L_p-Theory for Two-Phase Flows with Soluble Surfactant

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International Workshop on
Mathematical Fluid Dynamics

Tokyo, March 2010

Joint Work with D. Bothe and J. Prüß

Motivation

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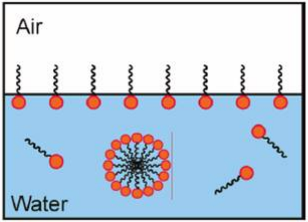
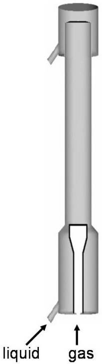
Surface Active Agent (Surfactant)

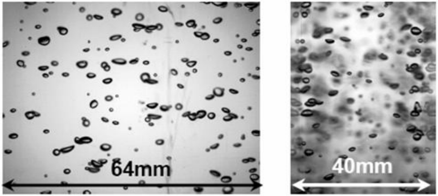
a substance which lowers the surface tension of the medium in which it is dissolved, and/or the interfacial tension with other phases, and, accordingly, is positively adsorbed at the liquid/vapour and/or at other interfaces



Example for an isothermal two-phase flow without phase change

bubbly flow through a column



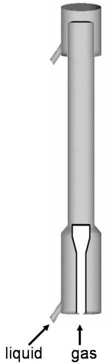


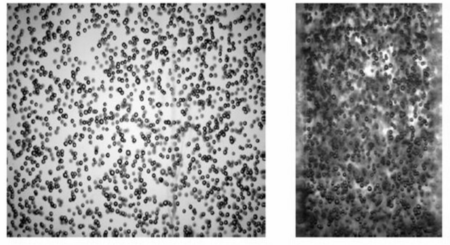
„clean system“
(experiments by Takagi, Tokyo University)

Motivation

Example for an isothermal two-phase flow without phase change

bubbly flow through a column





2ppm Triton-X100
(experiments by Takagi, Tokyo University)

Overview

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The Model

Local Well-Posedness

Transformation to a Fixed Domain

Linearization and L_p-Maximal Regularity

The Model

Modeling-Assumptions

We will use a sharp interface model for isothermal two-phase flows with surface tension and without phase-transition based on continuum mechanics, i. e.

- both fluids are contained in a domain $\Omega \subseteq \mathbb{R}^n$ and separated by an evolving interface $\Gamma(t) \subseteq \Omega$, where the evolution of this interface is part of the problem,
- the interface $\Gamma(t)$ moves with a velocity u_Γ and separates Ω into the two parts $\Omega_\pm(t)$, where $\partial\Omega \cap \Gamma(t) = \emptyset$, $\partial\Omega_+(t) = \partial\Omega \cup \Gamma(t)$ and $\partial\Omega_-(t) = \Gamma(t)$,
- the connected component $\Omega_+(t)$ and the interface $\Gamma(t)$ carry a surfactant, which is not soluble in the fluid occupying $\Omega_-(t)$.

The derivation of the model for the surfactant-evolution will employ

- continuum mechanical balance equations for the surfactant-mass,
- transport-theorems and divergence-theorems and
- constitutive equations.

The Model

The Two-Phase Navier-Stokes Equations with Surface-Tension

Assuming constant densities $\rho_\pm > 0$ and constant viscosities $\eta_\pm > 0$, the flow is determined by the two-phase Navier-Stokes equations with surface-tension:

evolutions of mass and momentum

$$\rho D_t^u u - \eta \Delta u + \nabla p = \rho f, \quad \nabla \cdot u = 0 \quad \text{in } \mathrm{gr}(\Omega_\pm)$$

$$[\![u]\!]_\Gamma = 0, \quad -[\![\eta(\nabla u + \nabla u^\mathsf{T}) - p]\!]_\Gamma n_\Gamma = \nabla_\Gamma \cdot (\sigma(c_\Gamma) P_\Gamma) \quad \text{on } \mathrm{gr}(\Gamma)$$

$$u\big|_{\partial\Omega} = 0 \quad \text{on } J \times \partial\Omega, \qquad u(0) = u_0 \quad \text{in } \Omega_\pm(0)$$

evolution of the interface

$$(\, u_\Gamma \mid n_\Gamma \,) = (\, u \mid n_\Gamma \,) \quad \text{on } \mathrm{gr}(\Gamma), \qquad \Gamma(0) = \Gamma_0$$

