On a Mathematical Analysis of a flow of Inhomogeneous Incompressible Fluid-like Bodies

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A Joint Work with Professor Atusi Tani at Keio University

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- Specific equation and boundary conditions in this problem
- Necessary condition
- Existence theorem

1. Introduction

- Granular Materials

  example - quick sand, avalanches, *etc.*

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Dynamics

⬇

n-body problem

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In general *n*-body problem cannot be solved…

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n-body problem

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In general *n*-body problem cannot be solved…

$n \sim 10^{12}/m^3$

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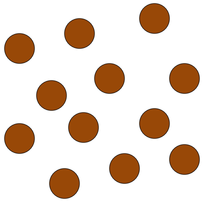
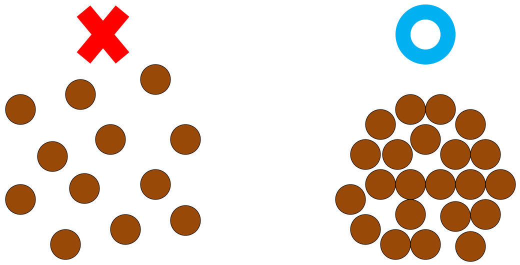


⟶ Void volume can vary through the process

⟶ The "density" should be regarded as the volume fraction

Incompressible model

- Incompressible motion



"sufficiently" dense

→ interstitial working is relatively negligible

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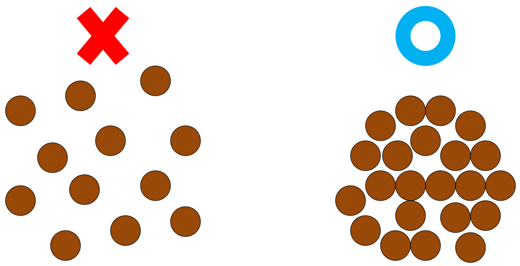

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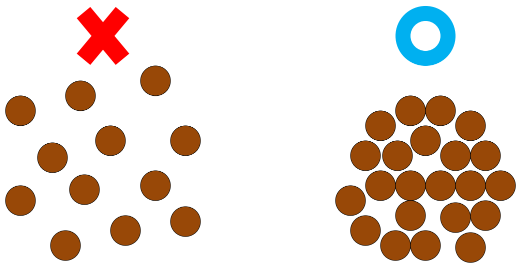

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➡ interstitial working is relatively negligible

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Volume fraction at each point still varies
(= regarded as <u>inhomogeneous</u> media)

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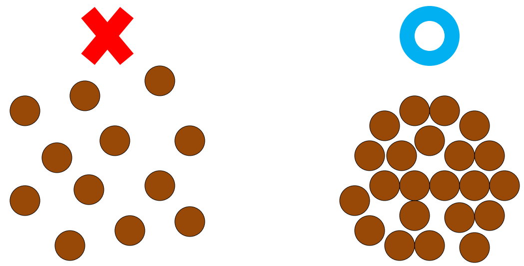

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➡ Consider a continuum model for Inhomogeneous Incompressible Fluid-like Bodies (IIFB).

Continuum Approximation

<u>Conservation laws in general</u>

- Conservation of mass: $\varrho_t + \mathrm{div}(\varrho \mathbf{v}) = 0$
- Conservation of linear momentum: $\varrho \left(\mathbf{v}_t + (\mathbf{v} \cdot \nabla) \mathbf{v} \right) = \mathrm{div}\mathbb{T} + \varrho \mathbf{b}$
- Conservation of angular momentum: $\mathbb{T} = \mathbb{T}^{\mathrm{T}}$

$$\varrho : \text{density}, \ \ \mathbf{v} : \text{velocity}, \ \ \mathbb{T} : \text{Cauchy stress}, \ \ \mathbf{b} : \text{external body force}.$$

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➡ IIFB model derived by Málek and Rajagopal (2006):

$$\mathbb{T} = -p\mathbb{I} + 2\nu\mathbb{D}(\mathbf{v}) - \beta\left(\mathbb{M} - \tfrac{1}{3}(\mathrm{tr}\mathbb{M})\mathbb{I}\right),$$

