The Navier-Stokes Equations: A Mathematical Analysis

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Article Outline

Glossary and Notation

Steady-State Flow Flow where both velocity and pressure fields are time-independent.

Three-dimensional (or 3D) flow Flow where velocity and pressure fields depend on all three spatial variables.

Two-dimensional (or planar, or 2D) flow Flow where velocity and pressure fields depend only on two spatial variables belonging to a portion of a plane, and the component of the velocity orthogonal to that plane is identically zero.

Local Solution Solution where velocity and pressure fields are known to exist only for a finite interval of time.

Global Solution Solution where velocity and pressure fields exist for all positive times.

Regular Solution Solution where velocity and pressure fields satisfy the Navier-Stokes equations and the corresponding initial and boundary conditions in the ordinary sense of differentiation and continuity.

At times, we may interchangeably use the words "flow" and "solution".

Basic Notation. $\mathbb{N}$ is the set of positive integers. $\mathbb{R}$ is the field of real numbers and $\mathbb{R}^N$, $N \in \mathbb{N}$, is the set of all N-tuple $\boldsymbol{x} = (x_1, \cdots, x_N)$. The canonical base in $\mathbb{R}^N$ is denoted by $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, \ldots, \boldsymbol{e}_N\} \equiv \{\boldsymbol{e}_i\}$. For $a, b \in \mathbb{R}$, $b > a$, we set $(a, b) = \{x \in \mathbb{R} : a < x < b\}$, $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$, $[a, b) = \{x \in \mathbb{R} : a \leq x < b\}$ and $(a, b] = \{x \in \mathbb{R} : a < x \leq b\}$. By $\overline{\mathcal{A}}$ we indicate the closure of the subset $\mathcal{A}$ of $\mathbb{R}^N$. A *domain* is an open connected subset of $\mathbb{R}^N$. Given a second-order tensor $\boldsymbol{A}$ and a vector $\boldsymbol{a}$, of components $\{A_{ij}\}$ and $\{a_i\}$, respectively, in the basis $\{\boldsymbol{e}_i\}$, by $\boldsymbol{a} \cdot \boldsymbol{A}$ [respectively, $\boldsymbol{A} \cdot \boldsymbol{a}$] we mean the vector with components $A_{ij} a_i$ [respectively, $A_{ij} a_j$]. (We use the Einstein summation convention over repeated indices, namely, if an index occurs twice in the same expression, the expression is implicitly summed over all possible values for that index.) Moreover, we set $|\boldsymbol{A}| = \sqrt{A_{ij}A_{ij}}$. If $\boldsymbol{h}(\boldsymbol{z}) \equiv \{h_i(\boldsymbol{z})\}$ is a vector field, by $\nabla \boldsymbol{h}$ we denote the second-order tensor field whose components $\{\nabla \boldsymbol{h}\}_{ij}$ in the given basis are given by $\{\partial h_j / \partial z_i\}$.

Function Spaces Notation. If $\mathcal{A} \subseteq \mathbb{R}^N$ and $k \in \mathbb{N} \cup \{0\}$, by $C^k(\mathcal{A})$ [respectively, $C^k(\overline{\mathcal{A}})$] we denote the class of functions which are continuous in $\mathcal{A}$ up to their k-th derivatives included [respectively, are bounded and uniformly continuous in $\mathcal{A}$ up to their k-th derivatives included]. The subset of $C^k(\mathcal{A})$ of functions vanishing outside a compact subset of $\mathcal{A}$ is indicated by $C_0^k(\mathcal{A})$. If $u \in C^k(\mathcal{A})$ for all $k \in \mathbb{N} \cup \{0\}$, we shall write $u \in C^\infty(\mathcal{A})$. In analogous way we define $C^\infty(\overline{\mathcal{A}})$ and $C_0^\infty(\mathcal{A})$. The symbols $L^q(\mathcal{A})$, $W^{m,q}(\mathcal{A})$, $m \geq 0$, $1 \leq q \leq \infty$, denote the usual Lebesgue and Sobolev spaces, respectively ($W^{0,q}(\mathcal{A}) = L^q(\mathcal{A})$). Norms in $L^q(\mathcal{A})$ and $W^{m,q}(\mathcal{A})$ are denoted by $\|\cdot\|_{q,\mathcal{A}}$, $\|\cdot\|_{m,q,\mathcal{A}}$. The trace space on the boundary, $\partial\mathcal{A}$, of $\mathcal{A}$ for functions from $W^{m,q}(\mathcal{A})$ will be denoted by $W^{m-1/q,q}(\partial\mathcal{A})$ and its norm by $\|\cdot\|_{m-1/q,q,\partial\mathcal{A}}$.

By $D^{k,q}(\mathcal{A})$, $k \geq 1$, $1 < q < \infty$, we indicate the homogeneous Sobolev space of order (m, q) on $\mathcal{A}$, that is, the class of functions u that are (Lebesgue) locally integrable in $\mathcal{A}$ and with $D^\beta u \in L^q(\mathcal{A})$, $|\beta| = k$, where $D^\beta = \partial^{|\beta|}/\partial x_1^{\beta_1}\partial x_2^{\beta_2}\cdots\partial x_N^{\beta_N}$, $|\beta| = \beta_1 + \beta_2 + \cdots + \beta_N$. For $u \in D^{k,q}(\mathcal{A})$, we put

$$|u|_{k,q,\mathcal{A}} = \left(\sum_{|\beta|=k} \int_{\mathcal{A}} |D^\beta u|^q\right)^{1/q} .$$

Notice that, whenever confusion does not arise, in the integrals we omit the infinitesimal volume or surface elements. Let

$$\mathcal{D}(\mathcal{A}) = \{\boldsymbol{\varphi} \in C_0^\infty(\mathcal{A}) : \operatorname{div} \boldsymbol{\varphi} = 0\} .$$

By $L_\sigma^q(\mathcal{A})$ we denote the completion of $\mathcal{D}(\mathcal{A})$ in the norm $\|\cdot\|_q$. If $\mathcal{A}$ is any domain in $\mathbb{R}^N$ we have $L^2(\mathcal{A}) = L_\sigma^2(\mathcal{A}) \oplus G(\mathcal{A})$, where $G(\mathcal{A}) = \{\boldsymbol{h} \in L^2(\mathcal{A}) : \boldsymbol{h} = \nabla p\,,\ \text{for some } p \in D^{1,2}(\mathcal{A})\}$; see [31, Section III.1]. We denote by P the orthogonal projection operator from $L^2(\mathcal{A})$ onto $L_\sigma^2(\mathcal{A})$. By $\mathcal{D}_0^{1,2}(\mathcal{A})$ we mean the completion of $\mathcal{D}(\mathcal{A})$ in the norm $|\cdot|_{1,2,\mathcal{A}}$. $\mathcal{D}_0^{1,2}(\mathcal{A})$ is a Hilbert space with scalar product $[\boldsymbol{v}_1, \boldsymbol{v}_2] := \int_{\mathcal{A}} (\partial \boldsymbol{v}_1/\partial x_i) \cdot (\partial \boldsymbol{v}_2/\partial x_i)$. Furthermore, $\mathcal{D}_0^{-1,2}(\mathcal{A})$ is the dual space of $\mathcal{D}_0^{1,2}(\mathcal{A})$ and $\langle \cdot, \cdot \rangle_{\mathcal{A}}$ is the associated duality pairing.

If $\boldsymbol{g} \equiv \{g_i\}$ and $\boldsymbol{h} \equiv \{h_i\}$ are vector fields on $\mathcal{A}$, we set

$$(\boldsymbol{g}, \boldsymbol{h})_{\mathcal{A}} = \int_{\mathcal{A}} g_i h_i\,,$$

whenever the integrals make sense.

In all the above notation, if confusion will not arise, we shall omit the subscript $\mathcal{A}$.

Given a Banach space X, and an open interval (a, b), we denote by $L^q(a, b; X)$ the linear space of (equivalence classes of) functions $f : (a, b) \to X$ whose X-norm is in $L^q(a, b)$. Likewise, for r a non-negative integer and I a real interval, we denote by $C^r(I; X)$ the class of continuous functions from I to X, which are differentiable in I up to the order r included.

If X denotes any space of real functions, we shall use, as a rule, the same symbol X to denote the corresponding space of vector and tensor-valued functions.

I Definition of the Subject: A Precious Tool in Real-Life Applications and an Outstanding Mathematical Challenge

The Navier-Stokes equations are a mathematical model aimed at describing the motion of an incompressible viscous fluid, like many common ones as, for instance, water, glycerine, oil and, under certain circumstances, also air. They were introduced in 1822 by the French engineer Claude Louis Marie Henri Navier and successively re-obtained, by different arguments, by a number of authors including Augustin-Louis Cauchy in 1823, Siméon Denis Poisson in 1829, Adhémar Jean Claude Barré de Saint-Venant in 1837, and, finally, George Gabriel Stokes in 1845. We refer the reader to the beautiful paper by Olivier Darrigol [17], for a detailed and thorough analysis of the history of the Navier-Stokes equations.

Even though, for quite some time, their significance in the applications was not fully recognized, the Navier-Stokes equations are, nowadays, at the foundations of many branches of applied sciences, including Meteorology, Oceanography, Geology, Oil Industry, Airplane, Ship and Car Industries, Biology and Medicine. In each of the above areas, these equations have collected many undisputed successes, which definitely place them among the most accurate, simple and beautiful models of mathematical physics.

Notwithstanding these successes, up to the present time, a number of unresolved basic mathematical questions remain open –mostly, but not only, for the physically relevant case of three-dimensional flow.

Undoubtedly, the most celebrated is that of proving or disproving the existence of global 3D regular flow for data of arbitrary "size", no matter how smooth (*global regularity problem*). Since the beginning of the 20th century, this notorious question has challenged several generations of mathematicians who have not been able to furnish a definite answer. In fact, to date, 3D regular flows are known to exist *either* for all times but for data of "small size", *or* for data of "arbitrary size" but for a finite interval of time only. The problem of global regularity has become so intriguing and compelling that, in the year 2000, it was decided to put a generous bounty on it. In fact, properly formulated, it is listed as one of the seven \$1M Millennium Prize Problems of the Clay Mathematical Institute.

However, the Navier-Stokes equations present also other fundamental open questions. For example, it not known whether, in the 3D case, the associated initial-boundary value problem is (in an appropriate function space) well-posed in the sense of Hadamard. Stated differently, in 3D it is open the question of whether solutions to this problem exist for all times, are unique and depend continuously upon the data, without being necessarily "regular".

Another famous, unsettled challenge is whether or not the Navier-Stokes equations are able to provide a rigorous model of turbulent phenomena. These phenomena occur when the magnitude of the driving mechanism of the fluid motion becomes sufficiently "large", and, roughly speaking, it consists of flow regimes characterized by chaotic and random property changes for velocity and pressure fields throughout the fluid. They are observed in 3D as well as in two-dimensional (2D) motions (*e.g.*, in flowing soap films). We recall that a 2D motion occurs when the relevant region of flow is contained in a portion of a plane, ϖ, and the component of the velocity field orthogonal to ϖ is negligible.

It is worth emphasizing that, in principle, the answers to the above questions may be unrelated. Actually, in the 2D case, the first two problems have long been solved in the affirmative, while the third one remains still open. Nevertheless, there is hope that proving or disproving the first two problems in 3D will require completely fresh and profound ideas that will open new avenues to the understanding of turbulence.

The list of main open problems can not be exhausted without mentioning another outstanding question pertaining the boundary value problem describing steady-state flow. The latter is characterized by time-independent velocity and pressure fields. In such a case, if the flow region, $\mathcal{R}$, is multiply-connected, it is not known (neither in 2D nor in 3D) if there exists a solution under a given velocity distribution at the boundary of $\mathcal{R}$ that *merely* satisfy the physical requirement of conservation of mass.

II Introduction

Fluid mechanics is a very intricate and intriguing discipline of the applied sciences. It is, therefore, not surprising that the mathematics involved in the study of its properties can be, often, extremely complex and difficult. Complexities and difficulties, of course, may be more or less challenging depending on the mathematical model chosen to describe the physical situation.

Among the many mathematical models introduced in the study of fluid mechanics, the Navier-Stokes equations can be considered, without a doubt, the most popular one. However, this does not mean that they can correctly model any fluid under any circumstance. In fact, their range of applicability is restricted to the class of so called *Newtonian fluids*; see Remark 3 in Section III. This class includes several common liquids like water, most salt solutions in water, aqueous solutions, some motor oils, most mineral oils, gasoline and kerosene.

As mentioned in the previous section, these equations were proposed in 1822 by the French engineer Claude Navier upon the basis of a suitable molecular model. It is interesting to observe, however, that the law of interaction between the molecules postulated by Navier were shortly recognized to be totally inconsistent from the physical point of view for several materials and, in particular, for liquids. It was only more than two decades later, in 1845, that the same equations were re derived by the twenty-six year old George Stokes in a quite general way, by means of the theory of continua.

The role of mathematics in the investigation of the fluid properties and, in particular, of the Navier-Stokes equations, is, as in most branches of the applied sciences, twofold and aims at the accomplishment of the following objectives. The first one, of a more fundamental nature, is the validation of the mathematical model, and consists in securing conditions under which the governing equations possess the essential requirements of well-posedness, that is, existence and uniqueness of corresponding solutions and their continuous dependence upon the data. The second one, of a more advanced character, is to prove that the model gives an adequate interpretation of the observed phenomena.

This paper is, essentially, directed toward the first objective. In fact, its goals consist of (i) formulating the primary problems, (ii) describing the related known results, and (iii) pointing out the remaining basic open questions, for both boundary and initial-boundary value problems. Of course, it seemed unrealistic to give a detailed presentation of such a rich and diversified subject in a few number of pages. Consequently, we had to make a choice which, we hope, will still give a fairly good idea of this fascinating and intriguing subject of applied mathematics.

The plan of the chapter is the following. In Section III, we shall first give a brief derivation of the Navier-Stokes equations from continuum theory, then formulate the basic problems and, further on, discuss some basic properties. Section IV is dedicated to the boundary value problem in both bounded and exterior domains. Besides existence and uniqueness, the main topics include the study of the structure of solution set at large Reynolds number, as well as a condensed treatment of bifurcation issues. Section V deals with the initial-boundary value problem in a bounded domain and with the related questions of well-posedness and regularity. Finally, in section Section VI we shall outline some future directions of research. Companion to this last section, is the list of a number of significant open questions that are mentioned throughout the paper.

