Nonlinear diffusion equations derived from nonreversible particle systems

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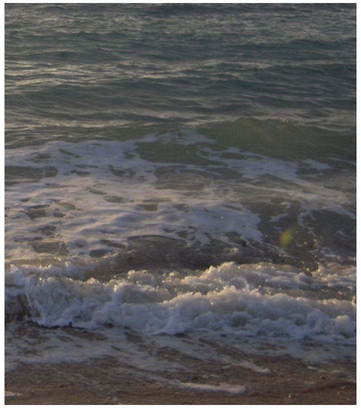
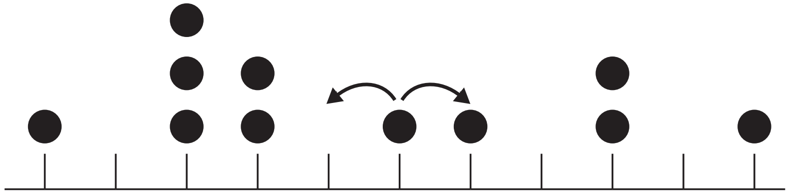
Backgrounds of interacting particle systems



Large-scale interacting systems

Scaling limit



Random walks of particles on a lattice space

Interaction is given by the jump rates of particles

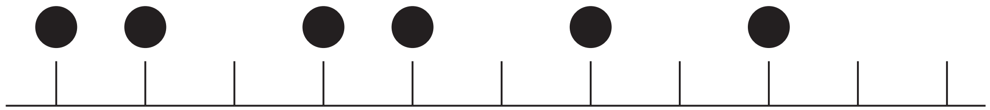
Interaction of the microscopic system $\Rightarrow$
The evolution equation of the macroscopic parameter

Simple Symmetric Exclusion Processes (SSEP)

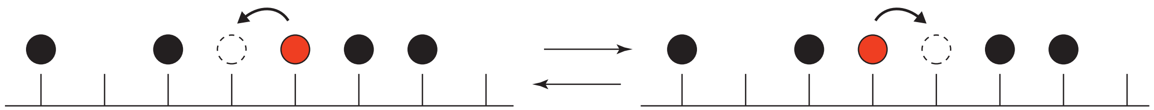
$\mathbb{T}_N^d := (\mathbb{Z}/N\mathbb{Z})^d = \{0, 1..., N-1\}^d$: discrete torus of size N^d

$\chi_N^d := \{0, 1\}^{\mathbb{T}_N^d}$: state space

$\eta = (\eta_x)_{x \in \mathbb{T}_N^d}$: element of χ_N^d



Dynamics of SSEP



jump to one of the neighboring sites with probability $\frac{1}{2d}$

exclusion rule

jump rate is a constant 1 (the inverse of the expectation value of random waiting time)

$\eta^N(t)$: Markov process on χ_N^d with generator

$$(L_N f)(\eta) = \frac{1}{2d} \sum_{|x-y|=1} 1_{\{(\eta_x,\eta_y)=(1,0)\}}(f(\eta^{x\to y}) - f(\eta))$$

The number of particles is a unique conserved quantity
$\Rightarrow$ The density of particles characterizes the equilibrium states
$\Rightarrow$ Derive an evolution equation of the density of particles

Denote by π_t^N the scaled empirical measure:

$$\pi_t^N(du) = \frac{1}{N^d} \sum_{x \in \mathbb{T}_N^d} \eta_x(N^2 t)\delta_{\frac{x}{N}}(du) \in \mathcal{M}(\mathbb{T}^d := [0,1)^d)$$

Hydrodynamic limit for the SSEP

Theorem 1. *(De Masi, et al. 1984) Assume*

$$\pi_0^N(du) \to \pi_0(du) = \rho_0(u)du \quad N \to \infty \quad \textit{in prob}$$

with some measurable function $\rho_0 : \mathbb{T}^d \to [0,1]$*. Then,* $\forall t > 0$,

$$\pi_t^N(du) \to \pi_t(du) = \rho(t,u)du \quad N \to \infty \quad \textit{in prob}$$

