Improved Moment Estimates for Invariant Measures of Semilinear SPDE

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Outline

The Stochastic Burgers Equation

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As a motivation consider

$$\partial_t u(t,\xi) = \nu \partial_\xi^2 u(t,\xi) + u(t,\xi)\partial_\xi u(t,\xi) + \eta(t,\xi) \quad t > 0, \xi \text{ in } U$$
$$u\big|_{t=0} = u_0.$$

- $U = [0, 1]$,
- $\nu > 0$,
- Dirichlet boundary conditions on ∂U,
- Noise $\eta(t,\xi)$, e. g. space-time white noise.

The Stochastic Burgers Equation

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$$u\big|_{t=0} = u_0.$$

- Existence of a unique solution and existence of a unique invariant measure μ (Da Prato, Debussche, Temam (NoDEA1994)).
- A priori estimates for μ: $\forall k, p \geq 1$

$$\int_{L^2(U)} \|x\|_{L^p}^k d\mu(x) < \infty$$

(Da Prato, Debussche (Potential Analysis 2007)).

Abstract Form: Semilinear SPDE

Reformulate as an abstract ODE on a function space

$$\begin{aligned} dX(t) &= \Big(AX(t) + B\big(X(t)\big)\Big)dt + dW(t) \quad t > 0 \\ X(0) &= x \in H. \end{aligned} \tag{1}$$

- $H = L^2(U)$,
- A : Dirichlet Laplacian,
- $B : D(B) \subset H \to H$,
- $Q \in \mathcal{L}(H)$ symmetric, positive definite,
- $\big(W(t)\big)_{t \geq 0}$ cylindrical Wiener Process on H.

Abstract Form: Semilinear SPDE

In the following: A general semilinear SPDE

$$\begin{aligned} dX(t) &= \Big(AX(t) + B\big(X(t)\big)\Big)dt + \sqrt{Q}dW(t) \quad t > 0 \\ X(0) &= x \in H. \end{aligned} \tag{1}$$

- H : separable Hilbert space.
- $A : D(A) \to H$ self-adjoint of negative type ω_0 and compact resolvent,
- $B : D(B) \subset H \to H$,
- $Q \in \mathcal{L}(H)$ symmetric, positive definite,
- $\big(W(t)\big)_{t \geq 0}$ cylindrical Wiener Process on H,
- $V_\gamma := D\big((-A)^\gamma\big)$, $\langle x, y \rangle_{V_\gamma} := \langle (-A)^\gamma x, (-A)^\gamma y \rangle_H$.

The Wiener Process

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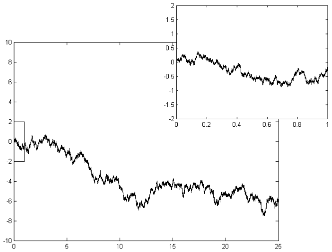
Formally

$$W(t) = \sum_{n=0}^{\infty} \beta_n(t) e_n.$$

- (e_n) complete ONS of eigenvectors of Q,
- (μ_n) corresponding eigenvalues,
- β_n are independent Brownian motions on $\mathbb{R}$.



The Wiener Process

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But

$$\mathbb{E}\left[\|\sum_{n=0}^{N}\beta_n(t)e_n\|_H^2\right] = \sum_{n=0}^{N}\mathbb{E}\left[\beta_n(t)^2\right] = Nt \to \infty$$

so the series does not converge in $L^2(H)$.