III Derivation of the Navier-Stokes Equations and Preliminary Considerations

In the continuum mechanics modeling of a fluid, $\mathcal{F}$, one assumes that, in the given time interval, $I \equiv [0, T]$, $T > 0$, of its motion, $\mathcal{F}$ continuously fills a region, Ω, of the three-dimensional space, $\mathbb{R}^3$. We call points, surfaces and volumes of $\mathcal{F}$, *material points* (or *particles*), *material surfaces* and *material volumes*, respectively. In most relevant applications, the region Ω does not depend on time. This happens, in particular, whenever the fluid is bounded by rigid walls like, for instance, in the case of a flow past a rigid obstacle, or a flow in a bounded container with fixed walls. However, there are also some significant situations where Ω depends on time as, for example, in the motion of a fluid in a pipe with elastic walls. Throughout this chapter, we shall consider flow of $\mathcal{F}$ where Ω *is time-independent*.

Balance Laws. In order to describe the motion of $\mathcal{F}$ it is convenient to represent the relevant physical quantities in the *Eulerian form*. Precisely, if $\boldsymbol{x} = (x_1, x_2, x_3)$ is a point in Ω and $t \in [0, T]$, we let $\rho = \rho(\boldsymbol{x}, t)$, $\boldsymbol{v} = \boldsymbol{v}(\boldsymbol{x}, t) = (v_1(\boldsymbol{x}, t), v_2(\boldsymbol{x}, t), v_3(\boldsymbol{x}, t))$ and $\boldsymbol{a} = \boldsymbol{a}(\boldsymbol{x}, t) = (a_1(\boldsymbol{x}, t), a_2(\boldsymbol{x}, t), a_3(\boldsymbol{x}, t))$ be the density, velocity and acceleration, respectively, of that particle of $\mathcal{F}$ that, at the time t, passes through the point $\boldsymbol{x}$. Furthermore, we denote by $\boldsymbol{f} = \boldsymbol{f}(\boldsymbol{x}, t) = (f_1(\boldsymbol{x}, t), f_2(\boldsymbol{x}, t), f_3(\boldsymbol{x}, t))$ the external force per unit volume (*body force*) acting on $\mathcal{F}$.

In the continuum theory of non-polar fluids, one postulates that in every motion the following equations must hold

$$\left.\begin{aligned} &\frac{\partial \rho}{\partial t}(\boldsymbol{x}, t) + \frac{\partial}{\partial x_i}(\rho(\boldsymbol{x}, t) v_i(\boldsymbol{x}, t)) = 0\,, \\ &\rho(\boldsymbol{x}, t)\, a_i(\boldsymbol{x}, t) = \frac{\partial T_{ji}}{\partial x_j} + \rho(\boldsymbol{x}, t)\, f_i(\boldsymbol{x}, t)\,, \quad i = 1, 2, 3 \\ &T_{ij}(\boldsymbol{x}, t) = T_{ji}(\boldsymbol{x}, t)\,, \quad i, j = 1, 2, 3\,, \end{aligned}\right\} \text{ for all } (\boldsymbol{x}, t) \in \Omega \times (0, T)\,. \tag{1}$$

These equations represent the local form of the *balance laws of $\mathcal{F}$ in the Eulerian description*. Specifically, $(1)_1$ expresses the *conservation of mass*, $(1)_2$ furnishes the *balance of linear momentum.*, while $(1)_3$ is equivalent to the *balance of angular momentum*. The function $\boldsymbol{T} = \boldsymbol{T}(\boldsymbol{x}, t) = \{T_{ji}(\boldsymbol{x}, t)\}$ is a second-order, symmetric tensor field, the *Cauchy stress tensor*, that takes into account the *internal forces* exerted by the fluid. More precisely, let S denote a (sufficiently smooth) fixed, closed surface in $\mathbb{R}^3$ bounding the region $\mathcal{R}_S$, let $\boldsymbol{x}$ be any point on S, and let $\boldsymbol{n} = \boldsymbol{n}(\boldsymbol{x})$ be the outward unit normal to S at $\boldsymbol{x}$. Furthermore, let $\mathcal{R}_S^{(e)}$ be the region exterior to $\mathcal{R}_S$, with $\mathcal{R}_S^{(e)} \cap \Omega \neq \emptyset$. Then the vector $\boldsymbol{t} = \boldsymbol{t}(\boldsymbol{x}, t)$ defined as

$$\boldsymbol{t}(\boldsymbol{x}, t) := \boldsymbol{n}(\boldsymbol{x}) \cdot \boldsymbol{T}(\boldsymbol{x}, t)\,, \tag{2}$$

represents the force per unit area exerted by the portion of the fluid in $\mathcal{R}_S^{(e)}$ on S at the point $\boldsymbol{x}$ and at time t; see Figure 1.

Constitutive Equation. An important kinematical quantity associated with the motion of $\mathcal{F}$ is the *stretching tensor* field, $\boldsymbol{D} = \boldsymbol{D}(\boldsymbol{x}, t)$, whose components, D_{ij}, are defined as follows:

$$D_{ij} = \tfrac{1}{2}\left(\frac{\partial v_j}{\partial x_i} + \frac{\partial v_i}{\partial x_j}\right)\,, \quad i, j = 1, 2, 3\,. \tag{3}$$

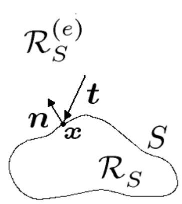


Figure 1. Stress vector at the point $\boldsymbol{x}$ of the surface S.

The stretching tensor is, of course, symmetric and, roughly speaking, it describes the rate of deformation of parts of $\mathcal{F}$. In fact, it can be shown that a necessary and sufficient condition for a motion of $\mathcal{F}$ to be *rigid* (namely, the mutual distance between two arbitrary particles of $\mathcal{F}$ does not change in time), is that $\boldsymbol{D}(\boldsymbol{x}, t) = 0$ at each $(\boldsymbol{x}, t) \in \Omega \times I$. Moreover, $\operatorname{div} \boldsymbol{v}(\boldsymbol{x}, t) \equiv \operatorname{trace} \boldsymbol{D}(\boldsymbol{x}, t) \equiv \dfrac{\partial v_i}{\partial x_i}(\boldsymbol{x}, t) = 0$ for all $(\boldsymbol{x}, t) \in \Omega \times I$, if and only if the motion is *isochoric*, namely, every material volume does not change with time. A noteworthy class of fluids whose generic motion is isochoric is that of fluids having constant density. In fact, if ρ is a positive constant, from $(1)_1$ we find $\operatorname{div} \boldsymbol{v}(\boldsymbol{x}, t) = 0$, for all $(\boldsymbol{x}, t) \in \Omega \times I$. Fluids with constant density are called *incompressible*. In this chapter, incompressible fluids will be referred to as *liquids*.

During the generic motion, the internal forces will, in general, produce a deformation of parts of $\mathcal{F}$. The relation between internal forces and deformation, namely, the functional relation between $\boldsymbol{T}$ and $\boldsymbol{D}$, is called *constitutive equation* and characterizes the physical properties of the fluid. A liquid is said to be *Newtonian* if and only if the relation between $\boldsymbol{T}$ and $\boldsymbol{D}$ is linear, that is, there exist a scalar function $p = p(\boldsymbol{x}, t)$ (the *pressure*) and a constant μ (the *shear viscosity*) such that

$$\boldsymbol{T} = -p\boldsymbol{I} + 2\mu\, \boldsymbol{D}\,, \tag{4}$$

where $\boldsymbol{I}$ denotes the identity matrix. In a *viscous* Newtonian liquid, one assumes that the shear viscosity satisfies the restriction

$$\mu > 0\,. \tag{5}$$

We will come back, in Remark 2, about the meaning of this assumption.

Navier-Stokes Equations. In view of the condition $\operatorname{div}\boldsymbol{v} = 0$, it easily follows from (4) that

$$\frac{\partial T_{ji}}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \mu\, \Delta v_i$$

where $\Delta := \dfrac{\partial^2}{\partial x_l \partial x_l}$ is the Laplace operator. Therefore, by (1) and (4) we deduce that the equations governing the motion of a Newtonian viscous liquid are furnished by

$$\left.\begin{aligned} &\rho\, a_i = -\frac{\partial p}{\partial x_i} + \mu \Delta v_i + \rho\, f_i\,, \quad i = 1,2,3 \\ &\frac{\partial v_i}{\partial x_i} = 0 \end{aligned}\right\} \quad \text{in } \Omega \times (0,T)\,. \tag{6}$$

It is interesting to observe that *both* equations in (6) are *linear* in all kinematical variables. However, in the Euler description, the acceleration is a *nonlinear* functional of the velocity, and we have

$$a_i = \frac{\partial v_i}{\partial t} + v_l \frac{\partial v_i}{\partial x_l}\,,$$

or, in a vector form,

$$\boldsymbol{a} = \frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v} \cdot \nabla \boldsymbol{v}\,.$$

Replacing this latter expression in (6) we obtain the *Navier-Stokes equations*:

$$\left.\begin{aligned} &\rho \left(\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v} \cdot \nabla \boldsymbol{v} \right) = -\nabla p + \mu \Delta \boldsymbol{v} + \rho\, \boldsymbol{f}\,, \\ &\operatorname{div}\boldsymbol{v} = 0 \end{aligned}\right\} \quad \text{in } \Omega \times (0,T)\,. \tag{7}$$

In these equations, the (constant) density ρ, the shear viscosity μ (satisfying (5)) and the force $\boldsymbol{f}$ are given quantities, while the *unknowns* are velocity $\boldsymbol{v} = \boldsymbol{v}(\boldsymbol{x}, t)$ and pressure $p = p(\boldsymbol{x}, t)$ fields.

Some preliminary comments about the above equations are in order. Actually, we should notice that the unknowns $\boldsymbol{v}, p$ do not appear in a "symmetric" way. In other words, the equation of conservation of mass $(7)_2$ does *not* involve the pressure field. This is due to the fact that, from the mechanical point of view, the pressure plays the role of *reaction force* (Lagrange multiplier) associated with the isochoricity constraint $\operatorname{div}\boldsymbol{v} = 0$. In other words, whenever a portion of the liquid "tries to change its volume" the liquid "reacts" with a suitable distribution of pressure to keep that volume constant. Thus, the pressure field must be generally deduced in terms of the velocity field, once this latter has been determined; see Remark 1.

Initial-Boundary Value Problem. In order to find solutions to the problem (7), we have to append suitable *initial*, at time $t = 0$, and *boundary* conditions. As a matter of fact, these conditions may depend on the specific physical problem we want to model. We shall assume that the region of flow, Ω, is bounded by rigid walls, $\partial\Omega$, and that the liquid does not "slip" at $\partial\Omega$. The appropriate initial and boundary conditions then become, respectively, the following ones

$$\begin{aligned} &\boldsymbol{v}(\boldsymbol{x},0) = \boldsymbol{v}_0(\boldsymbol{x})\,, \quad \boldsymbol{x} \in \Omega \\ &\boldsymbol{v}(\boldsymbol{x},t) = \boldsymbol{v}_1(\boldsymbol{x},t)\,, \quad (\boldsymbol{x},t) \in \partial\Omega \times (0,T)\,, \end{aligned} \tag{8}$$

where $\boldsymbol{v}_0$ and $\boldsymbol{v}_1$ are *prescribed* vector fields.

Steady-State Flow and Boundary-Value Problem. An important, special class of solutions to (7), called *steady-state solutions*, is that where velocity and pressure fields are independent of time. Of course, a necessary requirement for such solutions to exist is that $\boldsymbol{f}$ does not depend on time as well. From (7) we thus obtain that a generic steady-state solution, $(\boldsymbol{v} = \boldsymbol{v}(\boldsymbol{x}), p = p(\boldsymbol{x}))$, must satisfy the following equations

$$\left.\begin{aligned} &\rho\boldsymbol{v}\cdot\nabla\boldsymbol{v} = -\nabla p + \mu\Delta\boldsymbol{v} + \rho\,\boldsymbol{f}\,, \\ &\operatorname{div}\boldsymbol{v} = 0 \end{aligned}\right\}\ \text{in } \Omega\,. \tag{9}$$

Under the given assumptions on the region of flow, from $(8)_2$ it follows that the appropriate boundary conditions are

$$\boldsymbol{v}(\boldsymbol{x}) = \boldsymbol{v}_*(\boldsymbol{x})\,, \quad \boldsymbol{x} \in \partial\Omega\,, \tag{10}$$

where $\boldsymbol{v}_*$ is a prescribed vector field.

Two-Dimensional Flow. In several mathematical questions related to the unique solvability of problems (7)–(8) and (9)–(10), separate attention deserve two-dimensional solutions describing the *planar motions* of $\mathcal{F}$. For these solutions the fields $\boldsymbol{v}$ and p depend only on x_1, x_2 (say) (and t in the case (7)–(8)), and, moreover, $v_3 \equiv 0$. Consequently, the relevant (spatial) region of motion, Ω, becomes a subset of $\mathbb{R}^2$.

Remark 1 By formally operating with "div" on both sides of $(7)_1$ and by taking into account $(8)_2$, we find that, at each time $t \in (0,T)$ the pressure field $p = p(\boldsymbol{x},t)$ must satisfy the following Neumann problem

$$\begin{aligned} &\Delta p = \rho\,(\boldsymbol{v}\cdot\nabla\boldsymbol{v} - \boldsymbol{f}) \ \text{ in } \Omega \\ &\frac{\partial p}{\partial n} = -[\mu\Delta\boldsymbol{v} - \rho\,(\boldsymbol{v}_1\cdot\nabla\boldsymbol{v} - \boldsymbol{f})]\cdot\boldsymbol{n} \ \text{ at } \partial\Omega \end{aligned} \tag{11}$$

where $\boldsymbol{n}$ denotes the outward unit normal to $\partial\Omega$. It follows that the *prescription* of the pressure at the bounding walls or at the initial time *independently* of $\boldsymbol{v}$, could be incompatible with (8) and, therefore, could render the problem ill-posed. □

Remark 2 We can give a simple qualitative explanation of the assumption (5). To this end, let us consider a steady-state flow of a viscous liquid between two parallel, rigid walls Π_1, Π_2, set at a distance d apart and parallel to the $x_1 - x_2$ plane; see Figure 2. The flow is induced by a force per unit area $\boldsymbol{F} = F\boldsymbol{e}_1$, $F > 0$, applied to the wall Π_1, that moves Π_1 with a constant velocity $\boldsymbol{V} = V\,\boldsymbol{e}_1$, $V > 0$, while Π_2 is kept fixed; see Figure 2. No external force is acting on the liquid. It is then easily checked that the velocity and pressure fields $\boldsymbol{v} = V\,(x_2/d)\boldsymbol{e}_1$, $p = p_0$=const. (*pure shear* flow) satisfy (9) with $\boldsymbol{f} \equiv \boldsymbol{0}$, along with the appropriate boundary conditions $\boldsymbol{v}(0) = \boldsymbol{0}$, $\boldsymbol{v}(d) = V\,\boldsymbol{e}_1$.