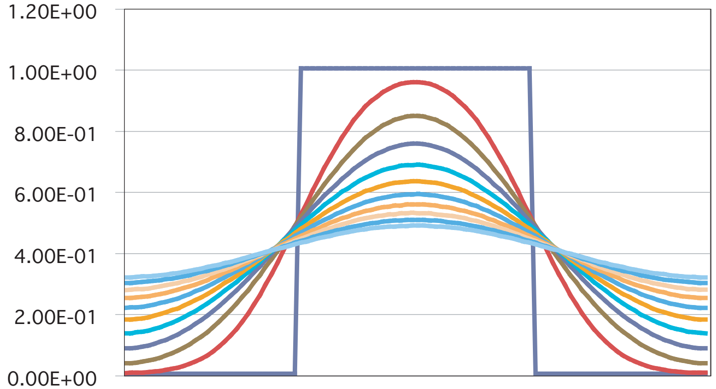
where $\rho(t,u)$ *is the unique solution of the heat equation:*

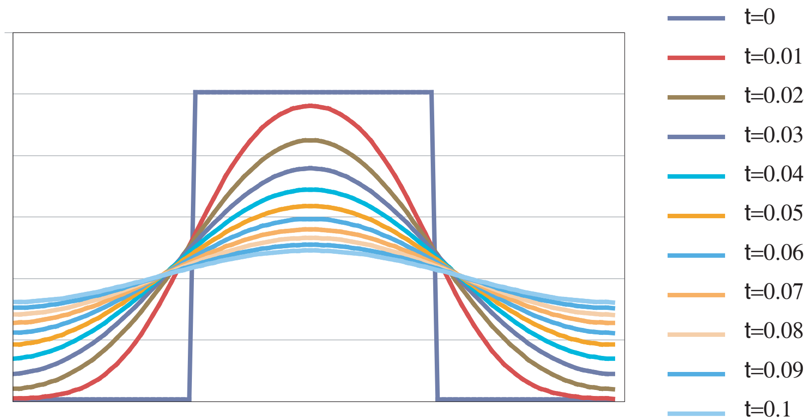
$$\begin{cases} \partial_t \rho(t,u) = \dfrac{1}{2d} \Delta \rho(t,u) \\ \rho(0,\cdot) = \rho_0(\cdot) \end{cases}$$

Remark 1. *For the totally asymmetric simple exclusion process* $(d = 1)$*, under the hyperbolic scaling, the hydrodynamic equation is*

$$\partial_t \rho(t,u) = \partial_u \{\rho(t,u)(1 - \rho(t,u))\}$$

Numerical simulation





*exact computation

*numerical simulation

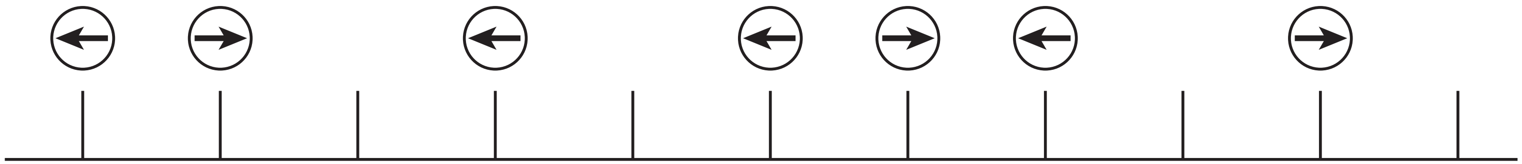
Exclusion processes with velocity

$\mathbb{T}_N := (\mathbb{Z}/N\mathbb{Z}) = \{0, 1..., N-1\}$: discrete torus of size N