- $[\![\phi]\!]_\Gamma$ denotes the jump of a quantity ϕ across $\Gamma(t)$, i. e.

$$[\![\phi]\!]_\Gamma(t, x) = \lim_{h \to 0+} \Big(\phi(t, x + h n_{\Gamma(t)}(x)) - \phi(t, x - h n_{\Gamma(t)}(x))\Big)\,.$$

- P_Γ the projection onto $T\Gamma$, hence $\nabla_\Gamma \cdot (\sigma(c_\Gamma) P_\Gamma) = \sigma(c_\Gamma)\kappa_\Gamma n_\Gamma + \sigma'(c_\Gamma)\nabla_\Gamma c_\Gamma$.

The Model

The Two-Phase Navier-Stokes Equations with Surface-Tension

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- Contributers:

  Denisova, Denisova–Solonnikov, Tanaka, Shibata–Shimizu, Prüß–Simonett, K.–Prüß-Wilke, . . .

- Recent Work employing L_p-maximal regularity:

  Shibata, Shimizu: Resolvent Estimates and Maximal Regularity of the Interface Problem for the Stokes System in a Bounded Domain (2009),

  K., Prüß, Wilke: Qualitative Behaviour of Solutions for the Two-Phase Navier-Stokes Equations with Surface Tension (preprint).

The Model

Balance Equation for the Surfactant-Mass in the Bulk-Phase

For a material volume $V(t) \subseteq \Omega_+(t)$ conservation of surfactant-mass with volume-specific density c implies

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V(t)} c \,\mathrm{d}x = -\int_{\partial V(t)} J^{mol} \cdot n_{\partial V(t)} \,\mathrm{d}o.$$

- J denotes the molecular surfactant-flux in the bulk-phase.
- No artificial sources and sinks are present.

The Model

Balance Equation for the Surfactant-Mass in the Bulk-Phase

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- No artificial sources and sinks are present.

The usual divergence-theorem and Reynolds' transport-theorem therefore imply

$$\int_{V(t)} D_t^u c + c \operatorname{div} u \,\mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}t} \int_{V(t)} c \,\mathrm{d}x = - \int_{V(t)} \operatorname{div} J^{mol} \,\mathrm{d}x.$$

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Hence, localization leads to

$$D_t^u c + c \operatorname{div} u + \operatorname{div} J^{mol} = 0 \quad \text{in } \operatorname{gr}(\Omega_+).$$

The Model

Balance Equation for the Surfactant-Mass in the Bulk-Phase

We assume the constitutive equation

$$J^{mol} = -d\nabla c,$$

i. e. the flux is modeled by Fick's law with a constant diffusion coefficient $d > 0$.

Hence, we derived the

Surfactant-Balance in the Bulk-Phase

$$D_t^u c - d\Delta c = 0 \qquad \text{in } \mathrm{gr}(\Omega_+)$$

The Model

Balance Equation for the Surfactant-Mass on the Interface

For a material volume $V(t) \subseteq \Omega$ with $\Sigma(t) = \Gamma(t) \cap V(t)$ conservation of surfactant-mass with surface-specific density c_Γ implies

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Sigma(t)} c_\Gamma \,\mathrm{d}o = - \int_{\partial\Sigma(t)} J_\Gamma^{mol} \cdot n_{\partial\Sigma(t)} \,\mathrm{d}s + \int_{\Sigma(t)} f^{sorp} \,\mathrm{d}o.$$

- J_Γ^{mol} denotes the interfacial molecular surfactant-flux, tangential to $\Gamma(t)$.
- f^{sorp} denotes the interfacial density of sources and sinks due to sorption.

