A reproductive property of the time dependent boudary value problem to the Navier-Stokes equations under the general flux condition

Takahiro Okabe

Waseda University

International Workshop on Mathematical Fluid Dynamics
8–16, March, 2010

Introduction

$\Omega \subset \mathbb{R}^3$: b'dd smooth domain, $\partial\Omega = \bigcup_{j=0}^{L} \Gamma_j$, where

(i) Γ_i : C^∞-surface,

(ii) $\Gamma_i \cap \Gamma_j = \emptyset$, $i \neq j$,

(iii) $\Gamma_1, \ldots, \Gamma_L$ lie inside of Γ_0 and outside of one another.

We consider the initial boundary value problem to the Navier-Stokes equations;

$$\text{(N-S)} \qquad \begin{cases} \partial_t u - \Delta u + u \cdot \nabla u + \nabla p = f, & \Omega \times (0, T) \\ \operatorname{div} u = 0, & \Omega \times (0, T) \\ u|_{\partial\Omega} = \beta, & \partial\Omega \times (0, T), \\ \big(u(0) = u_0, & \Omega\big). \end{cases}$$

where

$$\left.\begin{array}{ll} u = u(x,t) = (u^1(x,t), u^2(x,t), u^3(x,t)), & \text{velocity} \\ p = p(x,t), & \text{pressure} \end{array}\right\} \text{unknown}$$

external force : $f = f(x,t)$ boundary value : $\beta = \beta(x,t)$.

compatibility condition

$$\begin{cases} \operatorname{div} u = 0, & \Omega \times (0, T) \\ u|_{\partial\Omega} = \beta, & \partial\Omega \times (0, T). \end{cases}$$

(G.F.C.) $$\sum_{j=0}^{L} \int_{\Gamma_j} \beta(t) \cdot \nu \, dS = 0, \quad \text{for } t \geq 0.$$

$$0 = \int_{\Omega} \operatorname{div} u \, dx = \int_{\partial\Omega} u \cdot \nu \, dS = \sum_{j=0}^{L} \int_{\Gamma_j} \beta \cdot \nu \, dS.$$

The reproductive property

Definition (reproductive property)

we call that (N-S) has a reproductive property at $T > 0$, if there exist an initial value u_0 and a weak solution u of (N-S) such that

$$(*) \qquad u(T) = u_0 \quad \text{in } L^2$$

- If u is periodic with period T, then u satisfy the condition $(*)$.

Solenoidal extension

By the trace theorem and the Bogovskii theorem, for $\beta \in C^1([0,T); H^{1/2}(\partial\Omega))$ we have the solenoidal extension $b \in C^1([0,T); H^1(\Omega))$ of β, i.e.,

$$\operatorname{div} b(t) = 0, \quad b(t)|_{\partial\Omega} = \beta(t) \quad \text{for } t \geq 0.$$

$u - b =: w$

$$(\text{N-S}') \qquad \begin{cases} \partial_t w - \Delta w + b \cdot \nabla w + w \cdot \nabla b + w \cdot \nabla w + \nabla p = F, \\ \operatorname{div} w = 0, \\ w|_{\partial\Omega} = 0. \end{cases}$$

where $F = f - \partial_t b + \Delta b - b \cdot \nabla b$.

The convection term $b \cdot \nabla w + w \cdot \nabla b$

we have to handle the linear convection term to obtain a priori estimate:

$$\|w(t)\|_2^2 + \int_0^t \|\nabla w(\tau)\|_2^2 \, d\tau \le C \int_0^t (\|f\|_2^2 + \|\partial_t b\|_2^2 + \|\nabla b\|_2^2 + \|b\|_4^4) \, d\tau + C\|a\|_2^2$$

- the restricted flux condition

(R.F.C.) $$\int_{\Gamma_j} \beta \cdot \nu \, dS = 0, \quad j = 0, \dots, L.$$

Leray ('33)

- the symmetry of the domain
Kobayasi ('09), Morimoto ('09)

The convection term $b \cdot \nabla w + w \cdot \nabla b$

Kozono-Yanagisawa ('09)
$b \in W^{1,2}(\Omega)$ with $\mathrm{div}\, b = 0$ and $b|_{\partial\Omega} = \beta$

$$b = h + \mathrm{rot}\ \omega$$

where h is a harmonic vector field on Ω and $\omega \in W^{2,2}(\Omega)$
Furthermore,

$$h = \sum_{j=1}^{L} \Big(\int_{\Gamma_j} \beta \cdot \nu \, dS \Big) \phi_j$$

where $\{\phi_j\}_{j=1}^{L}$ are harmonic vecter fields determined by Ω.

Main theorem

Theorem

Let $f \in L^2_{loc}([0,\infty); L^2(\Omega))$. *Suppose that* $\beta \in C^1([0,\infty); H^{1/2}(\partial\Omega))$ *satisfy (G.F.C.) with a restriction*

$$(**) \qquad \sup_{t\geq 0} \Big\| \sum_{j=1}^{L} \Big(\int_{\Gamma_j} \beta(t)\cdot\nu \, dS \Big) \phi_j \Big\|_3 \leq \frac{1}{5C_s},$$

where $C_s = 3^{-1/2} 2^{2/3} \pi^{-2/3}$ *is the best constant of* $W^{1,2}_0 \hookrightarrow L^6$. *For every* $0 < T < \infty$, *there exist an initial value* $u_0 \in L^2_\sigma(\Omega)$ *and a weak solution* w *of (N-S′) such that*

$$w(T) = u_0 \quad \text{in } L^2$$

Remark

- We do not assume that f and β is periodic or small.

