Concentration and Diffusion Effects of Heat Convection in an Incompressible Fluid

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The Boussinesq Equations in $\mathbb{R}^d$

Heat convection in a viscous incompressible fluid under the influence of gravity is described through the Boussinesq Equations:

$$\begin{aligned} u_t - \Delta u + (u \cdot \nabla) u + \nabla p &= g\theta && \text{in} \quad [0, T) \times \mathbb{R}^d, \\ \theta_t - \Delta \theta + (u \cdot \nabla)\theta &= 0 && \text{in} \quad [0, T) \times \mathbb{R}^d, \\ \operatorname{div} u &= 0 && \text{in} \quad [0, T) \times \mathbb{R}^d, \\ u(0) &= u_0 && \text{in} \quad \mathbb{R}^d, \\ \theta(0) &= \theta_0 && \text{in} \quad \mathbb{R}^d. \end{aligned}$$

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In our case we assume the gravity g belongs to $L^\infty_{d-1}(\mathbb{R}^d)^d$, i.e.

$$\|g\|_{L^\infty_{d-1}} := \operatorname{ess\,sup}(1 + |x|)^{d-1} |g(x)| < \infty.$$

Theorem 1: (Existence and Uniqueness of mild solutions)
Let $(u_0, \theta_0) \in L^\infty_\delta(\mathbb{R}^d)^d \times L^\infty_\sigma(\mathbb{R}^d)$ such that
$\operatorname{div} u_0 = 0, \delta \in (0, d], \sigma \in (0, d+1]$ and $g \in L^\infty_{d-1}(\mathbb{R}^d)^d$.

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Then there exist $T > 0$ and a unique mild solution

$$(u, \theta) \in C((0, T); L^\infty_\delta(\mathbb{R}^d)^d) \times C((0, T); L^\infty_\sigma(\mathbb{R}^d)).$$

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In particular, we can choose T such that

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A mild solution solves the integral equations:

$$u(t) = e^{t\Delta}u_0 - \int_0^t e^{(t-s)\Delta}\mathbb{P}(u \cdot \nabla u)ds + \int_0^t e^{(t-s)\Delta}\mathbb{P}(g\theta)(s)ds,$$
$$\theta(t) = e^{t\Delta}\theta_0 - \int_0^t e^{(t-s)\Delta}(u \cdot \nabla\theta)(s)ds.$$

Spatial Asymptotic

Theorem 2: (Spatial Asymptotic Behaviour)
For $\delta \geq \frac{d+2}{2}, \sigma \geq 3$ and an initial data $(u_0, \theta_0) \in L^\infty_\delta(\mathbb{R}^d)^d \times L^\infty_\sigma(\mathbb{R}^d)$ with div $u_0 = 0$, let (u, θ) be the solution of the preceding theorem. The following profile holds for $|x| >> \sqrt{t}$:

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$$u(x,t) = e^{t\Delta}u_0(x)$$
$$+ \nabla \left[\gamma_{d,1} \sum_{h,k=1}^{d} \left(\frac{\delta_{h,k}}{d|x|^d} - \frac{x_h x_k}{|x|^{d+2}} \right) \cdot \int_0^t \int_{\mathbb{R}^d} (u_h u_k)(y,s)dyds \right]$$
$$+ \gamma_{d,2} \sum_{h=1}^{d} \left(\frac{d x_h x_j}{|x|^{d+2}} - \frac{\delta_{j,h}}{|x|^d} \right) \cdot \int_0^t \int_{\mathbb{R}^d} (g_h \theta)(y,s)dyds + \mathcal{O}_t(|x|^{-d-2}),$$

$$\theta(x,t) = e^{t\Delta}\theta_0(x) - \gamma_{d,3} \sum_{h=1}^{d} \frac{x_h}{|x|^{d+2}} \cdot \int_0^t \int_{\mathbb{R}^d} (u_h \theta)(y,s)dyds + \mathcal{O}_t(|x|^{-d-2}).$$

Sketch of Proof

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- We rewrite $e^{t\Delta}\mathbb{P}\,\mathrm{div}$ as the kernel of a convolution operator:

$$\gamma_{d,1}\left(\frac{\sigma_{j,h,k}(x)}{|x|^{d+2}}-(d+2)\frac{x_j x_h x_k}{|x|^{d+4}}\right)+|x|^{-d-1}\Psi(\frac{x}{\sqrt{t}}).$$

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- Seperation of Variables: Define $v_{h,k}$ such that

$$(u_h u_k)(x,t)=\mathcal{G}(x)\int_{R^d}(u_h u_k)(y,t)dy+v_{h,k}(x,t).$$

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- We estimate the remainder terms.