The Model

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- J_Γ^{mol} denotes the interfacial molecular surfactant-flux, tangential to $\Gamma(t)$.
- f^{sorp} denotes the interfacial density of sources and sinks due to sorption.

The surface-divergence-theorem and the surface-transport-theorem imply

$$\int_{\Sigma(t)} D_t^u c_\Gamma + c_\Gamma \, \mathrm{div}_\Gamma u \, \mathrm{d}o = \frac{\mathrm{d}}{\mathrm{d}t} \int_{\Sigma(t)} c_\Gamma \, \mathrm{d}o = - \int_{\Sigma(t)} \mathrm{div}_\Gamma \, J_\Gamma^{mol} \, \mathrm{d}o + \int_{\Sigma(t)} f^{sorp} \, \mathrm{d}o.$$

The Model

Balance Equation for the Surfactant-Mass on the Interface

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Hence, localization leads to

$$D_t^u c_\Gamma + c_\Gamma \, \mathrm{div}_\Gamma u + \mathrm{div} \, J_\Gamma^{mol} = f^{sorp} \quad \text{on } \mathrm{gr}(\Gamma).$$

The Model

Balance Equation for the Surfactant-Mass on the Interface

We assume the constitutive equations

$$J_\Gamma^{mol} = -d_\Gamma \nabla_\Gamma c_\Gamma, \qquad f^{sorp} = r(c|_\Gamma, c_\Gamma)$$

i. e. the flux is modeled is by Fick's law with a constant interfacial diffusion coefficient $d_\Gamma > 0$ and the sorption-rate depends on $c|_\Gamma$ and c_Γ.

Hence, we derived the

Surfactant-Balance on the Interface

$$D_t^u c_\Gamma + c_\Gamma \operatorname{div}_\Gamma u - d_\Gamma \Delta_\Gamma c_\Gamma = r(c|_\Gamma, c_\Gamma) \qquad \text{on } \operatorname{gr}(\Gamma)$$

The Model

Balance Equation for the Surfactant-Mass in the Coupling-Region

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For a material volume $V(t) \subseteq \Omega$ with $\Sigma(t) = \Gamma(t) \cap V(t)$ conservation of surfactant-mass with volume-specific density c and surface-specific density c_Γ implies

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_{V(t)} c \,\mathrm{d}x + \int_{\Sigma(t)} c_\Gamma \,\mathrm{d}o\right] = -\int_{\partial V(t)} J^{mol} \cdot n_{\partial V(t)} \,\mathrm{d}o - \int_{\partial \Sigma(t)} J_\Gamma^{mol} \cdot n_{\partial \Sigma(t)} \,\mathrm{d}s$$

The two-phase-divergence-theorem, the surface-divergence-theorem, Reynolds' transport-theorem and the surface-transport-theorem imply

$$\int_{V(t)} D_t^u c + c \operatorname{div} u \,\mathrm{d}x + \int_{\Sigma(t)} D_t^u c_\Gamma + c_\Gamma \operatorname{div}_\Gamma u \,\mathrm{d}o = \frac{\mathrm{d}}{\mathrm{d}t}\int_{V(t)} c \,\mathrm{d}x + \frac{\mathrm{d}}{\mathrm{d}t}\int_{\Sigma(t)} c_\Gamma \,\mathrm{d}o$$

$$= \quad -\int_{V(t)} \operatorname{div} J^{mol} \,\mathrm{d}x - \int_{\Sigma(t)} \operatorname{div}_\Gamma J_\Gamma^{mol} \,\mathrm{d}o - \int_{\Sigma(t)} [\![J^{mol}]\!]_\Gamma \cdot n_\Gamma \,\mathrm{d}o.$$