- Solution: Extend the definition via some Hilbert-Schmidt embedding.
- In particular: If tr $Q < \infty$, then

$$\sqrt{Q}W(t) = \sum_{n=0}^{\infty}\sqrt{\mu_n}\beta_n(t)e_n \in H.$$

Concepts of Solution

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A mild solution of (1) is

$$X(t,x) = e^{tA}x + \int_0^t e^{(t-s)A}B\big(X(s,x)\big)ds + \int_0^t e^{(t-s)A}\sqrt{Q}dW(s).$$

"New" object, so called stochastic convolution

$$W_A(t) = \int_0^t e^{(t-s)A}\sqrt{Q}dW(s) \quad \text{and} \quad \mathbb{E}\left[\|W_A(t)\|_H^2\right] = \int_0^t \|e^{sA}\sqrt{Q}\|_{HS}^2 ds.$$

- Wiener-Ito-Isometry
- Gaussian random variable.
- Cov $W_A(t) = \int_0^t e^{sA}Qe^{sA}ds$.

Invariant Measures

A mild solution of (1) is called stationary if

$$\mu = \mathbb{P} \circ X(t)^{-1} = \mathbb{P} \circ X(0)^{-1}$$

and μ its time invariant distribution.

Remark

Definition of the transition semigroup

$$P_t f(x) := \mathbb{E}\left[f\big(X(t,x)\big)\right]$$

A probability measure μ on $\big(H, \mathcal{B}(H)\big)$ is called invariant for P_t if

$$\int_H P_t f(x) d\mu(x) = \int_H f(x) d\mu(x), \quad \forall t \geq 0, f \in \mathcal{B}_b(H)$$

Conditions for Existence

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Usually proof of existence via Krylov-Bogoliubov theorem. If P_t Feller, then:

- Any weak limit point μ of
$$\mu_T(A) := \frac{1}{T}\int_0^T \mu_{X(t,x)}(A)dt$$
is invariant.
- Show tightness via some stability assumption on coefficients.

Assume the Lyapunov condition

$$\langle Ay + B(y + w), y\rangle_H \leq -\alpha_1 \|y\|^2_{V_{\gamma_1}} + \alpha_2 \|w\|^s_{V_{\gamma_2}} + \alpha_3$$

(see Stannat, Es-Sarhir (JEE2008))

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  is invariant.
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This implies

$$\sup_{T>1} \frac{1}{T}\int_0^T \mathbb{E}\left[\|X(t,x)\|^2_{V_{\gamma_1}}\right] < \infty$$

and therefore tightness. Note: μ has full support on V_γ, $\gamma < \gamma_1$.

A More General Case

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With the Lyapunov condition

$$\langle Ay + B(y+w), y\rangle_H \leq -\alpha_1 \|y\|^2_{V_{\gamma_1}} + \alpha_3 \|w\|^s_{V_{\gamma_2}} + \alpha_4$$

one can also prove a priori estimates of moments of any order.
Problem: Stochastic Burgers equation does not fit in this case.

A More General Case

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We now generalize to

$$\langle Ay + B(y + w), y\rangle_H \leq -\alpha_1 \|y\|^2_{V_{\gamma_1}} + \alpha_2 \|y\|^2_{V_{\gamma_1}} \|w\|^s_{V_{\gamma_2}} + \alpha_3 \|w\|^s_{V_{\gamma_2}} + \alpha_4$$

suitable for semilinear equations with quadratic nonlinearity.

- γ_1: controls regularity of solution,
- $\gamma_2 > \gamma_1$: regularity of stochastic convolution.

A Priori Estimates

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Theorem

Let μ be an invariant measure for (1)*. Then*

$$\int_H \|x\|_H^p d\mu(x) < \infty, \quad p \geq 0$$

and

$$\int_H \|x\|_H^p \|x\|_{V_\gamma}^2 d\mu(x) < \infty, \quad p \geq 0, \gamma < \gamma_1.$$

sketch of proof

Decompose solution into

$$Y_\lambda(t,x) := X(t,x) - W_{A-\lambda}(t)$$

with

$$W_{A-\lambda}(t) := \int_0^t e^{(t-s)(A-\lambda)} \sqrt{Q} dW(s).$$

Y_λ satisfies an evolution equation with random coefficients

$$\dot{Y}_\lambda(t,x) = AY_\lambda(t,x) + B(Y_\lambda(t,x) + W_{A-\lambda}(t) + \lambda W_{A-\lambda}(t)$$