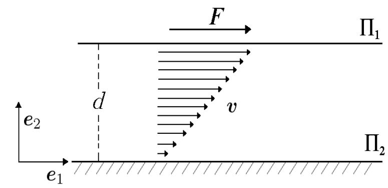


Figure 2. Pure shear flow between parallel plates induced by a force $\boldsymbol{F}$.

From (2) and (4), we obtain that the force $\boldsymbol{t}$ per unit area exerted by the fluid on Π_1 is given by

$$\boldsymbol{t}(x_1, d, x_3) = -\boldsymbol{e}_2 \cdot \boldsymbol{T}(x_1, d, x_3) = p_0 \boldsymbol{e}_2 - \mu \frac{V}{d} \boldsymbol{e}_1,$$

that is, $\boldsymbol{t}$ has a "purely pressure" component $\boldsymbol{t}_p = p_0\, \boldsymbol{e}_2$, and a "purely shear viscous" component $\boldsymbol{t}_v = -\mu(V/d)\boldsymbol{e}_1$. As expected in a viscous liquid, $\boldsymbol{t}_v$ is directed parallel to $\boldsymbol{F}$ and, of course, if it is not zero, it should also act *against* $\boldsymbol{F}$, that is, $\boldsymbol{t}_v \cdot \boldsymbol{e}_1 < 0$. However, $(V/d) > 0$, and so we must have $\mu > 0$. Since the physical properties of the fluid are independent of the particular flow, this simple reasoning justifies the assumption made in (5). □

Remark 3 As mentioned in the Introduction, the constitutive equation (4) and, as a consequence, the Navier-Stokes equations, provide a satisfactory model only for a certain class of liquids, while, for others, their predictions are at odds with experimental data. These latter include, for example, biological fluids, like blood or mucus, and aqueous polymeric solutions, like common paints, or even ordinary shampoo. In fact, these liquids, which, in contrast to those modeled by (4), are called *non-Newtonian*, exhibit a number of features that the linear relation (4) is not able to predict. Most notably, in liquids such as blood, the shear viscosity is no longer a constant and, in fact, it decreases as the magnitude of shear (proportional to $\sqrt{D_{ij}D_{ij}}$) increases. Furthermore, liquids like paints or shampoos show, under a given shear rate, a distribution of stress (other than that due to the pressure) in the direction orthogonal to the direction of shear (the so-called *normal stress effect*). Modeling and corresponding mathematical analysis of a variety of non-Newtonian liquids can be found, for example, in [42] □

IV Mathematical Analysis of the Boundary Value Problem

We begin to analyze the properties of solutions to the boundary-value problem (9)–(10). We shall divide our presentation into two parts, depending on the "geometry" of the region of flow Ω. Specifically, we shall treat the cases when Ω is either a bounded domain or an exterior domain (flow past an obstacle). For each case we shall describe methods, main results and fundamental open questions.

IV.1 Flow in Bounded Domains

In this section we shall analyze problem (9)–(10) where Ω is a bounded domain of $\mathbb{R}^3$. A similar (and simpler) analysis can be performed in the case of planar flow, with exactly the same results.

Variational Formulation and Weak Solutions. To show existence of solutions, one may use, basically, two types of methods that we shall describe in some detail. The starting point of both methods is the so-called *variational formulation* of (9)–(10). Let $\boldsymbol{\varphi} \in \mathcal{D}(\Omega)$. Since $\operatorname{div} \boldsymbol{\varphi} = \operatorname{div} \boldsymbol{v} = 0$ in Ω and $\boldsymbol{\varphi}|_{\partial\Omega} = \boldsymbol{0}$, we have (by a formal integration by parts)

$$\begin{gathered}(\boldsymbol{\varphi}, \nabla p) = \int_{\partial\Omega} p\, \boldsymbol{\varphi} \cdot \boldsymbol{n} = 0\,, \\ (\Delta \boldsymbol{v}, \boldsymbol{\varphi}) = -[\boldsymbol{v}, \boldsymbol{\varphi}] + \int_{\partial\Omega} \boldsymbol{n} \cdot \nabla \boldsymbol{v} \cdot \boldsymbol{\varphi} = -[\boldsymbol{v}, \boldsymbol{\varphi}] \qquad (12) \\ (\boldsymbol{v} \cdot \nabla \boldsymbol{v}, \boldsymbol{\varphi}) = \int_{\partial\Omega} \boldsymbol{v} \cdot \boldsymbol{n} \boldsymbol{v} \cdot \boldsymbol{\varphi} - (\boldsymbol{v} \cdot \nabla \boldsymbol{\varphi}, \boldsymbol{v}) = -(\boldsymbol{v} \cdot \nabla \boldsymbol{\varphi}, \boldsymbol{v})\end{gathered}$$

where $[\cdot,\cdot]$ is the scalar product in $\mathcal{D}_0^{1,2}(\Omega)$. Thus, if we dot-multiply both sides of $(9)_1$ by $\boldsymbol{\varphi} \in \mathcal{D}(\Omega)$ and integrate by parts over Ω, we (formally) obtain with $\nu := \mu/\rho$

$$\nu\,[\boldsymbol{v},\boldsymbol{\varphi}] - (\boldsymbol{v}\cdot\nabla\boldsymbol{\varphi},\boldsymbol{v}) = \langle \boldsymbol{f},\boldsymbol{\varphi}\rangle\,, \quad \text{for all } \boldsymbol{\varphi}\in\mathcal{D}(\Omega)\,, \tag{13}$$

where we assume that $\boldsymbol{f}$ belongs to $\mathcal{D}_0^{-1,2}(\Omega)$. Equation (13) is the *variational* (or *weak*) *form* of $(9)_1$. We observe that (13) does not contain the pressure. Moreover, every term in (13) is well-defined provided $\boldsymbol{v}\in W^{1,2}_{\rm loc}(\Omega)$.

Definition 1 *A function $\boldsymbol{v}\in W^{1,2}(\Omega)$ is called a* weak solution *to* (9)–(10) *if and only if:* (i) $\operatorname{div}\boldsymbol{v}=0$ *in Ω;* (ii) $\boldsymbol{v}|_{\partial\Omega}=\boldsymbol{v}_*$ *(in the trace sense);* (iii) $\boldsymbol{v}$ *satisfies* (13). *If, in particular, $\boldsymbol{v}_*\equiv\boldsymbol{0}$, we replace conditions* (i), (ii) *with the single requirement:* (i)$'$ $\boldsymbol{v}\in\mathcal{D}_0^{1,2}(\Omega)$.

Throughout this chapter we shall often use the following result, consequence of the Hölder inequality and of a simple approximating procedure.

Lemma 1 *The trilinear form*

$$(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c})\in L^q(\Omega)\times W^{1,2}(\Omega)\times L^r(\Omega)\mapsto(\boldsymbol{a}\cdot\nabla\boldsymbol{b},\boldsymbol{c})\in\mathbb{R}\,,\quad \frac{1}{q}+\frac{1}{r}=\frac{1}{2}\,,$$

is continuous. Moreover, $(\boldsymbol{a}\cdot\nabla\boldsymbol{b},\boldsymbol{c})=-(\boldsymbol{a}\cdot\nabla\boldsymbol{c},\boldsymbol{b})$, for any $\boldsymbol{b},\boldsymbol{c}\in D_0^{1,2}(\Omega)$, and for any $\boldsymbol{a}\in L^2_\sigma(\Omega)$. Thus, in particular, $(\boldsymbol{a}\cdot\nabla\boldsymbol{b},\boldsymbol{b})=0$.

Regularity of Weak Solutions. If $\boldsymbol{f}$ is sufficiently regular, the corresponding weak solution is regular as well and, moreover, there exists a scalar function, p, such that (9) is satisfied in the ordinary sense. Also, if $\partial\Omega$ and $\boldsymbol{v}_*$ are smooth enough, the solution $(\boldsymbol{v},p)$ is smooth up to the boundary and (10) is satisfied in the classical sense. A key tool in the proof of these properties is the following lemma, which is a special case of a more general result due to Cattabriga [12]; see also [31, Lemma IV.6.2 and Theorem IV.6.1].

Lemma 2 *Let Ω be a bounded domain of $\mathbb{R}^3$, of class C^{m+2}, $m\ge 0$, and let $\boldsymbol{g}\in\boldsymbol{W}^{m,q}(\Omega)$, $\boldsymbol{u}_*\in W^{m+2-1/q,q}(\partial\Omega)$, $1<q<\infty$, with $\int_{\partial\Omega}\boldsymbol{u}_*\cdot\boldsymbol{n}=0$. Moreover, let $\boldsymbol{u}\in W^{1,2}(\Omega)$ satisfy the following conditions*

(a) $\nu\,[\boldsymbol{u},\boldsymbol{\varphi}]=(\boldsymbol{g},\boldsymbol{\varphi})$ *for all $\boldsymbol{\varphi}\in\mathcal{D}(\Omega)$;*

(b) $\operatorname{div}\boldsymbol{u}=0$*;*

(c) $\boldsymbol{u}=\boldsymbol{u}_*$ *at $\partial\Omega$ in the trace sense.*

Then, $\boldsymbol{u}\in\boldsymbol{W}^{m+2,q}(\Omega)$ and there exists a unique $\phi\in\boldsymbol{W}^{m+1,q}(\Omega)$, with $\int_\Omega\phi=0$, such that the pair $(\boldsymbol{u},\phi)$ satisfies the following Stokes equations

$$\left.\begin{aligned} &-\nu\Delta\boldsymbol{u}=\nabla\phi+\boldsymbol{g}\\ &\operatorname{div}\boldsymbol{u}=0\end{aligned}\right\}\ \text{in } \Omega\,.$$

Furthermore, there exists a constant $C=C(\Omega,m,q)>0$ such that

$$\|\boldsymbol{u}\|_{m+2,q}+\|\phi\|_{m+1,q}\le C\,\left(\|\boldsymbol{g}\|_{m,q}+\|\boldsymbol{u}_*\|_{m+2-1/q,q,\partial\Omega}\right)\,.$$

To give an idea of how to prove regularity of weak solutions by means of Lemma 2, we consider the case $\boldsymbol{f}\in C^\infty(\overline{\Omega})$, Ω of class C^∞ and $\boldsymbol{v}_*\in C^\infty(\partial\Omega)$. (For a general regularity theory of weak solutions, see [32, Theorem VIII.52].) Thus, in particular, by the embedding [31, Theorem II.2.4]

$$W^{1,2}(\Omega)\subset L^q(\Omega)\,,\quad\text{for all } q\in[1,6]\,, \tag{14}$$

and by the Hölder inequality we have that $\boldsymbol{g} := \boldsymbol{f} - \boldsymbol{v}\cdot\nabla\boldsymbol{v} \in L^{3/2}(\Omega)$. From (13) and Lemma 2, we then deduce that $\boldsymbol{v} \in W^{2,3/2}(\Omega)$ and that there exists a scalar field $p \in W^{1,3/2}$ such that $(\boldsymbol{v}, p)$ satisfy (9) a.e. in Ω. Therefore, because of the embedding $W^{2,3/2}(\Omega) \subset W^{1,3}(\Omega) \subset L^r(\Omega)$, arbitrary $r \in [1,\infty)$, [31, Theorem II.2.4], we obtain the improved regularity property $\boldsymbol{g} \in W^{1,s}(\Omega)$, for all $s \in [1, 3/2)$. Using again Lemma 2, we then deduce $\boldsymbol{v} \in W^{3,s}(\Omega)$ and $p \in W^{2,s}(\Omega)$ which, in particular, gives further regularity for $\boldsymbol{g}$. By induction, we then prove $\boldsymbol{v}, p \in C^\infty(\overline{\Omega})$.

IV.1.1 Existence Results. Homogeneous Boundary Conditions

As we mentioned previously, there are, fundamentally, two kinds of approaches to show existence of weak solutions to (9)–(10), namely, the *finite-dimensional method* and the *function-analytic method*. We will first describe these methods in the case of homogeneous boundary conditions, $\boldsymbol{v}_* \equiv \boldsymbol{0}$, deferring the study of the non-homogeneous problem to subsection IV.1.2. In what follows, we shall refer to (9)–(10) with $\boldsymbol{v}_* = \boldsymbol{0}$ as (9)–$(10)_{\rm hom}$.