The Model

Balance Equation for the Surfactant-Mass in the Coupling-Region

For a material volume $V(t) \subseteq \Omega$ with $\Sigma(t) = \Gamma(t) \cap V(t)$ conservation of surfactant-mass with volume-specific density c and surface-specific density c_Γ implies

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_{V(t)} c \,\mathrm{d}x + \int_{\Sigma(t)} c_\Gamma \,\mathrm{d}o\right] = -\int_{\partial V(t)} J^{mol} \cdot n_{\partial V(t)} \,\mathrm{d}o - \int_{\partial \Sigma(t)} J_\Gamma^{mol} \cdot n_{\partial \Sigma(t)} \,\mathrm{d}s$$

The two-phase-divergence-theorem, the surface-divergence-theorem, Reynolds' transport-theorem and the surface-transport-theorem imply after subtraction of the bulk- and interface-balances and localization

$$[\![J^{mol}]\!]_\Gamma \cdot n_\Gamma + f^{sorp} = 0 \quad \text{on } \mathrm{gr}(\Gamma).$$

The Model

Balance Equation for the Surfactant-Mass in the Coupling-Region

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For the surfactant-mass this implies

$$r(c|_\Gamma,\, c_\Gamma) = -[\![J^{mol}]\!]_\Gamma \cdot n_\Gamma = d\nabla c|_\Gamma \cdot n_\Gamma \quad \text{on } \mathrm{gr}(\Gamma)$$

and hence, we derived the

Surfactant-Balance on the Interface

$$D_t^u c_\Gamma + c_\Gamma \operatorname{div}_\Gamma u - d_\Gamma \Delta_\Gamma c_\Gamma = d\nabla c|_\Gamma \cdot n_\Gamma \qquad \text{on } \mathrm{gr}(\Gamma)$$

The Model

Balance Equation for the Surfactant-Mass in the Coupling-Region

For the surfactant-mass this implies

$$r(c|_\Gamma, c_\Gamma) = -[\![J^{mol}]\!]_\Gamma \cdot n_\Gamma = d\nabla c|_\Gamma \cdot n_\Gamma \quad \text{on } \mathrm{gr}(\Gamma)$$

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Surfactant-Balance on the Interface

$$D_t^u c_\Gamma + c_\Gamma \operatorname{div}_\Gamma u - d_\Gamma \Delta_\Gamma c_\Gamma = d\nabla c|_\Gamma \cdot n_\Gamma \qquad \text{on } \mathrm{gr}(\Gamma)$$

Assuming that sorption is faster than diffusion we have the

Surfactant-Distribution at the Interface

$$c_\Gamma = \gamma(c|_\Gamma) \quad \text{on } \mathrm{gr}(\Gamma)$$

with the adsorption isotherm γ, e. g. the Langmuir adsorption isotherm.

The Model

The Two-Phase Navier-Stokes Equations with Soluble Surfactant

evolutions of mass and momentum

$$\rho D_t^u u - \eta \Delta u + \nabla p = \rho f, \quad \nabla \cdot u = 0 \quad \text{in } \mathrm{gr}(\Omega_\pm)$$

$$[\![u]\!]_\Gamma = 0, \quad -[\![\eta(\nabla u + \nabla u^\mathsf{T}) - p]\!]_\Gamma n_\Gamma = \nabla_\Gamma \cdot (\sigma(c_\Gamma) P_\Gamma) \quad \text{on } \mathrm{gr}(\Gamma)$$

$$u|_{\partial\Omega} = 0 \quad \text{on } J \times \partial\Omega, \qquad u(0) = u_0 \quad \text{in } \Omega_\pm(0)$$

evolution of the interface

$$(\, u_\Gamma \,|\, n_\Gamma \,) = (\, u \,|\, n_\Gamma \,) \quad \text{on } \mathrm{gr}(\Gamma), \qquad \Gamma(0) = \Gamma_0$$

