Nonlinear Scalar Field Equations in $\mathbf{R}^N$
- A Mountain Pass Approach -

Norihisa Ikoma

Graduate School of Fundamental Science and Technology,
Waseda University

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Based on a joint work with
J. Hirata (Waseda) and K. Tanaka (Waseda)

§1: Introduction

We consider the following nonlinear scalar field equations:

$$\begin{cases} -\Delta u = g(u) & \text{in } \mathbf{R}^N, \\ u \in H_r^1(\mathbf{R}^N). \end{cases} \tag{1}$$

$N \geq 2$ and $g \in C(\mathbf{R}, \mathbf{R})$, $g(-s) = -g(s)$ for all $s \in \mathbf{R}$,

$H_r^1(\mathbf{R}^N) = \{u \in H^1(\mathbf{R}^N) \mid u(x) = u(|x|)\}$.

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Concerning (1), there are famous papers.

[BL] Berestycki–Lions, I & II ('83). ($N \geq 3$)

[BGK] Berestycki–Gallouët–Kavian ('83). ($N = 2$)

§1: Conditioins in [BL] and [BGK]

(g0) $g \in C(\mathbf{R}, \mathbf{R})$ is an odd function.

(g1) • $(N \geq 3)$ $-\infty < \liminf_{s \to 0} \frac{g(s)}{s} \leq \limsup_{s \to 0} \frac{g(s)}{s} < 0$.

• $(N = 2)$ $-\infty < \lim_{s \to 0} \frac{g(s)}{s} < 0$.

(g2) • $(N \geq 3)$ $\lim_{s \to \infty} \frac{g(s)}{s^{2^*-1}} = 0$. Here $2^* = 2N/(N-2)$.

• $(N = 2)$ $\lim_{s \to \infty} \frac{g(s)}{\exp(\alpha s^2)} = 0$ for any $\alpha > 0$.

(g3) $\exists\ \zeta_0 > 0$ s.t. $G(\zeta_0) = \int_0^{\zeta_0} g(s)ds > 0$.

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[BL], [BGK] $\Rightarrow$ $\exists$ positive solution of (1).

§1: Ideas in [BL] and [BGK]

Consider the following minimizing problem:

$$(N \geq 3)\text{Minimize } \left\{ \|\nabla u\|_{L^2}^2 \;\Big|\; \int_{\mathbf{R}^N} G(u)dx = 1 \right\},$$

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Let v_0 be a minimizer. Then

$$\exists\ \lambda_0 > 0 \quad \text{s.t.} \quad u(x) = v_0(x/\lambda_0) \text{ is a pos. sol. of (1).}$$

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Pohozaev Identity. Let u be a sol. of (1). Then

$$P(u) = \frac{N-2}{2}\|\nabla u\|_{L^2}^2 - N\int_{\mathbf{R}^N} G(u)dx = 0.$$

§1: Motivation

Sol. of (1) is characterized by a critical pt. of

$$I(u) = \frac{1}{2}\|\nabla u\|_{L^2}^2 - \int_{\mathbf{R}^N} G(u)dx.$$

Motivation.

1 Can we obtain solutions of (1) by the mountain pass approach?

2 Extend the result of [BGK].

(g1') $-\infty < \liminf_{s\to 0} \frac{g(s)}{s} \leq \limsup_{s\to 0} \frac{g(s)}{s} < 0.$

§1: Main Result

Main Theorem. Suppose (g0),(g1'),(g2) and (g3). (1) has a positive least energy solution and infinitely many solutions, which are characterized by the mountain pass and symmetric mountain pass minimax arguments.

Remark. Brezis–Lieb ('84) considered the existence of a least energy solution to the more general settings. However, they did not obtain the multiplicity of solutions.

§2: Idea of proof

We seek critical points of

$$I(u) = \frac{1}{2}\|\nabla u\|_{L^2}^2 - \int_{\mathbf{R}^N} G(u)dx \in C^1(H_r^1(\mathbf{R}^N), \mathbf{R}).$$

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Proposition. Under (g0), (g1'), (g2) and (g3), I satisfies the mountain pass structure;

(i) $\exists\ \delta, \rho > 0$ s.t. $I(u) \geq \delta$ for all $\|u\|_{H^1} = \rho$.