A. The Finite-Dimensional Method. This popular approach, usually called *Galerkin method*, was introduced by Fujita [28] and, independently, by Vorovich and Yudovich [96]. It consists in projecting (13) on a suitable finite dimensional space, V_N, of $\mathcal{D}_0^{1,2}(\Omega)$ and then in finding a solution, $\boldsymbol{v}_N \in V_N$, of the "projected" equation. One then passes to the limit $N \to \infty$ to show, with the help of an appropriate uniform estimate, that $\{\boldsymbol{v}_N\}$ contains at least one subsequence converging to some $\boldsymbol{v} \in \mathcal{D}_0^{1,2}(\Omega)$ that satisfies condition (13). Precisely, let $\{\boldsymbol{\psi}_k\} \subset \mathcal{D}(\Omega)$ be an orthonormal basis of $\mathcal{D}_0^{1,2}(\Omega)$, and set $\boldsymbol{v}_N = \sum_{i=1}^N c_{iN}\boldsymbol{\psi}_i$, where the coefficients c_{iN} are requested to be solutions of the following nonlinear algebraic system

$$\nu[\boldsymbol{v}_N, \boldsymbol{\psi}_k] - (\boldsymbol{v}_N\cdot\nabla\boldsymbol{\psi}_k, \boldsymbol{v}_N) = \langle \boldsymbol{f}, \boldsymbol{\psi}_k\rangle\,, \quad k = 1, \cdots, N\,. \tag{15}$$

By means of the Brower fixed point theorem, it can be shown (see [32, Lemma VIII.3.2]) that a solution $(c_{1N}, \cdots, c_{NN})$ to (15) exists, provided the following estimate holds

$$|\boldsymbol{v}_N|_{1,2} \le M \tag{16}$$

where M is a finite, positive quantity *independent of* N. Let us show that (16) indeed occurs. Multiplying through both sides of (15) by c_{kN}, summing over k from 1 to N, and observing that, by Lemma 1, $(\boldsymbol{v}_N\cdot\nabla\boldsymbol{v}_N, \boldsymbol{v}_N) = 0$, we obtain

$$\nu\,|\boldsymbol{v}_N|_{1,2}^2 = \langle \boldsymbol{f}, \boldsymbol{v}_N\rangle \le |\boldsymbol{f}|_{-1,2}|\boldsymbol{v}_N|_{1,2}\,,$$

which proves the desired estimate (16) with $M := |\boldsymbol{f}|_{-1,2}$. Notice that the validity of this latter estimate can be obtained by formally replacing in (13) $\boldsymbol{\varphi}$ with $\boldsymbol{v} \in \mathcal{D}_0^{1,2}(\Omega)$ and by using Lemma 1. From classical properties of Hilbert spaces and from (16), we can select a subsequence $\{\boldsymbol{v}_{N'}\}$ and find $\boldsymbol{v} \in \mathcal{D}_0^{1,2}(\Omega)$ such that

$$\lim_{N'\to\infty}[\boldsymbol{v}_{N'}, \boldsymbol{\varphi}] = [\boldsymbol{v}, \boldsymbol{\varphi}]\,, \quad \text{for all } \boldsymbol{\varphi} \in \mathcal{D}_0^{1,2}(\Omega)\,. \tag{17}$$

By Definition 1, to prove that $\boldsymbol{v}$ is a weak solution to (9)–$(10)_{\rm hom}$ it remains to show that $\boldsymbol{v}$ satisfies (13). In view of the *Poincaré inequality* [31, Theorem II.4.1]:

$$\|\boldsymbol{\varphi}\|_2 \le c_P\,|\boldsymbol{\varphi}|_{1,2} \quad \boldsymbol{\varphi} \in \mathcal{D}_0^{1,2}(\Omega)\,, \quad c_P = c_P(\Omega) > 0\,, \tag{18}$$

by (16) it follows that $\{\boldsymbol{v}_N\}$ is bounded in $W_0^{1,2}(\Omega)$ and so, by Rellich compactness theorem (see [31, Theorem II.4.2]) we can assume that $\{\boldsymbol{v}_{N'}\}$ converges to $\boldsymbol{v}$ in $L^4(\Omega)$:

$$\lim_{N'\to\infty}\|\boldsymbol{v}_{N'} - \boldsymbol{v}\|_4 = 0\,. \tag{19}$$

We now consider (15) with $N = N'$ and pass to the limit $N' \to \infty$. Clearly, by (17), we have for each fixed k

$$\lim_{N' \to \infty} [\boldsymbol{v}_{N'}, \boldsymbol{\psi}_k] = [\boldsymbol{v}, \boldsymbol{\psi}_k] \,. \tag{20}$$

Moreover, by Lemma 1 and by (19),

$$\lim_{N' \to \infty} (\boldsymbol{v}_{N'} \cdot \nabla \boldsymbol{\psi}_k, \boldsymbol{v}_{N'}) = (\boldsymbol{v} \cdot \nabla \boldsymbol{\psi}_k, \boldsymbol{v}) \,,$$

and so, from this latter relation, from (20), and from (15) we conclude that

$$\nu \, [\boldsymbol{v}, \boldsymbol{\psi}_k] - (\boldsymbol{v} \cdot \nabla \boldsymbol{\psi}_k, \boldsymbol{v}) = \langle \boldsymbol{f}, \boldsymbol{\psi}_k \rangle \,, \quad \text{for all } k \in \mathbb{N} \,,$$

Since $\{\boldsymbol{\psi}_k\}$ is a basis in $\mathcal{D}_0^{1,2}(\Omega)$, this latter relation, along Lemma 1, immediately implies that $\boldsymbol{v}$ satisfies (13), which completes the existence proof.

Remark 4 The Galerkin method provides existence of a weak solution corresponding to *any* given $\boldsymbol{f} \in \mathcal{D}_0^{-1,2}(\Omega)$. Moreover, it is *constructive*, in the sense that the solution can be obtained as limit of a sequence of "approximate solutions" each of which can be, in principle, evaluated by solving the system of nonlinear algebraic equations (15). $\square$

B. The Function-Analytic Method. This approach, that goes back to the work of Leray [61, 63] and of Ladyzhenskaya [58], consists, first, in re-writing (13) as a *nonlinear equation* in the Hilbert space $\mathcal{D}_0^{1,2}(\Omega)$, and then using an appropriate topological degree theory to prove existence of weak solutions. Though more complicated than, and not constructive like the Galerkin method, this approach has the advantage of furnishing, as a byproduct, significant information on the solutions set.

By (14) and (18) we find

$$\mathcal{D}_0^{1,2}(\Omega) \subset L^4(\Omega) \,, \tag{21}$$

and so, from Lemma 1 and from the Riesz representation theorem, we have that, for each fixed $\boldsymbol{v} \in \mathcal{D}_0^{1,2}(\Omega)$, there exists $\boldsymbol{\mathcal{N}}(\boldsymbol{v}) \in \mathcal{D}_0^{1,2}(\Omega)$ such that

$$-(\boldsymbol{v} \cdot \nabla \boldsymbol{\varphi}, \boldsymbol{v}) = [\boldsymbol{\mathcal{N}}(\boldsymbol{v}), \boldsymbol{\varphi}] \,, \quad \text{for all } \boldsymbol{\varphi} \in \mathcal{D}_0^{1,2}(\Omega) \,. \tag{22}$$

Likewise, we have $\langle \boldsymbol{f}, \boldsymbol{\varphi} \rangle = [\boldsymbol{F}, \boldsymbol{\varphi}]$, for some $\boldsymbol{F} \in \mathcal{D}_0^{1,2}(\Omega)$ and for all $\boldsymbol{\varphi} \in \mathcal{D}_0^{1,2}(\Omega)$. Thus, equation (13) can be equivalently re-written as

$$\boldsymbol{N}(\nu, \boldsymbol{v}) = \boldsymbol{F} \,, \quad \text{in } \mathcal{D}_0^{1,2}(\Omega) \,, \tag{23}$$

where the map $\boldsymbol{N}$ is defined as follows

$$\boldsymbol{N} : \; (\nu, \boldsymbol{v}) \in (0, \infty) \times \mathcal{D}_0^{1,2}(\Omega) \mapsto \nu \, \boldsymbol{v} + \boldsymbol{\mathcal{N}}(\boldsymbol{v}) \in \mathcal{D}_0^{1,2}(\Omega) \,. \tag{24}$$

In order to show the above-mentioned property of solutions, we shall use some basic results related to Fredholm maps of index 0 [84]. This approach is preferable to that originally used by Leray –which applies to maps that are compact perturbations of homeomorphism– because it covers a larger class of problems, including flow in exterior domains [36].

Definition 2 *A map $M : X \mapsto Y$, X, Y Banach spaces, is* Fredholm*, if and only if:* (i) *M is of class C^1 (in the sense of Fréchet differentiability);* (ii) *its derivative, $D_x M(x)$, is , at each $x \in X$, a bounded, linear operator on X ;* (ii) *the integers $\alpha := \dim \{z \in X : [D_x M(x)](z) = 0\}$ and $\beta := \operatorname{codim} \{y \in Y : [D_x M(x)](z) = y \,,$ for some $z \in X\}$ are both finite.*

The integer $m := \alpha - \beta$ is independent of the particular $x \in U$ [100, §5.15], and is called the *index* of M.

Definition 3 *A map* $M : X \mapsto Y$, *is said* proper *if* $K_1 := \{x \in X : \ M(x) = y \,, y \in K\}$ *is compact in* X, *whenever* K *is compact in* Y.

By using the properties of proper Fredholm maps of index 0 [84], and of the associated *Caccioppoli-Smale degree* [10, 84], one can prove the following result [37, Theorem I.2.2]

Lemma 3 *Let* M *be a proper Fredholm map of index 0 and of class* C^2, *satisfying the following.*

(i) *There exists* $\overline{y} \in Y$ *such that the equation* $M(x) = \overline{y}$ *has one and only one solution* $\overline{x}$ *;*

(ii) $[D_x M(\overline{x})](z) = 0 \Rightarrow \ z = 0$.

Then:

(a) M *is surjective ;*

(b) *There exists an open, dense set* $Y_0 \subset Y$ *such that for any* $y \in Y_0$ *the solution set* $\{x \in X : \ M(x) = y\}$ *is finite and constituted by an odd number,* $\kappa = \kappa(y)$, *of points ;*

(c) *The integer* κ *is constant on every connected component of* Y_0.

The next result provides the required functional properties of the map $\boldsymbol{N}$.

Proposition 1 *The map* $\boldsymbol{N}$ *defined in* (24) *is of class* C^∞. *Moreover, for any* $\nu > 0$, $\boldsymbol{N}(\nu, \cdot) : \mathcal{D}_0^{1,2}(\Omega) \mapsto \mathcal{D}_0^{1,2}(\Omega)$ *is proper and Fredholm of index 0.*

Proof. It is a simple exercise to prove that $\boldsymbol{N}$ is of class C^∞ [37, Example I.1.6]. By the compactness of the embedding $W_0^{1,2}(\Omega) \subset L^4(\Omega)$ [31, Theorem II.4.2], from Lemma 1 and from the definition of the map $\boldsymbol{\mathcal{N}}$ (see (22)), one can show that $\boldsymbol{\mathcal{N}}$ is compact, that is, it maps bounded sequences of $\mathcal{D}_0^{1,2}(\Omega)$ into relatively compact sequences. Therefore, $\boldsymbol{N}(\nu, \cdot)$ is a compact perturbation of a multiple of the identity operator. Moreover, by (22) and by Lemma 1 we show that $[\boldsymbol{\mathcal{N}}(\boldsymbol{v}), \boldsymbol{v}] = 0$, which implies,

$$[\boldsymbol{N}(\nu, \boldsymbol{v}), \boldsymbol{v}] = \nu |\boldsymbol{v}|_{1,2}^2 \,. \tag{25}$$

Using the Schwartz inequality on the left-hand side of (25), we infer that $|\boldsymbol{N}(\nu, \boldsymbol{v})|_{1,2} \geq \nu |\boldsymbol{v}|_{1,2}$ and so, for each fixed $\nu > 0$ we have $|\boldsymbol{N}(\nu, \boldsymbol{v})|_{1,2} \to \infty$ as $|\boldsymbol{v}|_{1,2} \to \infty$, that is, $\boldsymbol{N}(\nu, \cdot)$ is (weakly) *coercive*. Consequently, $\boldsymbol{N}(\nu, \cdot)$ is proper [7, Theorem 2.7.2]. Finally, since the derivative of a compact map is compact [7, Theorem 2.4.6], the derivative map $D_{\boldsymbol{v}} \boldsymbol{N}(\nu, \boldsymbol{v}) \equiv \nu \boldsymbol{I} + D_{\boldsymbol{v}} \boldsymbol{\mathcal{N}}(\boldsymbol{v})$, $\boldsymbol{I}$ identity operator, is, for each fixed $\nu > 0$, a compact perturbation of a multiple of the identity, which implies that $\boldsymbol{N}(\nu, \cdot)$ is Fredholm of index 0 [100, Theorem 5.5.C]. ■

It is easy now to show that, for any fixed $\nu > 0$, $\boldsymbol{N}(\nu, \cdot)$ satisfies all the assumptions of Lemma 3. Actually, in view of Proposition 1, we have only to prove the validity of (i) and (ii). We take $\overline{y} \equiv \boldsymbol{0}$, and so, from (25) we obtain that the equation $\boldsymbol{N}(\nu, \boldsymbol{v}) = \boldsymbol{0}$ has only the solution $\overline{x} \equiv \boldsymbol{v} = \boldsymbol{0}$, so that (i) is satisfied. Furthermore, from (22) it follows that the equation $D_{\boldsymbol{v}} \boldsymbol{N}(\nu, 0)(\boldsymbol{w}) = 0$ is equivalent to $\nu \, \boldsymbol{w} = \boldsymbol{0}$, and so also condition (ii) is satisfied for $\nu > 0$. Thus, we have proved the following result (see also [26]).

Theorem 1 *For any* $\nu > 0$ *and* $\boldsymbol{F} \in \mathcal{D}_0^{1,2}(\Omega)$, *equation* (23) *has at least one solution* $\boldsymbol{v} \in \mathcal{D}_0^{1,2}(\Omega)$. *Moreover, for each fixed* $\nu > 0$, *there exists open and dense* $\mathcal{O} = \mathcal{O}(\nu) \subset \mathcal{D}_0^{1,2}(\Omega)$ *with the following properties:* (i) *For any* $\boldsymbol{F} \in \mathcal{O}$ *the number of solutions to* (23), $\mathfrak{n} = \mathfrak{n}(\boldsymbol{F}, \nu)$ *is finite and odd ;* (ii) *the integer* $\mathfrak{n}$ *is constant on each connected component of* $\mathcal{O}$.

Next, for a given $\boldsymbol{F} \in \mathcal{D}_0^{1,2}(\Omega)$, consider the solution manifold

$$\mathcal{S}(\boldsymbol{F}) = \Big\{(\nu, \boldsymbol{v}) \in (0, \infty) \times \mathcal{D}_0^{1,2}(\Omega) : \; N(\nu, \boldsymbol{v}) = \boldsymbol{F}\Big\} \,.$$

By arguments similar to those used in the proof of Theorem 1, one can show the following "generic" characterization of $\mathcal{S}(\boldsymbol{F})$ [37, Example I.2.4]; see also [93, Chapter 10.3] .