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Initial and Boundary Conditions

- <u>Initial conditions</u>

$$(\varrho, \mathbf{v})\big|_{t=0} = (\varrho_0, \mathbf{v}_0)$$

- <u>Boundary conditions</u>

  - Adherence condition: $\mathbf{v} = \mathbf{0}.$

  or

  - Generalized Navier's slip boundary conditions:

$$\mathbf{v} \cdot \mathbf{n} = 0, \quad \mathbf{v} + K\Pi\mathbb{T}\mathbf{n} = \mathbf{0},$$

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- $K \equiv 0 \quad \Longrightarrow \quad \mathbf{v} = 0$ on $\partial\Omega$ (no-slip case)
- $K \equiv \infty \quad \Longrightarrow \quad \Pi\mathbb{T}\mathbf{n} = 0$ on $\partial\Omega$ (perfect-slip case)

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(1)\quad \begin{cases} \varrho_t + (\mathbf{v}\cdot\nabla)\varrho = 0, \quad \operatorname{div}\mathbf{v} = 0 \quad \text{in } Q_T, \\ \varrho(\mathbf{v}_t + (\mathbf{v}\cdot\nabla)\mathbf{v}) = -\nabla p + \nu(\varrho)\Delta\mathbf{v} + 2\nu'(\varrho)\mathbb{D}(\mathbf{v})\nabla\varrho \\ \qquad\qquad - \frac{\beta}{3}\left(\frac{\partial^2\varrho}{\partial x_i \partial x_j}\right)\nabla\varrho - \beta\Delta\varrho\nabla\varrho + \varrho\mathbf{b} \quad \text{in } Q_T, \end{cases} \\
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When $\beta = 0$, problem (1) immediately becomes the usual Navier-Stokes equations.

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→ The terms related to β are originally derived from the Helmholtz free energy. In the case $\beta = 0$, the free energy of the body under consideration does not depend on $\nabla\varrho$.

→ β represents the magnitude of the influence of material inhomogeneity on the motion.

Eulerian and Lagrangian coordinates

The relationship between Eulerian coordinates X and Lagrangian coordinates x :

$$(2) \quad X = x + \int_0^t \mathbf{u}(x,\tau)\,\mathrm{d}\tau \equiv X_{\mathbf{u}}(x,t), \quad \mathbf{u}(x,t) = \mathbf{v}(X_{\mathbf{u}}(x,t),t),$$

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- $$\nabla_X F(X,t) \longleftrightarrow \nabla_{\mathbf{u}} F_{\mathbf{u}}(x,t) \quad \text{for } F_{\mathbf{u}}(x,t) = F(X_{\mathbf{u}}(x,t),t),$$

$$\nabla_{\mathbf{u}} = \mathbb{A}^{-\mathrm{T}}\nabla_x, \quad \mathbb{A} = (a_{ij}), \quad a_{ij} = \frac{\partial X^i_{\mathbf{u}}}{\partial x_j} = \delta_{ij} + \int_0^t \frac{\partial u_i}{\partial x_j}(x,\tau)\mathrm{d}\tau.$$

Transformed Problem in Lagrangian coordinates

$$
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\begin{cases}
\nabla_{\mathbf{u}} \cdot \mathbf{u} = 0 \quad \text{in } Q_T, \\
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q : pressure in the Lagrangian coordinate system, $\Delta_{\mathbf{u}} = \nabla_{\mathbf{u}}\cdot\nabla_{\mathbf{u}}$,

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$\mathbb{T}_{\mathbf{u}} = -q\,\mathbb{I} + 2\nu(\varrho_0)\mathbb{D}_{\mathbf{u}}(\mathbf{u}) - \beta\left(\nabla_{\mathbf{u}}\varrho_0 \otimes \nabla_{\mathbf{u}}\varrho_0 - \frac{1}{3}|\nabla_{\mathbf{u}}\varrho_0|^2\right)$.