evolution of the surfactant

$$D_t^u c - d \Delta c = 0 \quad \text{in } \mathrm{gr}(\Omega_+)$$

$$\gamma(c|_\Gamma) = c_\Gamma, \qquad D_t^u c_\Gamma + c_\Gamma \mathrm{div}_\Gamma u - d_\Gamma \Delta_\Gamma c_\Gamma = d \nabla c|_\Gamma \cdot n_\Gamma \quad \text{on } \mathrm{gr}(\Gamma)$$

$$\nabla c|_{\partial\Omega} \cdot n_{\partial\Omega} = 0 \quad \text{on } J \times \partial\Omega, \qquad c(0) = c_0 \quad \text{in } \Omega_+(0)$$

Local Well-Posedness

Theorem

Let $p > n + 2$ and let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain with boundary of class C^3. Let $\rho_\pm, \eta_\pm, d, d_\Gamma > 0$ and let $\sigma, \gamma \in C^{3-}(\mathbb{R}_+, \mathbb{R}_+)$ with $\gamma' > 0$. Suppose

$$u_0 \in W_p^{2-2/p}(\Omega \setminus \Gamma_0), \quad \Gamma_0 \in W_p^{3-2/p}, \quad c_0 \in W_p^{2-2/p}(\Omega_+(0), \mathbb{R}_+)$$

are subject to the following regularity and compatibility conditions

$$[\![u_0]\!]_{\Gamma_0} = 0, \quad u_0|_{\partial\Omega} = 0, \quad c_0|_{\Gamma_0} \in W_p^{2-2/p}(\Gamma_0), \quad \nabla c_0|_{\partial\Omega} \cdot n_{\partial\Omega} = 0,$$
$$\operatorname{div} u_0 = 0 \quad \text{in } \Omega \setminus \Gamma_0,$$
$$-P_{\Gamma_0}[\![\eta(\nabla u_0 + \nabla u_0^{\mathsf{T}})]\!]_{\Gamma_0} n_{\Gamma_0} = \sigma'(\gamma(c_0|_\Gamma))\nabla_\Gamma \gamma(c_0|_\Gamma) \quad \text{on } \Gamma_0$$

and $f \in L_p(\mathbb{R}_+ \times \Omega)$.

Then there exists $t_0 = t_0(u_0, \Gamma_0, c_0)$, such that the problem admits a unique strong solution $(u, p, \Gamma, c, c_\Gamma)$ with $c, c_\Gamma > 0$ on $(0, t_0)$.

Local Well-Posedness

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The proof employs

- the transformation to a fixed domain via the direct-mapping technique using a Hanzawa-transformation,
- the establishment of L_p-maximal regularity of a suitable linearization and
- a fixed-point-argument.

For a (bent) half-space result cf.

Bothe, Prüß, Simonett: Well-Posedness of a Two-Phase Flow with Soluble Surfactant, Prog. in Nonlinear Differential Equ. and Their Applications (64), 37–61

The Transformed System

evolutions of mass and momentum

$$\rho(\mathcal{D}_t^h + u \cdot \mathcal{G}^h)u - \eta \mathcal{L}^h u + \mathcal{G}^h p = \rho f, \quad \mathcal{D}^h u = 0 \quad \text{in } J \times \Omega \setminus \Sigma$$

$$[\![u]\!]_\Sigma = 0, \quad -[\![\eta(\mathcal{G}^h u + \mathcal{G}^h u^\mathsf{T}) - p]\!]_\Sigma n_\Gamma(h) = \mathcal{D}_\Sigma^h(\sigma(c_\Sigma) P_\Gamma(h)) \quad \text{on } J \times \Sigma$$

$$u\big|_{\partial\Omega} = 0 \quad \text{on } J \times \partial\Omega, \qquad u(0) = u_0 \quad \text{in } \Omega \setminus \Sigma$$

evolution of the interface

$$(\, n_\Sigma \mid n_\Gamma(h)\,)\partial_t h = (\, u \mid n_\Gamma(h)\,) \quad \text{on } J \times \Sigma, \qquad h(0) = h_0 \quad \text{on } \Sigma$$