$\chi_N := \{1, 0, -1\}^{\mathbb{T}_N}$

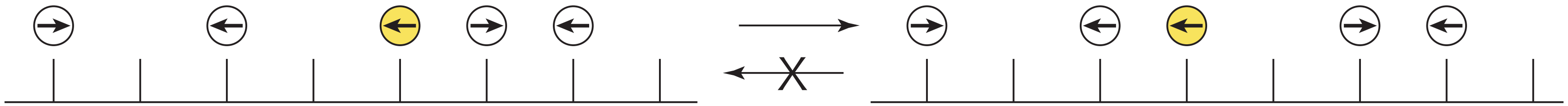
each particle has a velocity 1 or -1 (direction of jump)

one-dimensional periodic lattice

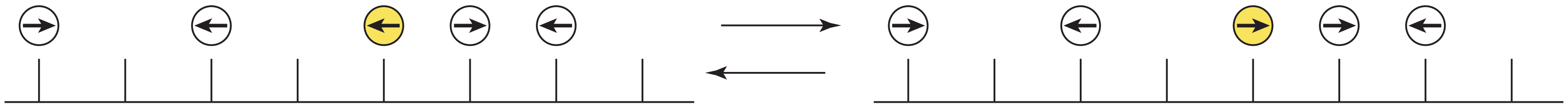


Dynamics

each particle jumps to the direction of its velocity with jump rate 1



velocity of each particle changes with rate $\gamma > 0$



Equilibrium states

The density of particles is a unique conserved quantity

In the equilibrium states, each particle has a velocity $+1$ and -1 with same "probability"

No drift $\Rightarrow$ Diffusive scaling limits

$$\pi_t^N(du) = \frac{1}{N} \sum_{x \in \mathbb{T}_N} \eta_x(N^2 t)\delta_{\frac{x}{N}}(du) \in \mathcal{M}(\mathbb{T} := [0,1))$$

$$\partial_t \rho(t,u) = \frac{1}{2}\Delta \rho(t,u) \quad ??$$

Main Theorem

Theorem 2. *(S. 2010) Assume*

$$\pi_0^N(du) \to \pi_0(du) = \rho_0(u)du \quad N \to \infty \quad \textit{in prob}$$

with some measurable function $\rho_0 : \mathbb{T}^d \to [0,1]$. *Then,* $\forall t > 0$,

$$\pi_t^N(du) \to \pi_t(du) = \rho(t,u)du \quad N \to \infty \quad \textit{in prob}$$

where $\rho(t,u)$ *is the unique weak solution of PDE:*

$$\begin{cases} \partial_t \rho(t,u) = \partial_u \{ D^\gamma(\rho(t,u)) \partial_u \rho(t,u) \} \\ \quad \rho(0,\cdot) = \rho_0(\cdot) \end{cases}$$

$$D^\gamma(\rho) = \frac{1}{\rho(1-\rho)} \inf_{g \in \mathcal{C}} \ll W - Lg \gg_\rho$$

Properties of $D^\gamma(\rho)$

$D^\gamma(\rho)$ is strictly bigger than $\frac{1}{2}$

$D^\gamma(\rho)$ is continuous

$\frac{1}{2} + \frac{1-\rho}{2\gamma} \leq D^\gamma(\rho) \leq \frac{1}{2} + \frac{2-\rho}{4\gamma}$

$D^\gamma(\rho)$ is not a constant function in ρ.