(ii) There exists a $w_0 \in H_r^1(\mathbf{R}^N)$ such that $\|w_0\|_{H^1} > \rho$ and $I(w_0) < 0$.

§2: Idea of proof

Define the mountain pass value b_{mp} as follows:

$$b_{mp} := \inf_{\gamma\in\Gamma} \max_{0\le t\le 1} I(\gamma(t)) \ge \delta > 0,$$

$$\Gamma := \{\gamma \in C([0,1], H^1_r(\mathbf{R}^N)) \mid \gamma(0) = 0,\ \gamma(1) = w_0\}.$$

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By the mountain pass theorem (Ambrosetti–Rabinowitz ('73)),

$$\exists\ (u_n) \subset H^1_r(\mathbf{R}^N) \quad \text{s.t.}$$

$$I(u_n) \to b_{mp},\ \ I'(u_n) \to 0 \quad \text{in } (H^1_r(\mathbf{R}^N))^*,\ \ \|u_n^-\|_{H^1} \to 0.$$

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Difficulty.

Boundedness of (u_n).

§2: Idea of proof

Define $\tilde{I}(\theta, u) \in C^1(\mathbf{R} \times H_r^1(\mathbf{R}^N), \mathbf{R}^N)$ by

$$\tilde{I}(\theta, u) := \frac{e^{(N-2)\theta}}{2} \|\nabla u\|_{L^2}^2 - e^{N\theta} \int_{\mathbf{R}^N} G(u)dx = I\left(u\left(\frac{x}{e^\theta}\right)\right).$$

(cf. Jeanjean ('97))

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Note $\tilde{I}(0,u) = I(u)$. Furthermore, $\exists\ \theta_0 > 0$ such that

$$\tilde{I}(\theta,u) \geq \delta > 0 \quad \text{for all } |\theta| \leq \theta_0, \|u\|_{H^1} = \rho,$$

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Thus we can define

$$\tilde{b}_{mp} := \inf_{\tilde{\gamma} \in \tilde{\Gamma}} \max_{0 \leq t \leq 1} \tilde{I}(\tilde{\gamma}(t)) \geq \delta > 0,$$

$$\tilde{\Gamma} := \{\tilde{\gamma} \in C([0, 1], \mathbf{R} \times H_r^1(\mathbf{R}^N)) \mid \tilde{\gamma}(0) = (0, 0), \\ \tilde{\gamma}(1) = (0, w_0)\}.$$

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Proposition.

$$b_{mp} = \tilde{b}_{mp}.$$

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By the mountain pass theorem, $\exists\ (\theta_n, v_n) \in \mathbf{R} \times H_r^1(\mathbf{R}^N)$ s.t.

$$\tilde{I}(\theta_n, v_n) \to b_{mp}, \quad D_u\tilde{I}(\theta_n, v_n) \to 0 \quad \text{in } (H_r^1(\mathbf{R}^N))^*,$$

$$\theta_n \to 0, \quad \|v_n^-\|_{H^1} \to 0, \quad D_\theta\tilde{I}(\theta_n, v_n) \to 0.$$

Here

$$D_\theta\tilde{I}(\theta_n, v_n) = e^{(N-2)\theta_n}\frac{N-2}{2}\|\nabla v_n\|_{L^2}^2 - e^{N\theta_n}N\int_{\mathbf{R}^N} G(v_n)dx$$

$$\to 0.$$

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Since $\theta_n \to 0$, (cf. Pohozaev Identity: $P(u) = 0$)

$$P(v_n) = \frac{N-2}{2}\|\nabla v_n\|_{L^2}^2 - N\int_{\mathbf{R}^N} G(v_n)dx \to 0.$$

Proposition. (v_n) is bounded in $H_r^1(\mathbf{R}^N)$.

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Therefore $\exists\ v_0(x) \geq 0$ s.t. $I(v_0) = b_{mp} > 0,\ I'(v_0) = 0$.

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Propositioin. (cf. Jeanjean–Tanaka (02')) For any non-trivial sol. $u \in H^1(\mathbf{R}^N)$ of (1),

$$I(u) \geq I(v_0).$$

Thank you for your attention!