Concentration and Diffusion Effects

Theorem 3: Let $d = 2, 3$, $g \in L^{\infty,*}_{d-1}$ with $g(\tilde{x}) = \tilde{g}(x)$,$0 < t_1 < ... < t_N$ be a finite sequence and $\varepsilon > 0$.

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Then there exists an initial data $(u_0, \theta_0) \in \mathcal{S}_\sigma(\mathbb{R}^d)^d \times \mathcal{S}(\mathbb{R}^d)$ and for each $i = 1, .., N$ there are $t_i`, t_i^*, \tilde{t}_i`, \tilde{t}_i^* \in (t_i - \varepsilon, t_i + \varepsilon)$ such that the corresponding unique solution (u, θ) of the Boussinesq Equations satisfies, for all $i = 1, ..., N$ and all $|x|$ large enough, the pointwise estimates

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Then there exists an initial data $(u_0, \theta_0) \in \mathcal{S}_\sigma(\mathbb{R}^d)^d \times \mathcal{S}(\mathbb{R}^d)$ and for each $i = 1, .., N$ there are $t_i‘, t_i^*, \tilde{t}_i‘, \tilde{t}_i^* \in (t_i - \varepsilon, t_i + \varepsilon)$ such that the corresponding unique solution (u, θ) of the Boussinesq Equations satisfies, for all $i = 1, \dots, N$ and all $|x|$ large enough, the pointwise estimates

$$|u(x, t_i^*)| \le C|x|^{-d-2},$$
$$|\theta(x, \tilde{t}_i^*)| \le C|x|^{-d-2},$$

and with $\omega = \frac{x}{|x|}$ there holds for almost all $|x|$ large enough

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Notation: For $x = (x_1, ..., x_d) \in \mathbb{R}^d$ we set $\tilde{x} := (x_2, ..., x_d, x_1)$.

Step I

Using Theorem 2 we have

$$u(x,t) = \nabla \left[\gamma_{d,1} \sum_{h,k=1}^{d} \left(\frac{\delta_{h,k}}{d|x|^d} - \frac{x_h x_k}{|x|^{d+2}} \right) \cdot \int_0^t \int_{\mathbb{R}^d} (u_h u_k)(y,s) dy ds \right]$$
$$+ \gamma_{d,2} \sum_{h=1}^{d} \left(\frac{d x_h x_j}{|x|^{d+2}} - \frac{\delta_{j,h}}{|x|^d} \right) \cdot \int_0^t \int_{\mathbb{R}^d} (g_h \theta)(y,s) dy ds + \mathcal{O}_t(|x|^{-d-2}),$$

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provided the initial data u_0 and θ_0 are Schwartz functions.

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provided the initial data u_0 and θ_0 are Schwartz functions.

At the moment t_i^* or $\tilde{t}_i^*$, respectively, we would like to have for all h,k:

$$\int_0^{t_i^*} \int_{\mathbb{R}^d} (u_h u_k)(y,s)dyds = \int_0^{t_i^*} \int_{\mathbb{R}^d} (g_h \theta)(y,s)dyds = 0,$$

$$\int_0^{\tilde{t}_i^*} \int_{\mathbb{R}^d} (u_h \theta)(y,s)dyds = 0.$$

Step II

If we assume a symmetry property of the gravity $g(\tilde{x}) = \tilde{g}(x)$ and the initial data

$$(u_0, \theta_0)(\tilde{x}) = (\tilde{u}_0, \theta_0)(x),$$

this propagates during the evolution:

$$(u, \theta)(\tilde{x}, t) = (\tilde{u}, \theta)(x, t).$$

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Furthermore, we would have in the case $d = 3$:

$$\int u_1 u_2 = \int u_2 u_3 = \int u_3 u_1, \qquad \int g_1 \theta = \int g_2 \theta = \int g_3 \theta,$$

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$$\text{and} \int u_1 \theta = \int u_2 \theta = \int u_3 \theta.$$

Thus all these terms vanish in t_i^* or $\tilde{t}_i^*$, respectively, if and only if

$$\int_0^{t_i^*} \int_{\mathbb{R}^d} (u_1 u_2)(y, s) dy ds = \int_0^{t_i^*} \int_{\mathbb{R}^d} (g_1 \theta)(y, s) dy ds = 0,$$

$$\text{resp.} \int_0^{\tilde{t}_i^*} \int_{\mathbb{R}^d} (u_1 \theta)(y, s) dy ds = 0.$$