evolution of the surfactant

$$(\mathcal{D}_t^h + u \cdot \mathcal{G}^h)c - d\mathcal{L}^h c = 0 \quad \text{in } J \times \Omega_+$$

$$\gamma(c|_\Sigma) = c_\Sigma, \qquad (\partial_t + u \cdot \mathcal{G}_\Sigma^h)c_\Sigma + c_\Sigma \mathcal{D}_\Sigma^h u - d_\Gamma \mathcal{L}_\Sigma^h c_\Sigma = d\mathcal{G}^h c|_\Sigma \cdot n_\Gamma(h) \quad \text{on } J \times \Sigma$$

$$\nabla c\big|_{\partial\Omega} \cdot n_{\partial\Omega} = 0 \quad \text{on } J \times \partial\Omega, \qquad c(0) = c_0 \quad \text{in } \Omega_+$$

The Linearized System

Linearization w. r. t. a Reference Solution $z^* = (u^*, p^*, h^*, c^*, c_\Sigma^*)$

evolutions of mass and momentum

$$\rho\partial_t u - \eta\Delta u + \nabla p = F(u, p, h) + \rho f, \qquad \operatorname{div} u = G(u, h) \quad \text{in } J \times \Omega \setminus \Sigma$$

$$-P_\Sigma[\![\eta(\nabla u + \nabla u^\mathsf{T})]\!]_\Sigma n_\Sigma - \sigma'(c_\Sigma^*)\nabla_\Sigma c_\Sigma = H_\tau(u, h, c_\Sigma; c_\Sigma^*),$$

$$(-[\![\eta(\nabla u + \nabla u^\mathsf{T})]\!]_\Sigma n_\Sigma \mid n_\Sigma) + [\![p]\!]_\Sigma - \sigma(c_\Sigma^*)\Delta_\Sigma h = H_\nu(u, h, c_\Sigma; c_\Sigma^*),$$

$$[\![u]\!]_\Sigma = 0 \quad \text{on } J \times \Sigma, \qquad u|_{\partial\Omega} = 0 \quad \text{on } J \times \partial\Omega, \qquad u(0) = u_0 \quad \text{in } \Omega \setminus \Sigma$$

evolution of the interface

$$\partial_t h - (u \mid n_\Sigma) + (u^* \mid \nabla_\Sigma h) = H_\gamma(u, h; u^*) \quad \text{on } J \times \Sigma, \quad h(0) = h_0 \quad \text{on } \Sigma$$

evolution of the surfactant

$$\partial_t c - d\Delta c = M(u, h, c) \quad \text{in } J \times \Omega_+$$

$$\gamma'(c^*|_\Sigma)c|_\Sigma - c_\Sigma = N_\nu(c, c_\Sigma; c^*), \quad \partial_t c_\Sigma - d_\Gamma \Delta_\Sigma c_\Sigma = N_\tau(u, h, c, c_\Sigma) \quad \text{on } J \times \Sigma$$

$$\nabla c|_{\partial\Omega} \cdot n_{\partial\Omega} = 0 \quad \text{on } J \times \partial\Omega, \qquad c(0) = c_0 \quad \text{in } \Omega_+$$

The Principal Linear Part

Linearization w. r. t. a Reference Solution $z^* = (u^*, p^*, h^*, c^*, c_\Sigma^*)$

evolutions of mass and momentum

$$\rho\partial_t u - \eta\Delta u + \nabla p = f, \qquad \operatorname{div} u = g \quad \text{in } J \times \Omega \setminus \Sigma$$

$$-P_\Sigma [\![\eta(\nabla u + \nabla u^{\mathsf{T}})]\!]_\Sigma n_\Sigma - \sigma'(c_\Sigma^*)\nabla_\Sigma c_\Sigma = h_\tau,$$