Theorem 2 *There exists dense $\mathcal{P} \subset \mathcal{D}_0^{1,2}(\Omega)$ such that, for every $\boldsymbol{F} \in \mathcal{P}$, the set $\mathcal{S}(\boldsymbol{F})$ is a C^∞ 1-dimensional manifold. Moreover, there exists an open and dense subset of $(0, \infty)$, $\Lambda = \Lambda(\boldsymbol{F})$, such that for each $\nu \in \Lambda$, equation* (23) *has a finite number $\mathfrak{m} = \mathfrak{m}(\boldsymbol{F}, \nu) > 0$ of solutions. Finally, the integer $\mathfrak{m}$ is constant on every open interval contained in Λ.*

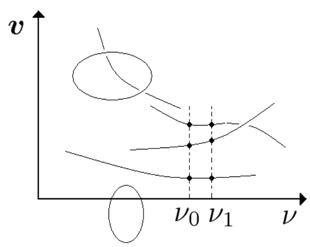


Figure 3. Sketch of the manifold $\mathcal{S}(\boldsymbol{F})$.

In other words, Theorem 2 expresses the property that, for every $\boldsymbol{F} \in \mathcal{P}$, the set $\mathcal{S}(\boldsymbol{F})$ is the union of smooth and *non-intersecting* curves. Furthermore, "almost all" lines $\nu = \nu_0 =$const. intersect these curves at a *finite* number of points, $\mathfrak{m}(\nu_0, \boldsymbol{F})$, each of which is a solution to (23) corresponding to ν_0 and to $\boldsymbol{F}$. Finally, $\mathfrak{m}(\nu_0, \boldsymbol{F}) = \mathfrak{m}(\nu_1, \boldsymbol{F})$ whenever ν_0 and ν_1 belong to an open interval of a suitable dense set of $(0, \infty)$. A sketch of the manifold $\mathcal{S}(\boldsymbol{F})$ is provided in Figure 3.

IV.1.2 Existence Results. Non-Homogeneous Boundary Conditions

Existence of solutions to (9)–(10) when $\boldsymbol{v}_* \not\equiv \boldsymbol{0}$ leads to one of the most challenging open questions in the mathematical theory of the Navier-Stokes equations, in the case when the boundary $\partial\Omega$ is constituted by more than one connected component. In order to explain the problem, let S_i, $i = i, \cdots, K$, $K \geq 1$, denote these components. Conservation of mass $(9)_2$ along with Gauss theorem imply the following *compatibility condition* on the data $\boldsymbol{v}_*$

$$\sum_{i=1}^{K} \int_{S_i} \boldsymbol{v}_* \cdot \boldsymbol{n}_i \equiv \sum_{i=1}^{K} \Phi_i = 0 \,, \tag{26}$$

where $\boldsymbol{n}_i$ denotes the outward unit normal to S_i. From the physical point of view, the quantity $\rho\,\Phi_i$ represents the mass flow-rate of the liquid through the surface S_i. Now, assuming $\boldsymbol{v}_*$ and Ω sufficiently smooth (for example, $\boldsymbol{v}_* \in W^{1/2,2}(\partial\Omega)$ and Ω locally Lipschitzian [32, §VIII.4]), we look for a weak solution to (9)–(10) in the form $\boldsymbol{v} = \boldsymbol{u} + \boldsymbol{V}$, where $\boldsymbol{V} \in W^{1,2}(\Omega)$ is an extension of $\boldsymbol{v}_*$ with $\operatorname{div} \boldsymbol{V} = 0$ in Ω. Thus, if we use, for example, the Galerkin method of Subsection IV.1.1(A), from (15) we obtain that the "approximate solution" $\boldsymbol{u}_N = \sum_{i=1}^{N} c_{iN} \boldsymbol{\psi}_i$ must satisfy the following equations ($k = 1, \cdots, N$)

$$\nu[\boldsymbol{u}_N, \boldsymbol{\psi}_k] - (\boldsymbol{u}_N \cdot \nabla\boldsymbol{\psi}_k, \boldsymbol{u}_N) - (\boldsymbol{u}_N \cdot \nabla\boldsymbol{\psi}_k, \boldsymbol{V}) - (\boldsymbol{V} \cdot \nabla\boldsymbol{\psi}_k, \boldsymbol{u}_N) - (\boldsymbol{V} \cdot \nabla\boldsymbol{\psi}_k, \boldsymbol{V}) + \nu[\boldsymbol{V}, \boldsymbol{\psi}_k] = \langle \boldsymbol{f}, \boldsymbol{\psi}_k \rangle. \tag{27}$$

Therefore, existence of a weak solution will be secured provided we show that the sequence $\{\boldsymbol{v}_N := \boldsymbol{u}_N + \boldsymbol{V}\}$ satisfies the bound (16). In turn, this latter is equivalent to showing

$$|\boldsymbol{u}_N|_{1,2} \leq M_1 \,, \tag{28}$$

where M_1 is a finite, positive quantity *independent of* N. Multiplying through both sides of (27) by c_{kN}, summing over k from 1 to N, and observing that, by Lemma 1, $(\boldsymbol{u}_N \cdot \nabla\boldsymbol{u}_N, \boldsymbol{u}_N) = (\boldsymbol{V} \cdot \nabla\boldsymbol{u}_N, \boldsymbol{u}_N) = 0$, we find

$$\nu|\boldsymbol{u}_N|_{1,2}^2 = (\boldsymbol{u}_N \cdot \nabla\boldsymbol{u}_N, \boldsymbol{V}) + (\boldsymbol{V} \cdot \nabla\boldsymbol{u}_N, \boldsymbol{V}) - [\boldsymbol{V}, \boldsymbol{u}_N] + \langle \boldsymbol{f}, \boldsymbol{u}_N \rangle \,.$$

By using (14), the Hölder inequality and the Cauchy-Schwartz inequality

$$a\,b \leq \varepsilon\,a^2 + (1/4\varepsilon)\,b^2, \quad a,b,\varepsilon > 0 \tag{29}$$

on the last three terms on the right-hand side of this latter equation, we easily find, for a suitable choice of ε,

$$\frac{\nu}{2}|\boldsymbol{u}_N|_{1,2}^2 \leq (\boldsymbol{u}_N \cdot \nabla \boldsymbol{u}_N, \boldsymbol{V}) + C\,, \tag{30}$$

where $C = C(\boldsymbol{V}, \boldsymbol{f}, \Omega) > 0$. From (30) it follows that, in order to obtain the bound (28) *without restrictions on the magnitude of* ν, it suffices that $\boldsymbol{V}$ meets the following requirement:

$$\text{Given } \varepsilon > 0, \text{ there is } \boldsymbol{V} = \boldsymbol{V}(\varepsilon, \boldsymbol{x}) \text{ such that } (\boldsymbol{\varphi} \cdot \nabla \boldsymbol{\varphi}, \boldsymbol{V}) \leq \varepsilon |\boldsymbol{\varphi}|_{1,2}^2 \text{ for all } \boldsymbol{\varphi} \in \mathcal{D}(\Omega)\,. \tag{31}$$

As indicated by Leray [61, pp. 28-30] and clarified by Hopf [52], if Ω is smooth enough and if $K = 1$, that is, if $\partial\Omega$ is constituted by only one connected component, S_1, it is possible to construct a family of extensions satisfying (31), with $\varepsilon = \nu/2$, say. Notice that, in such a case, condition (26) reduces to the single condition $\Phi_1 = 0$. If $K > 1$, the same construction is still possible but *with the limitation that* $\Phi_i = 0$*, for all* $i = 1, \cdots, K$. It should be emphasized that this condition is quite restrictive from the physical point of view, in that it does not allow for the presence, in the region of flow, of isolated "sources" and "sinks" of liquid. Nevertheless, one may wonder if, by using a different construction, it is still possible to satisfy (31). Unfortunately, as shown by Takeshita [90] by means of explicit examples, *in general*, *the existence of extensions satisfying* (31) *implies* $\Phi_i = 0$*, for all* $i = 1, \cdots, K$*;* see also [32, §VIII.4].

We wish to emphasize that the same type of conclusion holds if, instead of the Galerkin method, we use the function-analytic approach; see [61], [32, Notes to Chapter VIII].

Finally, it should be remarked that, in the special case of *two-dimensional* domains possessing suitable symmetry and of symmetric boundary data, Amick [1] and Fujita [29] have shown existence of corresponding symmetric solutions under the general assumption (26). However, we have the following.

Open Question. Let Ω be a smooth domain in $\mathbb{R}^n$, $n = 2, 3$, with $\partial\Omega$ constituted by $K > 1$ connected components, S_i, $i = 1, \cdots, K$, and let $\boldsymbol{v}_*$ be any smooth field satisfying (26). *It is not known if the corresponding problem* (9)–(10) *has at least one solution.*

IV.1.3 Uniqueness and Steady Bifurcation

It is a well-established experimental fact that a steady flow of a viscous incompressible liquid is "observed", namely, it is unique and stable, if the magnitude of the driving force, usually measured through a dimensionless number $\lambda \in (0, \infty)$, say, is below a certain threshold, λ_c. However, if $\lambda > \lambda_c$, this flow becomes unstable and another, different flow is instead observed. This latter may be steady or unsteady (typically, time-periodic). In the former case, we say that a *steady bifurcation* phenomenon has occurred. From the physical point of view, bifurcation happens because the liquid finds a more "convenient" motion (than the original one) to dissipate the increasing energy pumped in by the driving force. From the mathematical point of view, bifurcation occurs when, roughly speaking, two solution-curves (parameterized with the appropriate dimensionless number) intersect. (As we know from Theorem 2, the intersection of these curves is not "generic".) It can happen that one curve exists for all values of λ, while the other only exists for $\lambda > \lambda_c$ (*supercritical bifurcation*). The point of intersection of the two curves is called *bifurcation point*. Thus, in any neighborhood of the bifurcation point, we must have (at least) two distinct solutions and so a necessary condition for bifurcation is the occurrence of non-uniqueness. This section is dedicated to the above issues.

Uniqueness Results. It is simple to show that, if $|\boldsymbol{v}|_{1,2}$ is not "too large", then $\boldsymbol{v}$ is the only weak solution to (9)–(10) corresponding to the given data.

Theorem 3 (Uniqueness) *Let Ω be locally Lipschitzian and let*

$$|\boldsymbol{v}|_{1,2} < \nu/\kappa\,, \tag{32}$$

with $\kappa = \kappa(\Omega) > 0$. Then, there is only one weak solutions to (9)–(10).

Proof. Let $\boldsymbol{v}$, $\boldsymbol{v}_1 = \boldsymbol{v} + \boldsymbol{u}$ be two different solutions. From (13) we deduce

$$\nu[\boldsymbol{u}, \boldsymbol{\varphi}] = (\boldsymbol{u} \cdot \nabla\boldsymbol{\varphi}, \boldsymbol{v}) + (\boldsymbol{v}_1 \cdot \nabla\boldsymbol{\varphi}, \boldsymbol{u})\,, \quad \text{for all } \boldsymbol{\varphi} \in \mathcal{D}(\Omega). \tag{33}$$

If Ω is locally Lipschitzian, then $\boldsymbol{u} \in \mathcal{D}_0^{1,2}(\Omega)$ [31, Theorem II.3.2 and §II.3.5], and since $\mathcal{D}(\Omega)$ is dense in $\mathcal{D}_0^{1,2}(\Omega)$, with the help of Lemma 1 we can replace $\boldsymbol{\varphi}$ with $\boldsymbol{u}$ in (33) to get

$$\nu|\boldsymbol{u}|_{1,2}^2 = (\boldsymbol{u} \cdot \nabla\boldsymbol{u}, \boldsymbol{v})\,.$$

We now use (14) and Lemma 1 on the right-hand side of this equation to find

$$(\nu - \kappa\,|\boldsymbol{v}|_{1,2})|\boldsymbol{u}|_{1,2}^2 \le 0\,,$$

with $\kappa = \kappa(\Omega) > 0$, which proves $\boldsymbol{u} = 0$ in $\mathcal{D}_0^{1,2}(\Omega)$, namely, uniqueness, if $\boldsymbol{v}$ satisfies (33). ■

Remark 5 As we have seen previously, if $\boldsymbol{v}_* \equiv \boldsymbol{0}$, weak solutions satisfy $\nu|\boldsymbol{v}|_{1,2} \le |\boldsymbol{f}|_{-1,2}$. Thus, (32) holds if $|\boldsymbol{f}|_{-1,2} \le \nu^2/\kappa$. If $\boldsymbol{v}_* \neq \boldsymbol{0}$, then one can show that (32) holds if $|\boldsymbol{f}|_{-1,2} + (1+\nu)\|\boldsymbol{v}_*\|_{1/2,2,\partial\Omega} + \|\boldsymbol{v}_*\|_{1/2,2,\partial\Omega}^2 \le \nu^2/\kappa_1$, where κ_1 has the same properties as κ; see [32, Theorem VIII.4.2]. Notice that these conditions are satisfied if a suitable non-dimensional parameter $\lambda \sim \big(|\boldsymbol{f}|_{-1,2} + \|\boldsymbol{v}_*\|_{1/2,2,\partial\Omega}\big)$ is "sufficiently small". □

Remarkably enough, one can give *explicit examples of non-uniqueness*, if condition (32) is violated. More specifically, we have the following result [32, Theorem VIII.2.2].

Theorem 4 (Non-Uniqueness) *Let Ω be a bounded smooth body of revolution around an axis* r*, that does not include points of* r*. For example, Ω is a torus of arbitrary bounded smooth section. Then there are smooth fields $\boldsymbol{f}$ and $\boldsymbol{v}_*$ and a value of $\nu > 0$ such that problem* (9)–(10) *corresponding to these data admits at least two distinct and smooth solutions.*

Some Bifurcation Results. By using the functional setting introduced in Section IV.1.1(B), it is not difficult to show that steady bifurcation can be reduced to the study of a suitable nonlinear eigenvalue problem in the space $\mathcal{D}_0^{1,2}(\Omega)$. To this end, we recall certain basic definitions.

Let U be an open interval of $\mathbb{R}$ and let

$$M : (x, \mu) \in X \times U \mapsto Y\,. \tag{34}$$

Definition 4 *The point (x_0, μ_0) is called a* bifurcation point *of the equation*

$$M(x, \mu) = 0 \tag{35}$$

if and only if (a) *$M(x_0, \mu_0) = 0$, and* (b) *there are (at least) two sequences of solutions, $\{(x_m, \mu_m)\}$ and $\{(x_m^*, \mu_m)\}$, to* (35)*, with $x_m \neq x_m^*$, for all $m \in \mathbb{N}$, such that $(x_m, \mu_m) \to (x_0, \mu_0)$ and $(x_m^*, \mu_m) \to (x_0, \mu_0)$ as $m \to \infty$.*

If M is suitably smooth around (x_0, μ_0), a necessary condition for (x_0, μ_0) to be a bifurcation point is that $D_x M(x_0, \mu_0)$ is not a bijection. In fact, we have the following result which is an immediate corollary of the implicit function theorem (see, *e.g.*, [37, Lemma III.1.1]) .