sketch of proof

Decompose solution into

$$Y_\lambda(t,x) := X(t,x) - W_{A-\lambda}(t)$$

with

$$W_{A-\lambda}(t) := \int_0^t e^{(t-s)(A-\lambda)}\sqrt{Q}dW(s).$$

Applying the Lyapunov condition yields

$$\frac{1}{2}\frac{d}{dt}\|Y_\lambda(t,x)\|_H^2 \leq \|Y_\lambda(t,x)\|_{\gamma_1}^2 \underbrace{\left(-\frac{\alpha_1}{2} + \alpha_2\|W_{A-\lambda}(t)\|_{\gamma_2}^s\right)}_{=:\alpha} + R(\lambda,t).$$

A Pathwise Control of $W_{A-\lambda}(t)$

For simplicity define

$$W_{-\lambda}(t) := \int_0^t e^{-\lambda(t-s)} d\beta(s) \left[= \lambda \int_0^t \big(\beta(t) - \beta(s)\big) e^{-\lambda(t-s)} ds + e^{-\lambda t}\beta(t) \right]$$

where $\big(\beta(t)\big)_{t\geq 0}$ is a 1-dimensional Brownian motion. Then for $\delta \in (0, \frac{1}{2})$

$$\sup\left\{ |W_{-\lambda}(t)| : t \leq T \right\} \leq \lambda^{-\delta} M(\delta, T)$$

with

$$M(\delta, T) := \sup_{s,t \leq T} \frac{|\beta(t) - \beta(s)|}{|t - s|^{\delta}}$$

having finite moments of any order.

The General Case

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Now consider the V_γ valued process $W_{A-\lambda}(t)$. We get again for $\delta \in (0, \frac{1}{2})$

$$\sup\left\{\|W_{A-\lambda}(t)\|_{V_\gamma} : t \leq T\right\} \leq \lambda^{-2\varepsilon} \sum_{n=0}^{\infty} \mu_n \lambda_n^{2(\gamma-\delta+\varepsilon)} M_n(\delta, T)^2.$$

Condition for pathwise control

$$\exists \varepsilon > 0 : Z_{\delta,\gamma,\varepsilon} := \sum_{n=0}^{\infty} \mu_n \lambda_n^{2(\gamma-\delta+\varepsilon)} < \infty.$$

Here again

- $M_n(\delta, T) := \sup_{s,t \leq T} \frac{|\beta_n(t)-\beta_n(s)|}{|t-s|^\delta}$,
- $\beta_n(t) = \langle W(t), e_n \rangle$.
- Result is a generalization of Da Prato, Debussche (Potential Analysis 2007).

back to the proof

$$\frac{1}{2}\frac{d}{dt}\|Y_\lambda(t,x)\|_H^2 \le \|Y_\lambda(t,x)\|_{\gamma_1}^2 \underbrace{\left(-\frac{\alpha_1}{2} + \alpha_2\|W_{A-\lambda}(t)\|_{\gamma_2}^s\right)}_{=:\alpha} + R(\lambda,t).$$

Pathwise estimates imply $\alpha < C < 0$. “Reduced” to the case without quadratic term.

Corollary

For the stochastic Burgers equation with $Q = (-A)^{-2\beta}$, $\beta \in (\frac{1}{4}, 1]$, $\gamma \in [\frac{1}{2}, \frac{1}{4} + \beta)$ holds

$$\langle Au + B(u+v), u\rangle_H \le -\frac{1}{2}\|u\|_{V_{\frac{1}{2}}}^2 + K\|u\|_{V_{\frac{1}{2}}}^2\|v\|_{V_\gamma}^2 + \frac{1}{2}\|v\|_{V_\gamma}^2$$

and therefore the a priori estimates hold for the unique invariant measure.

Thank you for your attention.