$$(\, -[\![\eta(\nabla u + \nabla u^{\mathsf{T}})]\!]_\Sigma n_\Sigma \mid n_\Sigma \,) + [\![p]\!]_\Sigma - \sigma(c_\Sigma^*)\Delta_\Sigma h = h_\nu,$$

$$[\![u]\!]_\Sigma = 0 \quad \text{on } J \times \Sigma, \qquad u|_{\partial\Omega} = 0 \quad \text{on } J \times \partial\Omega, \qquad u(0) = u_0 \quad \text{in } \Omega \setminus \Sigma$$

evolution of the interface

$$\partial_t h - (\, u \mid n_\Sigma \,) + (\, u^* \mid \nabla_\Sigma h \,) = h_\gamma \quad \text{on } J \times \Sigma, \qquad h(0) = h_0 \quad \text{on } \Sigma$$

evolution of the surfactant

$$\partial_t c - d\Delta c = m \quad \text{in } J \times \Omega_+$$

$$\gamma'(c^*|_\Sigma)c|_\Sigma - c_\Sigma = n_\nu, \qquad \partial_t c_\Sigma - d_\Gamma \Delta_\Sigma c_\Sigma = n_\tau \quad \text{on } J \times \Sigma$$

$$\nabla c|_{\partial\Omega} \cdot n_{\partial\Omega} = 0 \quad \text{on } J \times \partial\Omega, \qquad c(0) = c_0 \quad \text{in } \Omega_+$$

The Principal Linear Part

L_p-maximal Regularity

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We denote the solution spaces by

$$u \in \mathbb{E}_u(a) = \left\{ \begin{array}{c} v \in H^1_p([0,a], L_p(\Omega, \mathbb{R}^n)) \cap L_p([0,a], H^2_p(\Omega \setminus \Sigma, \mathbb{R}^n)), \\ [\![v]\!]_\Sigma = 0, \quad v\big|_{\partial\Omega} = 0 \end{array} \right\}$$

$$p \in \mathbb{E}_p(a) = \left\{ \begin{array}{c} q \in L_p([0,a], \dot{H}^1_p(\Omega \setminus \Sigma)), \\ [\![q]\!]_\Sigma \in W_p^{1/2-1/2p}([0,a], L_p(\Sigma)) \cap L_p([0,a], W_p^{1-1/p}(\Sigma)) \end{array} \right\}$$

$$\begin{aligned} h \in \mathbb{E}_h(a) = {} & W_p^{2-1/2p}([0,a], L_p(\Sigma)) \cap H^1_p([0,a], W_p^{2-1/p}(\Sigma)) \\ & \cap L_p([0,a], W_p^{3-1/p}(\Sigma)) \end{aligned}$$

$$c \in \mathbb{E}_c(a) = \left\{ \begin{array}{c} e \in H^1_p([0,a], L_p(\Omega_+)) \cap L_p([0,a], H^2_p(\Omega_+)), \\ e\big|_\Sigma \in \mathbb{E}^\Sigma_c(a), \quad \nabla e\big|_{\partial\Omega} n_{\partial\Omega} = 0 \end{array} \right\}$$

$$c_\Sigma \in \mathbb{E}^\Sigma_c(a) = H^1_p([0,a], L_p(\Sigma)) \cap L_p([0,a], H^2_p(\Sigma))$$

and set $\mathbb{E}(a) = \mathbb{E}_u(a) \times \mathbb{E}_p(a) \times \mathbb{E}_h(a) \times \mathbb{E}_c(a) \times \mathbb{E}^\Sigma_c(a)$.