Transformed Problem in Lagrangian coordinates

$$
(3)\quad
\begin{cases}
\nabla_{\mathbf{u}} \cdot \mathbf{u} = 0 \quad \text{in } Q_T, \\
\varrho_0 \mathbf{u}_t = -\nabla_{\mathbf{u}} q + \nu(\varrho_0)\Delta_{\mathbf{u}}\mathbf{u} + 2\nu'(\varrho_0)\mathbb{D}_{\mathbf{u}}(\mathbf{u})\nabla_{\mathbf{u}}\varrho_0 \\
\qquad\qquad -\frac{\beta}{3}(\nabla^i_{\mathbf{u}}\nabla^j_{\mathbf{u}}\varrho_0)\nabla_{\mathbf{u}}\varrho_0 - \beta\Delta_{\mathbf{u}}\varrho_0\nabla_{\mathbf{u}}\varrho_0 + \varrho_0\mathbf{b}_{\mathbf{u}} \quad \text{in } Q_T, \\
\mathbf{u}|_{t=0} = \mathbf{v}_0 \quad \text{in } \Omega, \\
\mathbf{u}\cdot\underline{\mathbf{n}_{\mathbf{u}}} = 0, \quad \mathbf{u} + \underline{K_{\mathbf{u}}\Pi_{\mathbf{u}}\mathbb{T}_{\mathbf{u}}\mathbf{n}_{\mathbf{u}}} = \mathbf{0} \quad \text{on } G_T,
\end{cases}
$$

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$\longrightarrow (\mathbf{u}, q)$: unknown

Time-local existence theorem for problem (3)

Existence Theorem (NN-Tani, 2009)

Let Ω *be a bounded domain in* $\mathbb{R}^3$ *with the boundary* $\Gamma \in W_2^{7/2+l}$, $l \in (1/2, 1)$,

$\varrho_0 \in W_2^{2+l}(\Omega)$, $\varrho_0(x) \geq R_0 > 0$, $\nu \in C^2(\overline{\mathbb{R}}_+)$, $\inf \nu > 0$, $0 < T < +\infty$,

$\mathbf{b} \in W_2^{l,l/2}(Q_T)$, $\mathbf{v}_0 \in W_2^{1+l}(\Omega)$. *Assume that* $\mathbf{b}$ *and* K *are sufficiently smooth,*

$$(\textsc{Condition}\ K) \begin{cases} \bullet\ K(X,t) \equiv k : \text{constant},\ k \geq 0, \\ \text{or} \\ \bullet\ \inf\limits_{(X,t)\in G_T} K(X,t) > 0, \end{cases}$$

and that certain compatibility conditions for $\mathbf{v}_0$, ϱ_0 *and* K *are satisfied.*

Then problem (3) *has a unique solution* $(\mathbf{u}, \nabla q) \in W_2^{2+l,1+l/2}(Q_{T'}) \times W_2^{l,l/2}(Q_{T'})$ *on some interval* $(0, T')$ $(0 < T' \leq T)$, *whose magnitude* T' *depends on the data. Furthermore, if* $K \equiv k$, T' *can be taken uniformly in* $k \geq 0$.

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Furthermore, if $K \equiv k$, T' *can be taken uniformly in* $k \geq 0$.

Anisotropic Sobolev-Slobodetskiĭ spaces

$$\gamma > 0,\ \ f \in W_2^{\gamma,\gamma/2}(Q_T) \Longleftrightarrow \|f\|_{W_2^{\gamma,\gamma/2}(Q_T)} < +\infty$$

$$\|f\|^2_{W_2^{\gamma,\gamma/2}(Q_T)} = \|f\|^2_{W_2^{\gamma,0}(Q_T)} + \|f\|^2_{W_2^{0,\gamma/2}(Q_T)},$$

$$\|f\|^2_{W_2^{\gamma,0}(Q_T)} = \int_0^T \|f(\cdot,t)\|^2_{W_2^{\gamma}(\Omega)}dt,$$

$$\|f\|^2_{W_2^{0,\gamma/2}(Q_T)} = \int_\Omega \|f(x,\cdot)\|^2_{W_2^{\gamma/2}(0,T)}dx,$$