Lemma 4 *Suppose that $D_x M$ exists in a neighborhood of (x_0, μ_0), and that both M and $D_x M$ are continuous at (x_0, μ_0). Then, if (x_0, μ_0) is a bifurcation point of* (34)*, $D_x M(x_0, \mu_0)$ is not a bijection. If, in particular, $D_x M(x_0, \mu_0)$ is a Fredholm operator of index 0 (see* Definition 2*), then the equation $D_x M(x_0, \mu_0)x = 0$ has at least one nonzero solution.*

Let $\boldsymbol{v}_0 = \boldsymbol{v}_0(\nu)$, $\nu \in (0, \infty)$ be a family of weak solution to (9)–(10) corresponding to given data $\boldsymbol{f}$ and $\boldsymbol{v}_*$. We assume that $\boldsymbol{f}$ and $\boldsymbol{v}_*$ are fixed. Denoting by $\boldsymbol{v}$ any other solution corresponding to the same data, by an argument completely similar to that leading to (23), we find that $\boldsymbol{u} := \boldsymbol{v} - \boldsymbol{v}_0$, satisfies the following equation in $\mathcal{D}_0^{1,2}(\Omega)$

$$\nu \boldsymbol{u} + \boldsymbol{B}(\boldsymbol{v}_0)(\boldsymbol{u}) + \boldsymbol{\mathcal{N}}(\boldsymbol{u}) = \boldsymbol{0}\,, \tag{36}$$

with $\boldsymbol{B}(\boldsymbol{v}_0) := D_{\boldsymbol{v}}\boldsymbol{\mathcal{N}}(\boldsymbol{v}_0)$ and $\boldsymbol{\mathcal{N}}$ defined in (22). Obviously, $(\boldsymbol{v}_0(\nu_0), \nu_0)$ is a bifurcation point for the original equation if and only if $(\boldsymbol{0}, \nu_0)$ is such for (36). Thus, in view of Lemma 4, a necessary condition for $(\boldsymbol{0}, \nu_0)$ to be a bifurcation point for (36) is that the equation

$$\nu_0 \boldsymbol{v} + \boldsymbol{B}(\boldsymbol{v}_0(\nu_0))(\boldsymbol{v}) = \boldsymbol{0} \tag{37}$$

has at least one nonzero solution $\boldsymbol{v} \in \mathcal{D}_0^{1,2}(\Omega)$.

In several significant situations it happens that, after a suitable non-dimensionalization of (36), the family of solutions $\boldsymbol{v}_0(\nu)$, $\nu \in (0, \infty)$ is independent of the parameter ν which, this time, has to be interpreted as the inverse of an appropriate dimensionless number (*Reynolds number*), like, for instance, in the Taylor-Couette problem; see the following subsection. Now, from Proposition 1, we know that $\boldsymbol{B}(\boldsymbol{u})$ is compact at each $\boldsymbol{u} \in \mathcal{D}_0^{1,2}(\Omega)$, so that $\nu \boldsymbol{I} + \boldsymbol{B}(\boldsymbol{u})$ is Fredholm of index 0, at each $\boldsymbol{u} \in \mathcal{D}_0^{1,2}(\Omega)$, for all $\nu > 0$ [100, Theorem 5.5.C]. Therefore, whenever $\boldsymbol{v}_0$ does not depend on ν, in a neighborhood of ν_0, from Lemma 4 we find that a *necessary condition for $(\boldsymbol{v}_0, \nu_0)$ to be a bifurcation point for* (35) *is that ν_0 is an eigenvalue of the (compact) linear operator $\boldsymbol{B}(\boldsymbol{v}_0)$.*

The stated condition becomes also *sufficient*, provided we make the additional assumptions that ν_0 is a *simple eigenvalue* of $\boldsymbol{B}(\boldsymbol{v}_0)$. This is a consequence of the following theorem (see, *e.g.*, [37, Lemma III.1.2]).

Theorem 5 *Let $X \subset Y$ and let the operator M in* (34) *be of the form $M = \mu\, I + T$, where I is the identity in X and T is of class C^1. Furthermore, set $L := D_x T(0)$. Suppose that $\mu_0\, I + L$ is Fredholm of index 0 (*Definition 2*), for some $\mu_0 \in U$, and that $-\mu_0$ is a simple eigenvalue for L, namely, the equation $\mu_0\, x + L(x) = 0$ has one and only one (nonzero) independent solution, x_1, while the equation $\mu_0 x + L(x) = x_1$ has no solutions. Then, $(0, \mu_0)$ is a bifurcation point for the equation $M(x, \mu) = 0$.*

Bifurcation of Taylor-Couette Flow. A notable application of Theorem 5 is the *bifurcation of Taylor-Couette flow*. In this case the liquid fills the space between two coaxial, infinite cylinders, $\mathcal{C}_1$ and $\mathcal{C}_2$, of radii R_1 and $R_2 > R_1$, respectively. $\mathcal{C}_1$ rotates around the common axis, a, with constant angular velocity $\boldsymbol{\omega}$, while $\mathcal{C}_2$ is at rest. Denote by (r, θ, z) a system of cylindrical coordinates with z along a and oriented as $\boldsymbol{\omega}$ and let $(\boldsymbol{e}_r, \boldsymbol{e}_\theta, \boldsymbol{e}_z)$ the associated canonical base. The components of a vector $\boldsymbol{w}$ in such a base are denoted by w_r, w_θ and w_z, respectively. If we introduce the non-dimensional quantities

$$\boldsymbol{u} = \boldsymbol{v}/(\omega\, r_1)\,, \quad \overline{\boldsymbol{x}} = \boldsymbol{x}/r_1\,, \quad \mathsf{p} = p/(\rho\omega^2 r_1^2)\,, \quad R = R_2/R_1\,,$$

with $\omega = |\boldsymbol{\omega}|$, we see at once that the following velocity and pressure fields

$$\boldsymbol{u}_0 = (1 - R^2)^{-1}(r - R^2/r)\boldsymbol{e}_\theta\,, \quad \mathsf{p}_0 = |\boldsymbol{u}_0|^2 \ln r + \text{const}\,, \tag{38}$$

solve (9), with $\boldsymbol{f} = \boldsymbol{0}$, for all values of the Reynolds number $\lambda := \rho\,\omega\, r_1^2/\mu$. Moreover, $\boldsymbol{u}_0$ satisfies the boundary conditions

$$\boldsymbol{u}_0(r) = \boldsymbol{0} \text{ at } r = 1\,, \quad \boldsymbol{u}_0(r) = \boldsymbol{e}_\theta \text{ at } r = R\,. \tag{39}$$

Experiments show that, if λ exceeds a critical value, λ_c, a flow with entirely different features than the flow (38) is observed. In fact, this new flow is dominated by large toroidal vortices, stacked one on top of the other, called *Taylor vortices*. They are periodic in the z-direction and axisymmetric (independent of θ); see Figure 4. Therefore, we look for bifurcating solutions of the form $\boldsymbol{u}_0(r)+\boldsymbol{w}(r,z)$, $\mathsf{p}_0(r)+\mathsf{p}(r,z)$, satisfying (39). and where $\boldsymbol{w}$ and p are periodic in the z-direction with period $\mathfrak{P}$, and $\boldsymbol{w}$ satisfies the following parity conditions:

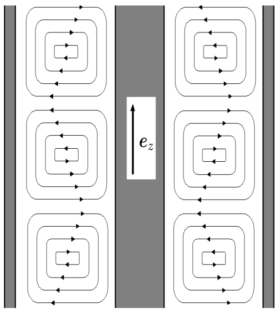


Figure 4. Sketch of the streamlines of Taylor vortices, at the generic section θ=const.

$$w_r(r,z)=w_r(r,-z)\,,\ w_\theta(r,z)=w_\theta(r,-z)\,,\ \ w_z(r,z)=-w_z(r,-z). \tag{40}$$

Moreover, the relevant region of flow becomes the *periodicity cell* $\Omega:=(1,R)\times(0,\mathfrak{P})$. If we now introduce the *stream function* ψ

$$\frac{\partial\psi}{\partial z}=w_r\,,\quad \frac{\partial(r\psi)}{\partial r}=-r\,w_z\,;\quad \psi(r,z)=-\psi(r,-z)\,, \tag{41}$$

and the vector $\boldsymbol{u}:=(\psi,w_\theta)$, it can be shown [99, §72.7] that $\boldsymbol{u}$ satisfies an equation of the type (36), namely,

$$\boldsymbol{u}+\lambda\left(\overline{\boldsymbol{B}}(\boldsymbol{v}_0)(\boldsymbol{u})+\overline{\boldsymbol{\mathcal{N}}}(\boldsymbol{u})\right)=\boldsymbol{0}\ \text{ in } \mathcal{H}(\Omega) \tag{42}$$

where the operators $\overline{\boldsymbol{B}}(\boldsymbol{v}_0)$ and $\overline{\boldsymbol{\mathcal{N}}}$ obey the same functional properties as $\boldsymbol{B}(\boldsymbol{v}_0)$ and $\boldsymbol{\mathcal{N}}$, and where $\mathcal{H}(\Omega)$ is the Banach space of functions $\boldsymbol{u}:=(\psi,w_\theta)\in C^{4,\alpha}(\Omega)\times C^{2,\alpha}(\Omega)$, $\alpha\in(0,1)$, such that: (i) $\boldsymbol{u}(r,0)=\boldsymbol{u}(r,\mathfrak{P})$ for all $r\in(1,R)$, (ii) $\boldsymbol{u}$ satisfies the parity conditions in (40), (41), and (iii) $\psi=\partial\psi/\partial r=w_\theta=0$ at $r=1,R$. Thus, in view of Theorem 5 and of the properties of the operators involved in (42), we will obtain that $(\boldsymbol{0},\lambda_0)$ is a bifurcation point for (41) if the following two conditions are met

$$\begin{aligned}&\text{(a)}\quad \boldsymbol{u}+\lambda_0\overline{\boldsymbol{B}}(\boldsymbol{v}_0)(\boldsymbol{u})=\boldsymbol{0}\ \text{has one and only one independent solution, } \boldsymbol{u}_1\in\mathcal{H}(\Omega)\,;\\ &\text{(b)}\quad \text{the equation } \boldsymbol{u}+\lambda_0\overline{\boldsymbol{B}}(\boldsymbol{v}_0)(\boldsymbol{u})=\boldsymbol{u}_1 \text{ has no solution in } \mathcal{H}(\Omega)\,.\end{aligned} \tag{43}$$

It can be shown [99, Lemma 72.14] that there exists a period $\mathfrak{P}$ for which both conditions in (43) are satisfied. In addition, for all $\lambda\in(0,\lambda_0)$ the equation $\boldsymbol{u}+\lambda\overline{\boldsymbol{B}}(\boldsymbol{v}_0)(\boldsymbol{u})=\boldsymbol{0}$ has only the trivial solution in $\mathcal{H}(\Omega)$, which, by Lemma 4, implies that no bifurcation occurs for $\lambda\in(0,\lambda_0)$, which, in turn, means that the bifurcation is *supercritical*.

IV.2 Flow in Exterior Domains

One of the most significant questions in fluid mechanics is to determine the properties of the steady flow of a liquid past a body $\mathcal{B}$ of simple symmetric shape (such as sphere or cylinder), over the entire range of the Reynolds number $\lambda:=\rho UL/\mu\in(0,\infty)$; see, *e.g.*, [4, §4.9]. Here L is a length scale representing the linear dimension of $\mathcal{B}$, while $\boldsymbol{v}_\infty=-U\boldsymbol{e}_1$, $U=\text{const}>0$, is the uniform velocity field of the liquid at large distances from $\mathcal{B}$. In the mathematical formulation of this problem, one assumes that the liquid fills the whole space, Ω, outside the closure of the domain $\mathcal{B}\subset\mathbb{R}^n$, $n=2,3$. Thus, by scaling $\boldsymbol{v}$ by U and $\boldsymbol{x}$ by L, from (9)–(10) we obtain that the steady flow past a body consists in solving the following non-dimensional *exterior boundary-value problem*

$$\left.\begin{aligned} &-\Delta \boldsymbol{w} - \lambda \frac{\partial \boldsymbol{w}}{\partial x_1} + \lambda \boldsymbol{w}\cdot\nabla\boldsymbol{w} = -\nabla p + \boldsymbol{f} \\ &\operatorname{div}\boldsymbol{w} = 0 \end{aligned}\right\} \text{ in } \Omega$$
$$\boldsymbol{w}(x) = \boldsymbol{e}_1\,,\ x\in\partial\Omega\,, \quad \lim_{|x|\to\infty}\boldsymbol{w}(x) = \boldsymbol{0}\,, \tag{44}$$

where $\boldsymbol{w} := \boldsymbol{v} + \boldsymbol{e}_1$ and where we assume, for simplicity, that $\boldsymbol{v}_* \equiv \boldsymbol{0}$. (However, all the stated results continues to hold, more generally, if $\boldsymbol{v}_*$ belongs to a suitable trace space.)

Whereas in the three-dimensional case (*e.g., flow past a sphere*) the investigation of (44) is, to an extent, complete, in the two-dimensional case (*e.g., flow past a circular cylinder*) there are still fundamental unresolved issues. We shall, therefore, treat the two cases separately.

We need some preliminary considerations.

Extension of the Boundary Data. If Ω is smooth enough (locally Lipschitzian, for example), since $\int_{\partial\Omega}\boldsymbol{e}_1\cdot\boldsymbol{n} = 0$, by what we observed in Section VI.1.2, for any $\lambda > 0$ we find $\boldsymbol{V} = \boldsymbol{V}(\lambda,\boldsymbol{x})$, with $\operatorname{div}\boldsymbol{V} = 0$ in Ω, and satisfying (31) with $\varepsilon = 1/(2\lambda)$, say. Actually, as shown in [36, Proposition 3.1], the extension $\boldsymbol{V}$ can be chosen, in particular, to be of class $C^\infty((0,\infty)\times\Omega)$ and such that the support of $V(\lambda,\cdot)$ is contained in a bounded set, independent of λ. The proof given in [36, loc.cit.] is for $\Omega\subset\mathbb{R}^3$, but it can be easily extended to the case $\Omega\subset\mathbb{R}^2$.