The Principal Linear Part

L_p-maximal Regularity

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We denote the data spaces by

$$
\begin{array}{lcccl}
f & \in & \mathbb{F}_f(a) & = & L_p([0,\,a] \times \Omega,\, \mathbb{R}^n) \\
g & \in & \mathbb{F}_g(a) & = & H_p^1([0,\,a],\, \dot{H}_p^{-1}(\Omega \setminus \Sigma)) \cap L_p([0,\,a],\, H_p^1(\Omega \setminus \Sigma)) \\
h_\tau & \in & \mathbb{F}_h^\tau(a) & = & W_p^{1/2-1/2p}([0,\,a],\, L_p(\Sigma,\, T\Sigma)) \cap L_p([0,\,a],\, W_p^{1-1/p}(\Sigma,\, T\Sigma)) \\
h_\nu & \in & \mathbb{F}_h^\nu(a) & = & W_p^{1/2-1/2p}([0,\,a],\, L_p(\Sigma)) \cap L_p([0,\,a],\, W_p^{1-1/p}(\Sigma)) \\
h_\gamma & \in & \mathbb{F}_h^\gamma(a) & = & W_p^{1-1/2p}([0,\,a],\, L_p(\Sigma,\, \mathbb{R}^n)) \cap L_p([0,\,a],\, W_p^{2-1/p}(\Sigma,\, \mathbb{R}^n)) \\
m & \in & \mathbb{F}_m(a) & = & L_p([0,\,a] \times \Omega_+) \\
n_\tau & \in & \mathbb{F}_n^\tau(a) & = & L_p([0,\,a] \times \Sigma) \\
n_\nu & \in & \mathbb{F}_n^\nu(a) & = & H_p^1([0,\,a],\, L_p(\Sigma)) \cap L_p([0,\,a],\, H_p^2(\Sigma))
\end{array}
$$

and set $\mathbb{F}(a) = \mathbb{F}_f(a) \times \mathbb{F}_g(a) \times \mathbb{F}_h^\tau(a) \times \mathbb{F}_h^\nu(a) \times \mathbb{F}_h^\gamma(a) \times \mathbb{F}_m(a) \times \mathbb{F}_n^\tau(a) \times \mathbb{F}_n^\nu(a)$.

Theorem

Let $p > n + 2$, $a > 0$. Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain with boundary of class C^3 and let $\Sigma \subseteq \Omega$ be a compact real-analytic hypersurface. Let $\rho_\pm$, $\eta_\pm$, d, $d_\Gamma > 0$ and $\sigma, \gamma \in C^{3-}(\mathbb{R}, \mathbb{R})$ with $\sigma, \gamma' > 0$. Let $z^* \in \mathbb{E}(a)$. Then the linear problem

$$L(z; z^*) = (f, g, h_\tau, h_\nu, h_\gamma, m, n_\tau, n_\nu), \qquad (u(0), h(0), c(0)) = (u_0, h_0, c_0)$$

admits a unique solution $z \in \mathbb{E}(a)$, if and only if the data is subject to the following regularity and compatibility conditions:

$$(f, g, h_\tau, h_\nu, h_\gamma, m, n_\tau, n_\nu) \in \mathbb{F}(a),$$
$$u_0 \in W_p^{2-2/p}(\Omega \setminus \Sigma, \mathbb{R}^n), \quad h_0 \in W_p^{3-2/p}(\Sigma), \quad c_0 \in W_p^{2-2/p}(\Omega_+),$$
$$[\![u_0]\!]_\Sigma = 0, \quad u_0|_{\partial\Omega} = 0, \quad c_0|_\Sigma \in W_p^{2-2/p}(\Sigma), \quad \nabla c_0|_{\partial\Omega} \cdot n_{\partial\Omega} = 0,$$
$$\operatorname{div} u_0 = g(0) \quad \text{in } \Omega \setminus \Sigma,$$
$$-P_\Sigma [\![\eta(\nabla u_0 + \nabla u_0^\mathsf{T})]\!]_\Sigma n_\Sigma - \sigma'(c_\Sigma^*(0)) \nabla_\Sigma \gamma(c_0|_\Sigma) = h_\tau(0) \quad \text{on } \Sigma.$$

Moreover, the solution map is continuous between the corresponding spaces.

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Thank You for Your Attention!