The Dirichlet Laplace operator on Sobolev spaces of higher order

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Manuel Nesensohn
Fachbereich Mathematik
TU Darmstadt

Motivation

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Recall:

- For $s \geq 0$ and $1 < p < \infty$ is the H_p^s-realization of the Laplacian in the whole-space

$$\Delta_s \colon H_p^{s+2}(\mathbb{R}^n) \subset H_p^s(\mathbb{R}^n) \to H_p^s(\mathbb{R}^n), \quad u \mapsto \Delta u = -\sum_{i=1}^{n} \partial_j^2 u$$

  a sectorial operator, admitting maximal regularity and a bounded $\mathcal{H}^\infty$-calculus.

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- For $1 < p < \infty$ is the L_p-realization of the Dirichlet-Laplacian in the half-space

$$\Delta_{D,0} \colon H_p^2(\mathbb{R}_+^n) \cap H_{p,0}^1(\mathbb{R}_+^n) \subset L_p(\mathbb{R}_+^n) \to L_p(\mathbb{R}_+^n), \quad u \mapsto \Delta u$$

  also a sectorial operator, admitting maximal regularity and a bounded $\mathcal{H}^\infty$-calculus.

Main Results

This talk is about the H_p^1-realization of the Dirichlet-Laplacian in the half-space

$$\Delta_{D,1} : H_p^3(\mathbb{R}_+^n) \cap H_{p,0}^1(\mathbb{R}_+^n) \subset H_p^1(\mathbb{R}_+^n) \to H_p^1(\mathbb{R}_+^n), \quad u \mapsto \Delta u.$$

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- This operator is not the generator of a C_0-semigroup, especially not a sectorial operator.
- An optimal resolvent estimate is

$$\|\lambda(\lambda + \Delta_{D,1})^{-1}\|_{L(H^1_p(\mathbb{R}^n_+))} \leq C|\lambda|^{(p-1)/2p}, \quad \lambda \in \overline{\mathbb{C}_+} \setminus \{0\}.$$

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- A resolvent estimate in parameter-dependend norms.
- A positive result in spaces with additional boundary conditions.

Optimal Resolvent Estimate

Theorem

1. *There exists* $C > 0$ *such that for* $\lambda \in \overline{\mathbb{C}_+} \setminus \{0\}$

$$\|\lambda(\lambda + \Delta_{D,1})^{-1}\|_{L(H^1_p(\mathbb{R}^n_+))} \leq C|\lambda|^{(p-1)/2p}.$$

2. *There exits* $f \in \mathcal{S}(\mathbb{R}^n_+)$ *and* $C(f), \lambda_0 > 0$ *such that for* $\lambda > \lambda_0$

$$\|\lambda(\lambda + \Delta_{D,1})^{-1} f\|_{H^1_p(\mathbb{R}^n_+)} \geq C(f)\lambda^{(p-1)/2p}\|f\|_{H^1_p(\mathbb{R}^n_+)}.$$

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The resolvent problem is given by:

$$(RHE)_f \begin{cases} (\lambda + \Delta)u &= f \quad \text{in} \quad \mathbb{R}_+^n, \\ u &= 0 \quad \text{on} \quad \mathbb{R}^{n-1}. \end{cases}$$

Optimal Resolvent Estimate

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- An explicit solution formula of $(RHE)_f$ is:

$$u(x', x_n) = \int_0^\infty k_-(x_n, s) f(x', s) ds, \quad k_\pm(x_n, s) = \frac{e^{-\sqrt{\lambda+\Delta_0'}|x_n - s|} \pm e^{-\sqrt{\lambda+\Delta_0'}(x_n - s)}}{2\sqrt{\lambda + \Delta_0'}}.$$

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- To obtain estimates in H_p^1 we have to calculate the derivative of the solution:

$$\partial_n u(x', x_n) = k_+(x_n, 0) f(x', 0) + \int_0^\infty k_+(x_n, s) \partial_s f(x', s)\, ds.$$

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- Using dual-pairing and Placherel's theorem, we can show the exists of $f \in \mathcal{S}(\mathbb{R}^n_+)$ and $C > 0$ such that for $\lambda > 0$

$$\left(\int_0^\infty \|k_+(x_n, 0) \gamma f\|^p_{L_p(\mathbb{R}^{n-1})} dx_n \right)^{1/p} \geq \frac{C(f)}{\lambda^{(p+1)/2p}} \|f\|_{H^1_p(\mathbb{R}^n_+)}.$$

