy equality** *is valid for smooth solutions.*
(b) The **equilibria** *are zero velocities, constant pressures in the phase-components, the dispersed phase is a union of nonintersecting balls.*
(c) The **energy functional** *is a strict Ljapunov-functional.*
(d) The **critical points** *of the energy functional for constant phase volumes are precisely the equilibria.*

The Stability Result

Assuming, for simplicity, that the phases are connected, we denote by

$$\mathcal{E} := \{(0, S_R(x_0)) : x_0 \in \Omega, R > 0, S_R(x_0) \subset \Omega\}$$

the set of equilibria without boundary contact. Note that $\mathcal{E}$ forms a real analytic manifold of dimension $n+1$.
Fix any such equilibrium $(0, \Sigma) \in \mathcal{E}$. We consider the behaviour of the solutions near this steady state.
Here we have to use the *full linearization* of the problem at an equilibrium $(0, \Sigma)$ i.e. at $(u, h) = (0, 0)$, and for this reason we have to replace Δ_Σ in the linear problem (3) by

$$\mathcal{A}_\Sigma = H'_\Gamma(0) = \frac{n-1}{R^2} + \Delta_\Sigma.$$

It is well-known that $\mathcal{A}_\Sigma$ is *selfadjoint*, *negative semidefinite* and has *compact resolvent* in $L_2(\Sigma)$; $\lambda_0 = 0$ is an eigenvalue of dimension n.

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Theorem

The equilibrium $(0, \Sigma)$ *is stable in the sense that for each* $\varepsilon \in (0, \varepsilon_0]$ *there exists* $\delta(\varepsilon) > 0$ *such that for all initial values* (u_0, Γ_0) *subject to*

$$\mathrm{dist}_{W_p^{3-2/p}}(\Gamma_0, \Sigma) \leq \delta(\varepsilon) \quad \textit{and} \quad \|u_0\|_{W_p^{2-2/p}(\Omega\setminus\Gamma_0)} \leq \delta(\varepsilon)$$

there exists a unique global solution $(u(t), \Gamma(t))$ *of the problem, and it satisfies*

$$\mathrm{dist}_{W_p^{3-2/p}}(\Gamma(t), \Sigma) \leq \varepsilon \quad \textit{and} \quad \|u(t)\|_{W_p^{2-2/p}(\Omega\setminus\Gamma(t))} \leq \varepsilon, \ \ t \geq 0.$$

Moreover, as $t \to \infty$ *each of these solutions* $(u(t), \Gamma(t))$ *converges to a probably different equilibrium* $(0, \Sigma_\infty)$ *in the same topology, i.e.*

$$\lim_{t\to\infty} \left(\mathrm{dist}_{W_p^{3-2/p}}(\Gamma(t), \Sigma_\infty) + \|u(t)\|_{W_p^{2-2/p}(\Omega\setminus\Gamma(t))} \right) = 0.$$

The convergence is at exponential rate.

Sketch of the Proof

As a base space we use

$$X_0 = L_{p,\sigma}(\Omega)^n \times W_p^{2-1/p}(\Sigma),$$

and we set

$$\bar{X}_1 = W_p^2(\Omega \setminus \Sigma)^n \times W_p^{3-1/p}(\Sigma)$$

Define a closed linear operator in X_0 by means of

$$A(v,h) = (-(\mu/\rho)\Delta v + \rho^{-1}\nabla q, -(v|\nu_\Sigma)),$$

with domain $X_1 := D(A) \subset \bar{X}_1$

$$D(A) = \{(v,h) \in \bar{X}_1 \cap X_0 : v = 0 \text{ on } \partial\Omega,\ [\![v]\!] = 0 \text{ and } [\![\mathcal{P}_\Sigma \mu(\nabla v + \nabla v^{\mathsf{T}})\nu_\Sigma]\!] = 0 \text{ on } \Sigma\}.$$

Here $q \in \dot{W}_p^1(\Omega \setminus \Sigma)$ is determined as the solution of the *weak transmission problem*

$$(\rho^{-1}\nabla q|\nabla\phi)_{L_2} = ((\mu/\rho)\Delta v|\nabla\phi)_{L_2}, \quad \phi \in W_{p'}^1(\Omega),$$

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Then with $z = (v, h)$ and $f = (f_v, g_h)$ as well as $z_0 = (v_0, h_0)$, system (3) can be rewritten as the abstract evolution equation

$$\dot{z} + Az = f, \quad t > 0, \quad z(0) = z_0, \tag{5}$$

provided $f_d = 0$ and $g = 0$.
Since (3) has maximal L_p-regularity, the abstract problem (5) has maximal L_p-regularity, as well. In particular, $-A$ generates an analytic C_0-semigroup in X_0.
We can show the following properties:

(i) The set of equilibria $\mathcal{E}$ is an $(n+1)$-dimensional smooth manifold.
(ii) The kernel $N(A)$ is isomorphic to the tangent space $T_{(0,\Sigma)}\mathcal{E}$.
(iii) $N(A) \oplus R(A) = X_0$, i.e. the eigenvalue $\lambda_0 = 0$ of A is semi-simple.
(iv) $\sigma(A) \setminus \{0\} \subset \mathbb{C}_+$, i.e. the semigroup $e^{-At}|_{R(A)}$ is exponentially stable.

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Global Existence and Convergence

There are basically two obstructions against global existence:

- **regularity**: the norms of either $u(t)$ or $\Gamma(t)$ become unbounded;
- **geometry**: the topology of the interface changes or the interface touches the boundary of Ω.

Note that the *phase volumes* are conserved by the semiflow!

We say that a solution (u, Γ) satisfies a uniform ball condition, if there is a radius $r > 0$ such that for each $t \in [0, t_*)$ and at every point $p \in \Gamma(t)$ we have

$$\bar{B}_r(p \pm r\nu_{\Gamma(t)}(p)) \cap \Gamma(t) = \{p\}.$$

Combining the above results, we obtain the following theorem on the asymptotic behavior of solutions.

Theorem

Let $p > n+2$. Suppose that (u, Γ) is a solution of the two-phase Navier-Stokes problem with surface tension on the maximal time interval $[0, t_*)$. Assume the following on $[0, t_*)$:
(i) $\|u(t)\|_{W_p^{2-2/p}} + \|\Gamma(t)\|_{W_p^{3-2/p}} \leq M < \infty$;
(ii) (u, Γ) *satisfies a uniform ball condition.*
Then $t_* = \infty$, i.e. the solution exists globally, and it converges in $\mathcal{PM}$ at exponential rate to an equilibrium $(0, \Gamma_\infty) \in \mathcal{E}$. The converse is also true.

The idea of the proof is as follows. Assuming (i) and (ii) we show that the solution is global. The energy is a strict Ljapunov functional, hence the limit set $\omega(u, \Gamma)$ of a solution is contained in the set $\mathcal{E}$ of equilibria. A compactness argument shows $\omega(u, \Gamma) \neq \emptyset$, hence the solution comes close to $\mathcal{E}$. Then we may apply the local convergence result.

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