$$\|f\|^2_{W_2^{\gamma}(\Omega)} = \sum_{|\alpha|<\gamma} \|D^\alpha f\|^2_{L^2(\Omega)} + \|f\|^2_{\dot{W}_2^{\gamma}(\Omega)}$$

$$\|f\|^2_{\dot{W}_2^{s}(\Omega)} = \begin{cases} \displaystyle\sum_{|\alpha|=[\gamma]} \int_\Omega\int_\Omega \frac{|D^\alpha f(x) - D^\alpha f(y)|^2}{|x-y|^{n+2(\gamma-[\gamma])}}dxdy & (\gamma \notin \mathbb{Z}_+), \\ \displaystyle\sum_{|\alpha|=[\gamma]} \|D^\alpha f\|^2_{L^2(\Omega)} & (\gamma \in \mathbb{Z}_+), \end{cases}$$

Convergence results concerning problem (3)

Theorem 1 (NN-Tani) —concerning the limit of slip rate as $k \to 0,\ k \to \infty$ —

Let $\Omega,\ \Gamma,\ l,\ \varrho_0,\ \mathbf{v}_0,\ \nu,\ \beta,\ T,\ T'$ *and* $\mathbf{b}$ *be the same as in Existence Theorem. Suppose that* $K(X,t) \equiv k,\ k \geq 0,$ *and denote the solutions of problem* (3) *with* k *by* $(\mathbf{u}^{(k)}, \nabla q^{(k)})$, *i.e.,* $K_{\mathbf{u}} \equiv k$ *in* (3). *Then the sequence of the solutions of Navier's slip problems*

$$\left\{(\mathbf{u}^{(k)}, \nabla q^{(k)})\right\}_{k>0} \ \textit{converges to} \ \begin{cases} (\mathbf{u}^{(0)}, \nabla q^{(0)}) \ \textit{as} \ k \downarrow 0 \quad (\textit{no-slip case}), \\ (\mathbf{u}^{(\infty)}, \nabla q^{(\infty)}) \ \textit{as} \ k \uparrow \infty \quad (\textit{perfect-slip case}) \end{cases}$$

in $W_2^{2+l,1+l/2}(Q_{T'}) \times W_2^{l,l/2}(Q_{T'})$, *where* $(\mathbf{u}^{(0)}, \nabla q^{(0)})$ *and* $(\mathbf{u}^{(\infty)}, \nabla q^{(\infty)})$ *are solutions of* (3) *with the adherence boundary condition* $(k \equiv 0)$ *and the perfect-slip boundary condition* $(k \equiv \infty)$, *respectively.*

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According to this result, the Navier's slip boundary condition is the reasonable generalization of the usual adherence boundary conditions asigned in problems for viscous fluids.

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in $W_2^{2+l,1+l/2}(Q_{T'}) \times W_2^{l,l/2}(Q_{T'}),$ *where* $(\mathbf{u}^{(0)}, \nabla q^{(0)})$ *and* $(\mathbf{u}^{(\infty)}, \nabla q^{(\infty)})$ *are solutions of* (3) *with the adherence boundary condition* $(k \equiv 0)$ *and the perfect-slip boundary condition* $(k \equiv \infty)$, *respectively.*

Remark

According to this result, the Navier's slip boundary condition is the reasonable generalization of the usual adherence boundary conditions asigned in problems for viscous fluids.

Moreover, the general Navier's slip condition connects the no-slip case to the perfect-slip case through slip coefficient k.

Convergence results concerning problem (3)

Theorem 2 (NN-Tani) — concerning the limit of slip rate as $\beta \to 0$ —

Let $\Omega,\ \Gamma,\ l,\ \varrho_0,\ \mathbf{v}_0,\ \nu,\ \beta,\ T,\ T',\ \mathbf{b}$ and K *be the same as in Existence Theorem, denote the solutions of problem* (3) *with* β *by* $(\mathbf{u}^{(\beta)}, \nabla q^{(\beta)})$.