Resolvent Estimate in Parameter-Dependent Norms

Let $f \in H_p^k(\mathbb{R}_+^n)$. Define the parameter-dependent norm

$$|||f|||_{H_p^k(\mathbb{R}_+^n)} = |\lambda|^{k/2} \|f\|_{L_p(\mathbb{R}_+^n)} + \|f\|_{H_p^k(\mathbb{R}_+^n)}.$$

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Theorem

For $f \in H_p^k(\mathbb{R}_+^n)$ and $\lambda \in \overline{\mathbb{C}_+} \setminus \{0\}$ there exists a unique solution $u \in H_p^{k+2}(\mathbb{R}_+^n)$ of $(RHE)_f$ and constants $C_1, C_2 > 0$ (independent of f and λ) such that

$$C_1|||f|||_{H_p^k(\mathbb{R}_+^n)} \le |||u|||_{H_p^{k+2}(\mathbb{R}_+^n)} \le C_2|||f|||_{H_p^k(\mathbb{R}_+^n)}.$$

Thus for $k = 1$

$$|\lambda|^{1/2}\|f\|_{L_p(\mathbb{R}_+^n)} + \|f\|_{H_p^1(\mathbb{R}_+^n)} \le |\lambda|^{3/2}\|u\|_{L_p(\mathbb{R}_+^n)} + \|u\|_{H_p^3(\mathbb{R}_+^n)} \le |\lambda|^{1/2}\|f\|_{L_p(\mathbb{R}_+^n)} + \|f\|_{H_p^1(\mathbb{R}_+^n)}.$$

Resolvent Estimate in Parameter-Dependent Norms

- Define the scaling operator

$$J\colon H_p^k(\mathbb{R}_+^n) \to H_p^k(\mathbb{R}_+^n), \quad (Jf)(x) = f(\frac{x}{|\lambda|^{1/2}}).$$

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- For $\lambda \in \overline{\mathbb{C}_+} \setminus \{0\}$ and $\lambda_0 > 0$ there exists constants $C_1, C_2 > 0$ such that for $f \in H_p^k(\mathbb{R}_+^n)$

$$J\left((\lambda + \Delta_{D,1})^{-1} f\right) = \frac{1}{|\lambda|}\left(\frac{\lambda}{|\lambda|} + \Delta_{D,1}\right)^{-1} Jf$$

  and $|\lambda| > \lambda_0$

$$C_1 \|Jf\|_{H_p^k(\mathbb{R}_+^n)} \leq |\lambda|^{n/(2p)-k/2} \|f\|_{H_p^k(\mathbb{R}_+^n)} \leq C_2 \|Jf\|_{H_p^k(\mathbb{R}_+^n)}.$$

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- Use the estimate for $|\lambda| = 1$ and the scaling properties.

Result in Spaces with Additional Boundary Conditions

Theorem

Let $s < 1/p$. The operators

$$\begin{aligned} \{f \in H^3_p(\mathbb{R}^n_+) \colon \gamma f = \gamma \Delta f = 0\} &\subset H^1_{p,0}(\mathbb{R}^n_+) \to H^1_{p,0}(\mathbb{R}^n_+), \quad f \mapsto \Delta f \\ \{f \in H^{s+2}_p(\mathbb{R}^n_+) \colon \gamma f = 0\} &\subset H^s_p(\mathbb{R}^n_+) \to H^s_p(\mathbb{R}^n_+), \quad f \mapsto \Delta f \end{aligned}$$

are sectorial operators.

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Let X, Y be Banach spaces with $D(A) \overset{d}{\hookrightarrow} Y \overset{d}{\hookrightarrow} X$ and A a sectorial operator in X. We define the Y-realization A_Y of the linear operator A:

$$A_Y \colon Y \to Y, \quad D(A_Y) = \{x \in D(A) \cap Y \colon Ax \in Y\}, \quad A_Y x = Ax.$$

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Define:

- $X_k \overset{def}{=} D(A^k)$ and $X_{k+\theta} \overset{def}{=} [X_k, X_{k+1}]_\theta$.

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Then: $A_\alpha : D(A_\alpha) = X_{\alpha+1} \subset X_\alpha \to X_\alpha$ is a sectorial operator.
If $A \in \mathcal{BIP}(X_0)$, then

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$$D(\Delta_{D,0}^{k+\frac{1}{2}}) = [D(\Delta_{D,0}^2), \Delta_{D,0}]_{1/2} = \begin{cases} \left\{ f \in H_p^1 : \gamma f = 0 \right\}, & k = 0 \\ \left\{ f \in H_p^3 : \gamma f \gamma \Delta f = 0 \right\}, & k = 1 \end{cases}$$

THANK YOU FOR YOUR ATTENTION