Variational Formulation and Weak Solutions. Setting $\boldsymbol{u} = \boldsymbol{w} - \boldsymbol{V}$ in $(44)_1$, dot-multiplying through both sides of this equation by $\boldsymbol{\varphi}\in\mathcal{D}(\Omega)$, integrating by parts and taking into account (12), we find

$$\begin{aligned}&[\boldsymbol{u},\boldsymbol{\varphi}] - \lambda\left(\frac{\partial\boldsymbol{u}}{\partial x_1},\boldsymbol{\varphi}\right) - \lambda\,(\boldsymbol{u}\cdot\nabla\boldsymbol{\varphi},\boldsymbol{u}) - \lambda\{(\boldsymbol{u}\cdot\nabla\boldsymbol{\varphi},\boldsymbol{V}) + (\boldsymbol{V}\cdot\nabla\boldsymbol{\varphi},\boldsymbol{u})\} + (\boldsymbol{H},\boldsymbol{\varphi}) \\ &\qquad = \langle\boldsymbol{f},\boldsymbol{\varphi}\rangle\,, \quad \text{for all } \boldsymbol{\varphi}\in\mathcal{D}(\Omega)\,.\end{aligned} \tag{45}$$

where

$$\boldsymbol{H} = \boldsymbol{H}(\boldsymbol{V}) := \Delta\boldsymbol{V} + \lambda\frac{\partial\boldsymbol{V}}{\partial x_1} - \lambda\boldsymbol{V}\cdot\nabla\boldsymbol{V}\,.$$

Definition 5 *A function $\boldsymbol{w}\in D^{1,2}(\Omega)$ is called a* weak solution *to* (44) *if and only if $\boldsymbol{w} = \boldsymbol{u} + \boldsymbol{V}$ where $\boldsymbol{u}\in\mathcal{D}_0^{1,2}(\Omega)$ and $\boldsymbol{u}$ satisfies* (45).

Regularity of Weak Solutions. (A) *Local Regularity.* The proof of differentiability properties of weak solutions can be carried out in a way similar to the case of a bounded domain. In particular, if $\boldsymbol{f}$ and Ω are of class C^∞, one can show that $\boldsymbol{w}\in C^\infty(\overline{\Omega})$ and there exists a scalar field $p\in C^\infty(\overline{\Omega})$ such that $(44)_{1,2,3}$ holds in the ordinary sense. For this and other intermediate regularity results, we refer to [32, Theorem IX.1.1].

(B) *Regularity at Infinity*. The study of the validity of $(44)_4$ and, more particularly, of the *asymptotic structure* of weak solutions, needs a much more involved treatment, and the results depend on whether the flow is three- or two-dimensional. In the *three-dimensional case*, if $\boldsymbol{f}$ satisfies suitable summability assumptions outside a ball of large radius, by using a *local* representation of a weak solution, (namely, at all points of a ball of radius 1 and centered at $\boldsymbol{x}\in\Omega$, $\operatorname{dist}(\boldsymbol{x},\partial\Omega) > 1$) one can prove that $\boldsymbol{w}$ and all its derivatives tend to zero at infinity uniformly pointwise and that a similar property holds for p [32, Theorem IX.6.1]. The starting point of this analysis is the crucial fact that

$$\text{if } \Omega\subset\mathbb{R}^3,\ \mathcal{D}_0^{1,2}(\Omega) \text{ is continuously embedded in } L^6(\Omega)\,, \tag{46}$$

which implies that $\boldsymbol{w}$ tends to zero at infinity in a suitable sense. However, the existence of the "wake" behind the body along with the *sharp* order of decay of $\boldsymbol{w}$ and p requires a more complicated analysis based on the *global* representation of a weak solution (namely, at all points outside a ball of sufficiently large radius) by means of the *Oseen fundamental tensor*, along with maximal regularity estimates for the solution of the linearized *Oseen problem*, this latter being obtained by suppressing the nonlinear term $\boldsymbol{w}\cdot\nabla\boldsymbol{w}$ in (44) [32, §§IX.6, IX.7 and IX.8]. In particular, one can prove the following result concerning the behavior of $\boldsymbol{w}$; see [32, Theorems IX.7.1, IX.8.1 and Remark IX.8.1].

Theorem 6 *Let Ω be a three-dimensional exterior domain of class C^2, and let $\boldsymbol{w}$ be a weak solution to* (44) *corresponding to a given $\boldsymbol{f}\in L^q(\Omega)$, for all $q\in(1,q_0]$, $q_0>3$. Then*

$$\boldsymbol{w}\in L^r(\Omega)\,, \quad \textit{if and only if } r>2\,. \tag{47}$$

If, in addition, $\boldsymbol{f}$ is of bounded support, then, denoting by θ the angle made by a ray starting from the origin of coordinates (taken, without loss of generality, in $\mathbb{R}^3-\overline{\Omega}$) with the positively directed x_1-axis, we have

$$|\boldsymbol{w}(\boldsymbol{x})|\le\frac{M}{|\boldsymbol{x}|\,[1+|\boldsymbol{x}|(1+\cos\theta)]}\,,\quad \boldsymbol{x}\in\Omega\,, \tag{48}$$

where $M=M(\lambda,\Omega)>0$.

Remark 6 Two significant consequences of Theorem 6 are: (i) *the total kinetic energy of the flow past an obstacle* ($\equiv\frac{1}{2}\rho\|\boldsymbol{w}\|_2^2$) *is infinite*, see (47); and (ii) *the asymptotic decay of $\boldsymbol{w}$ is faster outside any semi-infinite cone with its axis coinciding with the negative x_1-axis* (*existence of the "wake"*); see (48).□

The study of the asymptotic properties of solutions in the two-dimensional case is deferred till Section IV.2.3.

IV.2.1 Three-Dimensional Flow. Existence of Solutions and Related Properties

As in the case of a bounded domain, we may use two different approaches to the study of existence of weak solutions.

A. Finite-Dimensional Method. Assume $\boldsymbol{f}\in\mathcal{D}_0^{-1,2}(\Omega)$. With the same notation as in Subsection IV.1.1(A), we look for "approximate solutions" to (45) of the form $\boldsymbol{u}_N=\sum_{i=1}^N c_{iN}\boldsymbol{\psi}_i$, where

$$\begin{aligned}[]
[\boldsymbol{u}_N,\boldsymbol{\psi}_k]-\lambda\left(\frac{\partial\boldsymbol{u}_N}{\partial x_1},\boldsymbol{\psi}_k\right)-\lambda\,(\boldsymbol{u}_N\cdot\nabla\boldsymbol{\psi}_k,\boldsymbol{u}_N)-\lambda\{(\boldsymbol{u}_N\cdot\nabla\boldsymbol{\psi}_k,\boldsymbol{V})\\
+\,(\boldsymbol{V}\cdot\nabla\boldsymbol{\psi}_k,\boldsymbol{u}_N)\}+(\boldsymbol{H},\boldsymbol{\psi}_k)=\langle\boldsymbol{f},\boldsymbol{\psi}_k\rangle\,,\quad k=1,\cdots,N\,.
\end{aligned}\tag{49}$$

As in the case of a bounded domain, existence to the algebraic system (49), in the unknowns c_{iN}, will be achieved if we show the uniform estimate (28). By dot-multiplying through both sides of (49) by c_{kN}, by summing over k between 1 and N and by using Lemma 1 we find

$$|\boldsymbol{u}_N|_{1,2}^2=\lambda\,(\boldsymbol{u}_N\cdot\nabla\boldsymbol{u}_N,\boldsymbol{V})-(\boldsymbol{H},\boldsymbol{u}_N)+\langle\boldsymbol{f},\boldsymbol{u}_N\rangle\,. \tag{50}$$

From the properties of the extension of the boundary data, we have that $\boldsymbol{V}$ satisfies (31) with $\varepsilon=1/(2\lambda)$ and that, moreover $-(\boldsymbol{H},\boldsymbol{u}_N)\le C|\boldsymbol{u}_N|_{1,2}$, for some $C=C(\Omega,\lambda)>0$. Thus, from this and from (50) we deduce the uniform bound (28). This latter implies the existence of a subsequence $\{\boldsymbol{u}_{N'}\}$ converging to some $\boldsymbol{u}\in\mathcal{D}_0^{1,2}(\Omega)$ weakly. Moreover, by Rellich theorem, $\boldsymbol{u}_{N'}\to\boldsymbol{u}$ in $L^4(K)$, where $K:=\Omega\cap\{|\boldsymbol{x}|>\rho\}$, all sufficiently large and finite $\rho>0$ [31, Theorem II.4.2]. Consequently, recalling that $\boldsymbol{\psi}_k$ is of compact support in Ω, we proceed as in the bounded domain case and take the limit $N \equiv N' \to \infty$ in (49) to show that $\boldsymbol{u}$ satisfies (45) with $\boldsymbol{\varphi} \equiv \boldsymbol{\psi}_k$. Successively, by taking into account that every $\boldsymbol{\varphi} \in \mathcal{D}(\Omega)$ can be approximated in $L^3(\Omega)$ by linear combinations of $\boldsymbol{\psi}_k$ (see [31, Lemma VII.2.1]) by (46) and by Lemma 1 we conclude that $\boldsymbol{u}$ satisfies (45). Notice that, as in the case of flow in bounded domains, this method furnishes existence for any $\lambda > 0$ and any $\boldsymbol{f} \in \mathcal{D}_0^{-1,2}(\Omega)$.

B. The Function-Analytic Method. As in the case of flow in a bounded domain, Subsection IV.1.(B), our objective is to rewrite (45) as a nonlinear operator equation in an appropriate Banach space, where the relevant operator satisfies the assumptions of Lemma 3. In this way, we may draw the same conclusions of Theorem 1 also in the case of a flow past an obstacle. However, *unlike the case of flow in a bounded domain*, the map

$$\boldsymbol{\varphi} \in \mathcal{D}(\Omega) \mapsto (\boldsymbol{u} \cdot \nabla \boldsymbol{\varphi}, \boldsymbol{u}) \in \mathbb{R}$$

can *not* be extended to a linear, bounded functional in $\mathcal{D}_0^{1,2}(\Omega)$, if $\boldsymbol{u}$ *merely belongs to* $\mathcal{D}_0^{1,2}(\Omega)$. Analogous conclusion holds for the map $\boldsymbol{\varphi} \in \mathcal{D}(\Omega) \mapsto (\partial \boldsymbol{u}/\partial x_1, \boldsymbol{\varphi}) \in \mathbb{R}$. The reason is because, in an exterior domain, the Poincaré inequality (18) and, consequently, the embedding (21) are, in general, not true. It is thus necessary to consider the above functionals for $\boldsymbol{u}$ in a space *strictly contained* in $\mathcal{D}_0^{1,2}(\Omega)$. Set

$$[\![\boldsymbol{u}]\!] := \sup_{\boldsymbol{\varphi} \in \mathcal{D}(\Omega)} \frac{\left| \left(\dfrac{\partial \boldsymbol{u}}{\partial x_1}, \boldsymbol{\varphi} \right) \right|}{|\boldsymbol{\varphi}|_{1,2}}$$

and let

$$X = X(\Omega) := \left\{ \boldsymbol{u} \in \mathcal{D}_0^{1,2}(\Omega) : \ [\![\boldsymbol{u}]\!] < \infty \right\}.$$

Clearly, $X(\Omega)$ endowed with the norm $|\cdot|_{1,2} + [\![\cdot]\!]$ is a Banach space. Moreover, $X(\Omega) \subset L^4(\Omega)$, continuously; see [36, Proposition 1.1]. We may thus conclude, by Riesz theorem, by Hölder inequality and by the properties of the extension $\boldsymbol{V}$ that, for any $\boldsymbol{u} \in X(\Omega)$, there exist $\boldsymbol{L}(\boldsymbol{u})$, $\boldsymbol{\mathcal{M}}(\boldsymbol{u})$, $\boldsymbol{\mathcal{V}}(\boldsymbol{u})$, and $\boldsymbol{\mathcal{H}}(\lambda)$ in $\mathcal{D}_0^{1,2}(\Omega)$ such that, for all $\boldsymbol{\varphi} \in \mathcal{D}(\Omega)$,

$$-\left(\frac{\partial \boldsymbol{u}}{\partial x_1}, \boldsymbol{\varphi} \right) = [\boldsymbol{L}(\boldsymbol{u}), \boldsymbol{\varphi}]\,; \quad -(\boldsymbol{u} \cdot \nabla \boldsymbol{\varphi}, \boldsymbol{u}) = [\boldsymbol{\mathcal{M}}(\boldsymbol{u}), \boldsymbol{\varphi}]\,;$$

$$-(\boldsymbol{u} \cdot \nabla \boldsymbol{\varphi}, \boldsymbol{V}) - (\boldsymbol{V} \cdot \nabla \boldsymbol{\varphi}, \boldsymbol{u}) = [\boldsymbol{\mathcal{V}}(\boldsymbol{u}), \boldsymbol{\varphi}]\,; \quad (\boldsymbol{H}, \boldsymbol{\varphi}) = [\boldsymbol{\mathcal{H}}(\lambda), \boldsymbol{\varphi}]\,.$$

Consequently, we obtain that (45) is equivalent to the following equation

$$\boldsymbol{M}(\lambda, \boldsymbol{u}) = \boldsymbol{F} \quad \text{in } \mathcal{D}_0^{1,2}(\Omega)\,, \tag{51}$$

where

$$\boldsymbol{M} : \ (\lambda, \boldsymbol{u}) \in (0, \infty) \times X(\Omega) \mapsto \boldsymbol{u} + \lambda \left(\boldsymbol{L}(\boldsymbol{u}) + \boldsymbol{\mathcal{V}}(\boldsymbol{u}) + \boldsymbol{\mathcal{M}}(\boldsymbol{u}) \right) + \boldsymbol{\mathcal{H}}(\lambda) \in \mathcal{D}_0^{1,2}(\Omega)\,.$$

A detailed study of the properties of the operator $\boldsymbol{M}$ is done in [36, §5], where, in particular, the following result is proved.