Then the sequence of the solutions $\left\{(\mathbf{u}^{(\beta)}, \nabla q^{(\beta)})\right\}_{\beta>0}$ *converges to* $(\mathbf{U}, \nabla Q)$ *as* $\beta \downarrow 0$ *in* $W_2^{2+l,1+l/2}(Q_{T'}) \times W_2^{l,l/2}(Q_{T'})$, *where* $(\mathbf{U}, \nabla Q)$ *is a solution of the Navier-Stokes equations* ($\beta = 0$ *in* (3)).

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It does not matter how large β is to consider the time-local analysis.

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Remark

It does not matter how large β is to consider the time-local analysis.
(Of course, it is completely different when we consider the time-global behaviour.)

Related Results

Concerning Inhomogeneous fluids (problem (1) or (3))**:**

- Abels and Terasawa (2009): L^q-framework
- Bulíček, Feireisl, Málek and Shvydkoy (2010): inhomogeneous, invicid, time-local sol.

Concerning Navier's slip problem:

- Tani-Itoh-Tanaka (1994) : Time-local solution for homogeneous incompressible Navier-Stokes equations with $K(x,t) \geq 0$.
- Itoh-Tani (2000) : Time-local solution in 3D and Time global solution for large data in 2D for inhomogeneous incompressible Navier-Stokes equations with $K(x,t) \geq 0$.
- Bulícek-Málek-Rajagopal (2008) : Time-global weak solution for large data for Navier-Stokes-like equations with pressure- and shear-dependent viscosity and $K > 0$ constant.

Out line of the Proof

- Linear problem with constant coefficients in the half space and the whole space

  $\sim$ Fourier-Laplace transform & estimates in weighted Sobolev-Slobodetskiĭ spaces

- Linear problem with function coefficients in a general bounded domain

  $\sim$ Regularizer method & estimates in Sobolev-Slobodetski spaces

- Quasi-linear problem (3)