Step III

We represent the solution as a limit of an iteration as follows:

$$T_1(u_0,\theta_0) := e^{t\Delta}u_0, \qquad \tilde{T}_1(u_0,\theta_0) := e^{t\Delta}\theta_0,$$

$$T_k(u_0,\theta_0) := \sum_{l=1}^{k-1} \mathcal{B}\big(T_l(u_0,\theta_0), T_{k-l}(u_0,\theta_0)\big) + \mathcal{C}\big(\tilde{T}_k(u_0,\theta_0)\big),$$

$$\tilde{T}_k(u_0,\theta_0) := \sum_{l=1}^{k-1} \mathcal{D}\big(T_l(u_0,\theta_0), \tilde{T}_{k-l}(u_0,\theta_0)\big), \quad k \geq 2.$$

Under smallness assumption on the initial data the series

$$\phi(u_0,\theta_0) := \sum_{k=1}^{\infty} T_k(u_0,\theta_0) \quad \text{and} \quad \psi(u_0,\theta_0) := \sum_{k=1}^{\infty} \tilde{T}_k(u_0,\theta_0)$$

are absolutely convergent and $(\phi,\psi)(u_0,\theta_0)$ is a solution of the equations

$$u = e^{t\Delta}u_0 + \mathcal{B}(u,u) + \mathcal{C}(\theta), \quad \theta = e^{t\Delta}\theta_0 + \mathcal{D}(u,\theta).$$

Step IV

With this representation of the solution and $\eta > 0$ sufficiently small our considered terms with respect to the initial data $(\eta u_0, \eta^2\theta_0)$ behave like:

$$\begin{aligned}\int_0^t \int_{\mathbb{R}^d} (u_1 u_2)(y,s)dyds &\approx \eta^2 \int_0^t \int_{\mathbb{R}^d} e^{s\Delta}u_{0,1}(y)e^{s\Delta}u_{0,2}(y)dyds \\ &\quad + 2\eta^3 \int_0^t \int_{\mathbb{R}^d} \mathcal{C}(u_0,\theta_0)(y,s)dyds, \\ \int_0^t \int_{\mathbb{R}^d} (g_1\theta)(y,s)dyds &\approx \eta^2 \int_0^t \int_{\mathbb{R}^d} g_1(y)e^{s\Delta}\theta_0(y)dyds, \\ \int_0^t \int_{\mathbb{R}^d} (u_1\theta)(y,s)dyd &\approx \eta^3 \int_0^t \int_{\mathbb{R}^d} e^{s\Delta}u_{0,1}(y)e^{s\Delta}\theta_0(y)dyds.\end{aligned}$$

Step V

What still left is to construct such an initial data (u_0, θ_0) as provided above with the following properties:

- $u_0 \in \mathcal{S}(\mathbb{R}^d)^d$ with $\operatorname{div} u_0 = 0$ and $\theta_0 \in \mathcal{S}(\mathbb{R}^d)$,

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- $(u_0, \theta_0)(\tilde{x}) = (\tilde{u}_0, \theta_0)(x)$,
- $$\int_0^t \int_{\mathbb{R}^d} e^{s\Delta} u_{0,1}(y) e^{s\Delta} u_{0,2}(y) dyds = 0 \quad \text{for all} \quad t > 0,$$
- $$\int_0^t \int_{\mathbb{R}^d} g_1(y) e^{s\Delta} \theta_0(y) dyds \quad \text{and} \quad \int_0^t \int_{\mathbb{R}^d} e^{s\Delta} u_{0,1}(y) e^{s\Delta} \theta_0(y) dyds$$

  changes sign inside $(t_i - \varepsilon, t_i + \varepsilon)$.

Thank you for your attention!
Arigatoo gozaimasu.

Appendix

In Theorem 3 the gravity g belongs to $L^{\infty,*}_{d-1}$. The *-Notation means a quite weak symmetry property of the Fourier transform of $\frac{g(x)}{1+|x|^2}$:
Either for all c there is $\xi \in \mathbb{R}^d$ with $|\xi| = c$ such that

$$\mathfrak{g}_{neven}(\xi) := g^*(\xi) - g^*(-\xi) + g^*(\tilde{\xi}) - g^*(-\tilde{\xi}) \neq 0,$$

or for all c there is $\xi \in \mathbb{R}^d$ with $|\xi| = c$ such that

$$\mathfrak{g}_{nodd}(\xi) := g^*(\xi) + g^*(-\xi) + g^*(\tilde{\xi}) + g^*(-\tilde{\xi}) \neq 0,$$

with $g^*(\xi) := \mathcal{F}^{-1}(\frac{g(x)}{1+|x|^2})(\xi)$.
This strange property is used in Step V to construct the certain initial data.