Remark

- We do not assume that f and β is periodic or small.
- If $\beta(T) = \beta(0)$ for some $T > 0$, $\exists u$:a weak solution of (N-S) and $\exists u_0$ such that $u(T) = u_0$.

Remark

- We do not assume that f and β is periodic or small.
- If $\beta(T) = \beta(0)$ for some $T > 0$, $\exists u$:a weak solution of (N-S) and $\exists u_0$ such that $u(T) = u_0$.
- If we have the uniqueness theorem, then we obtain time periodic solutions form reproductive property.

Outline of proof

Key Lemma

Suppose that $\beta \in C^1([0,\infty); H^{1/2}(\partial\Omega))$ *satisfy (G.F.C.) with a restriction*

$$(**) \qquad \sup_{t\geq 0} \Big\| \sum_{j=1}^{L} \Big(\int_{\Gamma_j} \beta(t)\cdot \nu \, dS \Big) \phi_j \Big\|_3 \leq \frac{1}{5C_s}.$$

Then there exists $0 < \varepsilon_0 < 1/5$ *and* $b_{\varepsilon_0} \in C^1([0,\infty); W^{1,2})$ *with* $\mathrm{div}\, b_{\varepsilon_0} = 0$ *and* $b_{\varepsilon_0}|_{\partial\Omega} = \beta$ *such that*

$$|(w\cdot\nabla b_{\varepsilon_0}, w)| \leq \varepsilon_0 \|\nabla w\|_2^2, \quad \text{for } w \in W^{1,2}_{0,\sigma}, \quad t \geq 0.$$

Let w_m be approximation solution by the Galerkin method. By the key lemma, we have the energy inequatily:

$$\frac{d}{dt}\|w_m(t)\|_2^2 + 2(1-5\varepsilon)\|\nabla w_m(\tau)\|_2^2 \\ \leq C\|f\|_2^2 + \|\partial_t b\|_2^2 + \|\nabla b\|_2^2 + \|b\|_4^4 := K(t)$$

Furthermore by the Poincaré inequality,

$$\frac{d}{dt}\|w_m\|_2^2 + \alpha\|w_m\|_2^2 \leq K(t)$$

$\alpha > 0$. Hence

$$\|w_m(T)\|_2^2 \leq e^{-\alpha T}\|w_m(0)\|_2^2 + \int_0^T e^{-\alpha(T-t)}K(t)\,dt$$

$$\|w_m(T)\|_2^2 \le e^{-\alpha T}\|w_m(0)\|_2^2 + \int_0^T e^{-\alpha(T-t)}K(t)\,dt$$

Choose $R > 0$ so that

$$(1) \qquad \int_0^T e^{-\alpha(T-t)}K(t)\,dt \le R^2(1-e^{-\alpha T})$$

So from (1),

$$\|w_m(T)\|_2 \le R, \quad \text{if } \|w_m(0)\|_2 \le R.$$

On the other hand, the map $w_m(0) \mapsto w_m(T)$ is continuous.
By the Brower fix point theorem. we have

$$w_m(0) = w_m(T)$$

Outline of proof of Key lemma

Since $\beta \in C^1([0,T); H^{1/2}(\partial\Omega))$, by the trace theorem and the Bogovskii theorem, $\exists b \in C^1([0,T); H^1(\Omega))$ with $\mathrm{div}\, b = 0$ and $b|_{\partial\Omega} = \beta$. Then by the decomposition theorem,

$$b(t) = h(t) + \mathrm{rot}\ \omega(t).$$

For any $\varepsilon > 0$ we can take the cut-off function $\theta_\varepsilon(x,t)$ with $\theta_\varepsilon(\cdot,t) \equiv 1$ near $\partial\Omega$ such that

$$|(w \cdot \nabla \mathrm{rot}\ (\theta_\varepsilon \omega), w)| \le \varepsilon \|\nabla w\|_2^2 \quad w \in W^{1,2}_{0,\sigma} \quad t \ge 0.$$

$$b_\varepsilon := h + \mathrm{rot}\ (\theta_\varepsilon \omega)$$

So the convection term is estimated;

$$|(w \cdot \nabla b_\varepsilon, w)| \le C_s \|h\|_3 \|\nabla w\|_2^2 + \varepsilon \|\nabla w\|_2^2.$$

Thank you very much.

Theorem

Let Ω *be a general bounded domain in* $\mathbb{R}^3$ *with smooth boundary* $\partial\Omega$*. Let* $f \in L^2_{loc}([0,\infty); L^2(\Omega))$ *and let* $\beta \in C^1([0,\infty); H^{1/2}(\partial\Omega))$ *with (G.F.C.). Then there exists* $T_* > 0$ *such that for every* $0 < T < T_*$*, there exist a initial velocity* a *and a weak solution* v *of (N-S′) such that* $v(T) = a$ *in* $L^2_\sigma(\Omega)$*.*