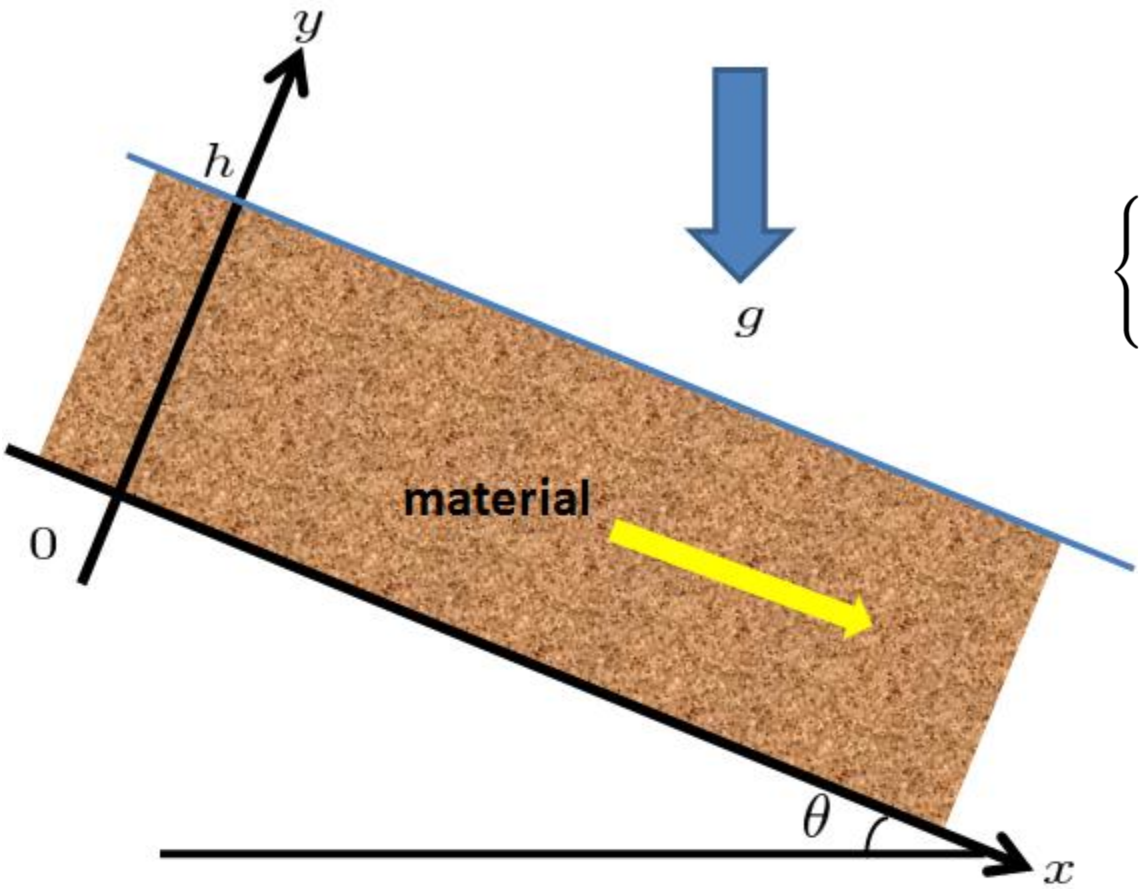
  $\sim$ Succesive approximation

Key Lemma

$$
(3)\quad \left\{\begin{array}{l}
\nabla_{\mathbf{u}} \cdot \mathbf{u} = 0 \quad \text{in } Q_T, \\
\varrho_0 \mathbf{u}_t = -\nabla_{\mathbf{u}} q + \nu(\varrho_0)\Delta_{\mathbf{u}}\mathbf{u} + 2\nu'(\varrho_0)\mathbb{D}_{\mathbf{u}}(\mathbf{u})\nabla_{\mathbf{u}}\varrho_0 \\
\qquad\qquad -\frac{\beta}{3}(\nabla^i_{\mathbf{u}}\nabla^j_{\mathbf{u}}\varrho_0)\nabla_{\mathbf{u}}\varrho_0 - \beta\Delta_{\mathbf{u}}\varrho_0\nabla_{\mathbf{u}}\varrho_0 + \varrho_0\mathbf{b}_{\mathbf{u}} \quad \text{in } Q_T, \\
\mathbf{u}|_{t=0} = \mathbf{v}_0 \quad \text{in } \Omega, \\
\mathbf{u} \cdot \mathbf{n}_{\mathbf{u}} = 0, \quad \mathbf{u} + K_{\mathbf{u}}\Pi_{\mathbf{u}}\mathbb{T}_{\mathbf{u}}\mathbf{n}_{\mathbf{u}} = \mathbf{0} \quad \text{on } G_T,
\end{array}\right.
$$

$$
(\mathrm{L})\quad \left\{\begin{array}{l}
\varrho_0(x)\mathbf{u}_t - \nu_1(x)\Delta\mathbf{u} + \nabla q = \varrho_0(x)\mathbf{f}, \quad \nabla \cdot \mathbf{u} = g \quad \text{in } Q_T, \\
\mathbf{u}|_{t=0} = \mathbf{v}_0 \quad \text{in } \Omega, \quad \mathbf{u} \cdot \mathbf{n} = \mathbf{b} \cdot \mathbf{n} \quad \text{on } G_T, \\
\Pi\mathbf{u} + 2\nu_1(x)K(x,t)\Pi\mathbb{D}(\mathbf{u})\mathbf{n} = K(x,t)\mathbf{d} \quad \text{on } G_T.
\end{array}\right.
$$

3. Steady Simple Shear Flow (1)



$$\begin{cases} \mathbf{v} = (u(y), 0, 0)^{\mathrm{T}} \\ \varrho = \varrho(y) \\ \mathbf{b} = (g\sin\theta, -g\cos\theta, 0)^{\mathrm{T}} \end{cases}$$

$$\Longrightarrow \quad \mathbb{T} = \begin{pmatrix} -p(\varrho) + \frac{\beta}{3}(\varrho')^2 & \nu(\varrho)u' & 0 \\ \nu(\varrho)u' & -p(\varrho) - \frac{2\beta}{3}(\varrho')^2 & 0 \\ 0 & 0 & -p(\varrho) + \frac{\beta}{3}(\varrho')^2 \end{pmatrix}$$