$D^\gamma(0) = \frac{1}{2} + \frac{1}{2\gamma}$

$\lim_{\gamma\to 0} D^\gamma(\rho) = \infty$ for $\rho \in [0,1)$

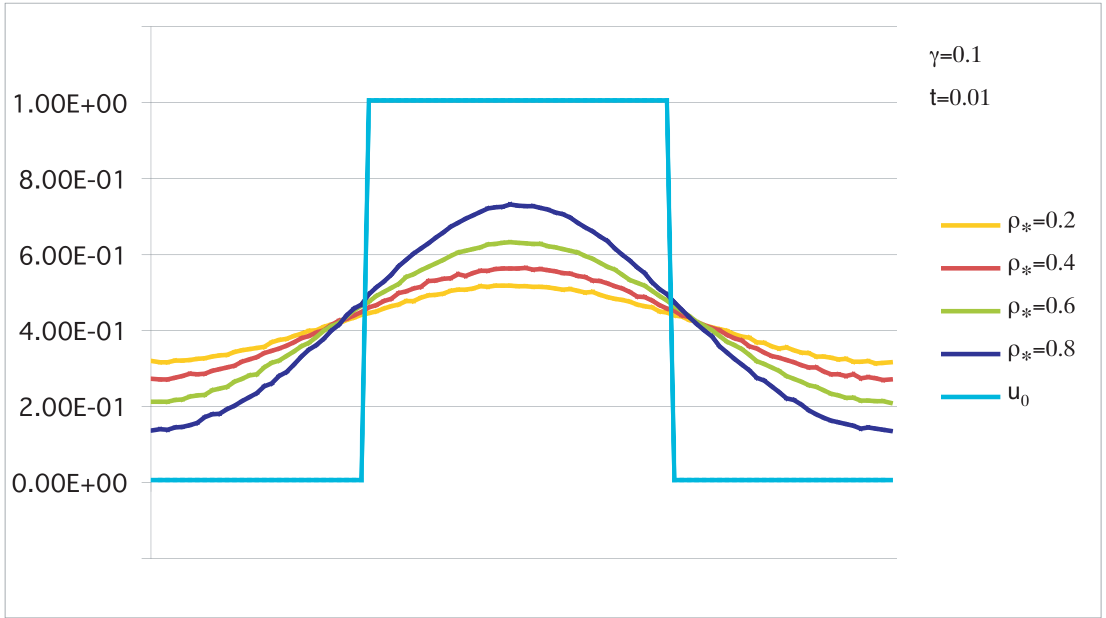
$\lim_{\gamma\to \infty} D^\gamma(\rho) = \frac{1}{2}$ for $\rho \in [0,1]$

$\gamma \to 0$: System goes to hyperbolic scaling

$\gamma \to \infty$: System goes to SSEP

Conjecture I from numerical simulation

$D^{\gamma}(\rho)$ is strictly decreasing as a function of ρ

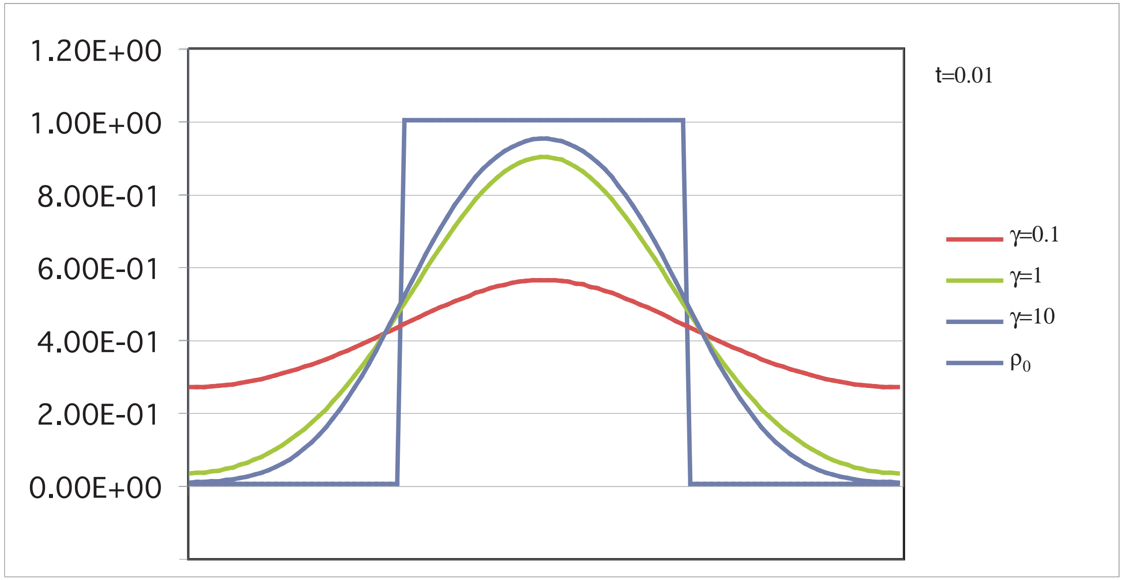


* graph of u_t

$$\rho_0(\cdot) = \frac{1}{10} u_0(\cdot) + \rho_*$$
$$\rho_t(\cdot) = \frac{1}{10} u_t(\cdot) + \rho_*$$

Conjecture II from numerical simulation

$D^{\gamma}(\rho)$ is strictly decreasing as a function of γ



$$\rho_0(\cdot) = 1_{[0.3,0.7]}(\cdot)$$

Thank you