3. Steady Simple Shear Flow (2)

- **Governing equations:**

$$(4)\begin{cases} (\nu(\varrho(y))u'(y))' + \varrho(y)g\sin\theta = 0 & \text{for } 0 < y < h, \\ \left\{-p(\varrho(y)) - \frac{2\beta}{3}(\varrho'(y))^2\right\}' - \varrho(y)g\cos\theta = 0 & \text{for } 0 < y < h. \end{cases}$$

- **Boundary conditions:**

$$\text{surface } (y = h) \qquad \mathbb{T}\mathbf{n} = -p_{\mathrm{e}}\mathbf{n},$$

$$\text{bottom } (y = 0) \qquad \mathbf{v} + k\Pi\mathbb{T}\mathbf{n} = \mathbf{0},$$

where $p_{\mathrm{e}}(> 0)$ is the external pressure, $k(\geq 0)$ the slip constant, $\Pi\mathbf{f} = \mathbf{f} - (\mathbf{f}\cdot\mathbf{n})\mathbf{n}$ the tangential part, $\mathbf{n}$ the unit outward normal, *i.e.*,

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$$
\mathbf{n} = \begin{cases} (0, 1, 0)^{\mathrm{T}} & \text{at } y = h, \\ (0, -1, 0)^{\mathrm{T}} & \text{at } y = 0. \end{cases}
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$$\mathbf{n} = \begin{cases} (0,1,0)^{\mathrm{T}} & \text{at } y = h, \\ (0,-1,0)^{\mathrm{T}} & \text{at } y = 0. \end{cases}$$

$$\Longrightarrow (5)\begin{cases} \nu(\varrho(h))u'(h) = 0, \\ -p(\varrho(h)) - \frac{2\beta}{3}(\varrho'(h))^2 = -p_{\mathrm{e}}, \\ u(0) - k\nu(\varrho(0))u'(0) = 0. \end{cases}$$

Decoupling

Integrating $(4)_1$ from y to h and integrating from 0 to y, we have

$(5)_1$ ⇒

$$\nu(\varrho(y))u'(y) = \int_y^h \varrho(s) g \sin\theta \mathrm{d}s.$$

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$$\nu(\varrho(y))u'(y) = \int_y^h \varrho(s) g \sin\theta \mathrm{d}s.$$

$(5)_3$ $\Rightarrow$
$$u(y) = k\int_0^h \varrho(s) g \sin\theta \mathrm{d}s + \int_0^y \frac{\mathrm{d}\eta}{\nu(\varrho(\eta))}\int_\eta^h \varrho(s) g \sin\theta \mathrm{d}s.$$

$\therefore$ u can be uniquely determined by ϱ.

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<u>Existence Theorem</u> (NN-Tani)

Suppose $p(\varrho) = p_0 \varrho$ with $p_0 > 0$, and let β and $g > 0$ and $\theta \in [0, \pi/2)$.

If β is sufficiently large, then there exist a solution of $(4)_2 - (5)_2$ satisfying $\varrho'(y) < 0$.

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If β is sufficiently large, then there exist a solution of $(4)_2 - (5)_2$ satisfying $\varrho'(y) < 0$.

Thus the problem (4) has a solution.

Concluding Remarks

- **<u>Key word</u>: density gradient** $\nabla\varrho \sim \textsc{Inhomogeneity}$
- A slow and dense granular flow can be described by **IIFB** model.
- The difficulties arising from such terms can be removed by using the **Lagrangian coordinates**.
- When the slip coefficient is constant, i.e., $K(X,t) = k$, the estimates used here can be deduced independently of k . Then one can see that the solution obtained here converges to the solution in the case $k = 0$.
- Even for a steady simple shear flow down an inclined plane there is interesting difference from the usual Newtonian fluid. (=> necessary condition)
- In some special cases we can prove the solvability of the steady problem above.

Thank you for your attention

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