Lemma 5 *The operator $\boldsymbol{M}$ is of class C^∞. Moreover, for each $\lambda > 0$, $\boldsymbol{M}(\lambda, \cdot) : X(\Omega) \mapsto \mathcal{D}_0^{1,2}(\Omega)$ is proper and Fredholm of index 0, and the two equations $\boldsymbol{M}(\lambda, \boldsymbol{u}) = \boldsymbol{\mathcal{H}}(\lambda)$ and $D_{\boldsymbol{u}} \boldsymbol{M}(\lambda, \boldsymbol{0})(\boldsymbol{w}) = \boldsymbol{0}$ only have the solutions $\boldsymbol{u} = \boldsymbol{w} = \boldsymbol{0}$.*

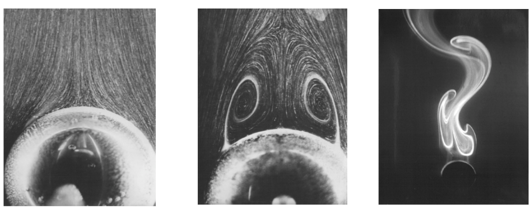
From this lemma and with the help of Lemma 3, we obtain the following result analogous to Theorem 1.

Theorem 7 *For any $\lambda > 0$ and $\boldsymbol{F} \in \mathcal{D}_0^{1,2}(\Omega)$ the equation* (51) *has at least one solution $\boldsymbol{u} \in X(\Omega)$. Moreover, for each fixed $\lambda > 0$, there exists open and dense $\mathcal{Q} = \mathcal{Q}(\lambda) \subset \mathcal{D}_0^{1,2}(\Omega)$ with the following properties:* (i) *For any $\boldsymbol{F} \in \mathcal{Q}$ the number of solutions to* (51)*, $\mathfrak{n} = \mathfrak{n}(\boldsymbol{F}, \lambda)$ is finite and odd;* (ii) *the integer $\mathfrak{n}$ is finite on each connected component of $\mathcal{Q}$.*

Finally, concerning the geometric structure of the set of pairs $(\lambda, \boldsymbol{u}) \in (0, \infty) \times X(\Omega)$ satisfying (51) for a fixed $\boldsymbol{F}$, a result entirely similar to Theorem 2 continues to hold; see [36, Theorem 6.2].

IV.2.2 Three-Dimensional Flow. Uniqueness and Steady Bifurcation

There is both experimental [91, 97] and numerical [68, 95] evidence that a steady flow past a sphere is unique (and stable) if the Reynolds number λ is sufficiently small. Moreover, experiments report that a closed recirculation zone first appears at λ around 20-25, and the flow stays steady and axisymmetric up to at least $\lambda \simeq 130$. This implies that the first bifurcation occurs through a steady motion. For higher values of λ, the wake behind the sphere becomes time-periodic, thus suggesting the occurence of unsteady (Hopf) bifurcation; see Figure 5.



$\lambda = 17.9$ $\quad$ $\lambda = 73.6$ $\quad$ $\lambda = 266$

Figure 5. Visualization of a flow past a sphere at increasing Reynolds numbers λ; after S. Taneda [91]

In this section we shall collect the relevant results available for uniqueness and steady bifurcation of a flow past a three-dimensional obstacle.

Uniqueness Results. Unlike the case of a flow in a bounded domain where uniqueness in the class of weak solutions is simply established (see Theorem 3), in the situation of a flow past an obstacle, the uniqueness proof requires the detailed study of the *asymptotic behavior* of a weak solution mentioned in Section IV.2; see also Theorem 6. Precisely, we have the following result, for whose proof we refer to [32, Theorem IX.5.3].

Theorem 8 *Suppose $\boldsymbol{f} \in L^{6/5}(\Omega) \cap L^{3/2}(\Omega)$, $\lambda \in (0, \overline{\lambda}]$, for some $\overline{\lambda} > 0$, and let $\boldsymbol{w}$ be the corresponding weak solution. (Notice that, under the given assumptions, $\boldsymbol{f} \in \mathcal{D}_0^{-1,2}(\Omega)$.) There exists $C = C(\Omega, \overline{\lambda}) > 0$ such that, if*

$$\|\boldsymbol{f}\|_{6/5} + \lambda < C\,,$$

then $\boldsymbol{w}$ is the only weak solution corresponding to $\boldsymbol{f}$.

Some Bifurcation Results. The rigorous study of steady bifurcation of a flow past a body is a very challenging mathematical problem. However, the function-analytic framework developed in the previous section allows us to formulate sufficient conditions for steady bifurcation, that are *formally* analogous to those discussed in Section IV.1.3 for the case of a flow in a bounded domain; see [36, §VII]. To this end, fix $\boldsymbol{f} \in \mathcal{D}_0^{-1,2}(\Omega)$, once and for all, and let $\boldsymbol{u}_0 = \boldsymbol{u}_0(\lambda)$, λ in some open interval $I \subseteq (0, \infty)$, be a *given* curve in $X(\Omega)$, constituted by solutions to (51) corresponding to the prescribed $\boldsymbol{f}$. If $\boldsymbol{u} + \boldsymbol{u}_0$, is another solution, from (51) we easily obtain that $\boldsymbol{u}$ satisfies the following equation in $\mathcal{D}_0^{1,2}(\Omega)$

$$\boldsymbol{u} + \lambda\left(\boldsymbol{L}(\boldsymbol{u}) + \boldsymbol{\mathcal{B}}(\boldsymbol{u}_0(\lambda))(\boldsymbol{u}) + \boldsymbol{\mathcal{M}}(\boldsymbol{u})\right) = \boldsymbol{0}\,, \quad \boldsymbol{u} \in X(\Omega)\,, \tag{52}$$

where $\boldsymbol{\mathcal{B}}(\boldsymbol{u}_0) := D_{\boldsymbol{u}}\boldsymbol{\mathcal{M}}(\boldsymbol{u}_0)$. In this setting, the branch $\boldsymbol{u}_0(\lambda)$ becomes the solution $\boldsymbol{u} \equiv \boldsymbol{0}$ and the bifurcation problem thus reduces to find a nonzero branch of solutions $\boldsymbol{u} = \boldsymbol{u}(\lambda)$ to (52) in every neighborhood of some *bifurcation point* $(\boldsymbol{0}, \lambda_0)$; see Definition 4. Define the map

$$F : (\lambda, \boldsymbol{u}) \in (0, \infty) \times X(\Omega) \mapsto \boldsymbol{u} + \lambda\left(\boldsymbol{L}(\boldsymbol{u}) + \boldsymbol{\mathcal{B}}(\boldsymbol{u}_0(\lambda))(\boldsymbol{u}) + \boldsymbol{\mathcal{M}}(\boldsymbol{u})\right) \in \mathcal{D}_0^{1,2}(\Omega)\,. \tag{53}$$

In [36, §VII] the following result is shown.

Lemma 6 *The map F is of class C^∞. Moreover, the derivative*

$$D_{\boldsymbol{u}}F(\lambda,\boldsymbol{0})(\boldsymbol{w}) = \boldsymbol{w} + \lambda\left(\boldsymbol{L}(\boldsymbol{w}) + \boldsymbol{\mathcal{B}}(\boldsymbol{u}_0(\lambda))(\boldsymbol{w})\right)\,,$$

is Fredholm of index 0

Therefore, from this lemma and from Lemma 4 we obtain that a *necessary condition* for $(\boldsymbol{0},\lambda_0)$ to be a bifurcation point to (52) is that the linear problem

$$\boldsymbol{w}_1 + \lambda_0\left(\boldsymbol{L}(\boldsymbol{w}_1) + \boldsymbol{\mathcal{B}}(\boldsymbol{u}_0(\lambda_0))(\boldsymbol{w}_1)\right) = \boldsymbol{0}\,, \quad \boldsymbol{w}_1 \in X(\Omega)\,, \tag{54}$$

has a non-zero solution $\boldsymbol{w}_1$. Once this necessary condition is satisfied, one can formulate several sufficient conditions for the point $(\boldsymbol{0},\lambda_0)$ to be a bifurcation point. For a review of different criteria for global and local bifurcation for Fredholm maps of index 0, we refer to [41, Section 6]. Here we wish to use the criterion of Theorem 5 to present a very simple (in principle) and familiar sufficient condition in the particular case when the given curve $\boldsymbol{u}_0$ can be made (locally, in a neighborhood of λ_0) independent of λ. This may depend on the particular non-dimensionalization of the Navier-Stokes equations and on the special form of the family of solutions $\boldsymbol{u}_0$. In fact, there are several interesting problems formulated in exterior domains where this circumstance takes place, like, for example, the problem of steady bifurcation considered in the previous section and the one studied in [40, Section 6]. Now, if $\boldsymbol{u}_0$ does not depend on λ, from Theorem 5 and from (54) we immediately find that a sufficient condition in order that $(\boldsymbol{0},\lambda_0)$ be a bifurcation point is that the following problem

$$\boldsymbol{w} + \lambda_0\boldsymbol{\mathcal{L}}(\boldsymbol{w}) = \boldsymbol{w}_1\,, \quad \boldsymbol{w} \in X(\Omega)\,, \tag{55}$$

with $\boldsymbol{\mathcal{L}} := \boldsymbol{L} + \boldsymbol{\mathcal{B}}(\boldsymbol{u}_0)$ and $\boldsymbol{w}_1$ solving (54), has no solution. In different words, *a sufficient condition for* $(\boldsymbol{0},\lambda_0)$ *to be a bifurcation point is that* $-1/\lambda_0$ *is a simple eigenvalue of the operator* $\boldsymbol{\mathcal{L}}$. It is interesting to observe that this condition is *formally* the same as the one arising in steady bifurcation problems for flow in a *bounded* domain; see Section IV.1.3. However, while in this latter case $\boldsymbol{\mathcal{L}}$ ($\equiv \boldsymbol{B}(\boldsymbol{v}_0)$) is *compact* and defined on the whole of $\mathcal{D}_0^{1,2}(\Omega)$, in the present situation $\boldsymbol{\mathcal{L}}$, with domain $D := X(\Omega) \subset \mathcal{D}_0^{1,2}(\Omega)$, is an *unbounded* operator. As such, we can not even be sure that $\boldsymbol{\mathcal{L}}$ has real *simple* eigenvalues. Nevertheless, since, by Lemma 6, $\boldsymbol{I} + \lambda\boldsymbol{\mathcal{L}}$ is Fredholm of index 0 for all $\lambda \in (0,\infty)$, and since one can show [36, Lemma 7.1] that $\boldsymbol{\mathcal{L}}$ is graph-closed, if $\boldsymbol{u}_0 \in L^3(\Omega)$ (this latter condition is satisfied under suitable hypotheses on $\boldsymbol{f}$; see Theorem 6) from well-known results of spectral theory [46, Theorem XVII.2.1], it follows that the set Λ of real eigenvalues of $\mathcal{L}$ satisfies the following properties: (a) Λ is at most countable; (b) Λ is constituted by isolated points of finite algebraic and geometric multiplicities, and (c) points in Λ can only cluster at 0. Consequently, the bifurcation condition requiring the simplicity of $-1/\lambda_0$ (namely, algebraic multiplicity 1) is perfectly meaningful.

IV.2.2 Two-Dimensional Flow. The Problem of Existence

The planar motion of a viscous liquid past a cylinder is among the oldest problems to have received a systematic mathematical treatment. Actually, in 1851, it was addressed by Sir George Stokes in his work on the motion of a pendulum in a viscous liquid [89]. In the wake of his successful study of the flow past a sphere in the limit of vanishing λ (*Stokes approximation*), Stokes looked for solutions to (44), with $\boldsymbol{f} \equiv \boldsymbol{0}$ and $\lambda = 0$, in the case when Ω is the exterior of a circle. However, to his surprise, he found that this linearized problem has *no* solution, and he concluded with the following (wrong) statement [89], p. 63,

"It appears that the supposition of steady motion is inadmissible."

Such an observation constitutes what we currently call *Stokes Paradox*.

This is definitely a very intriguing starting point for the resolution of the boundary-value problem (44), in that it suggests that, if the problem has a solution, the nonlinear terms have to play a major role. In this regard, by using, for instance, the Galerkin method and proceeding exactly as in Subsection IV.2.1(A), we can prove the existence of a weak solution to (44), *for any* $\lambda > 0$ *and* $\boldsymbol{f} \in \mathcal{D}_0^{-1,2}(\Omega)$. In addition, this solution is as smooth as allowed by the regularity of Ω and $\boldsymbol{f}$ (see Section IV.2) and, in such a case, it satisfies $(44)_{1,2,3}$ in the ordinary sense. However, unlike the three-dimensional case (see eq. (46)), the space $\mathcal{D}_0^{1,2}(\Omega)$ is *not embedded in any* L^q*-space* and, therefore, we can not be sure that, even in an appropriate generalized sense, this solution vanishes at infinity, as requested by $(44)_4$. Actually, if $\Omega \subset \mathbb{R}^2$ there are functions in $\mathcal{D}_0^{1,2}(\Omega)$ becoming *unbounded* at infinity. Take, for example, $\boldsymbol{w} = (\ln|\boldsymbol{x}|)^\alpha\, \boldsymbol{e}_\theta$, $\alpha \in (0,1)$, and Ω the exterior of the unit circle. In this sense, we call these solutions *weak*, and not because of lack of local regularity (they are as smooth as allowed by the smoothness of Ω and $\boldsymbol{f}$). This problem was first pointed out by Leray [61, pp. 54-55].

The above partial results leave open the worrisome possibility that a Stokes paradox could also hold for the *fully nonlinear* problem (45). If this chance turned out to be indeed true, it would cast serious doubts on the Navier-Stokes equations as reliable fluid model, in that they would not be able to catch the physics of a very elementary phenomenon, easily reproduced experimentally.

The possibility of a nonlinear Stokes paradox was ruled out by Finn and Smith in a deep paper published in 1967 [19], where it is shown that if $\boldsymbol{f} \equiv \boldsymbol{0}$ and if Ω is sufficiently regular, then (45) has a solution, at least for "small" (but nonzero!) λ . The method used by these authors is based on the representation of solutions and careful estimates of the Green tensor of the Oseen problem. Another approach to existence for small λ, relying upon the L^q theory of the Oseen problem, was successively given by Galdi [30] where one can find the proof of the following result.

Theorem 9 *Let* Ω *be of class* C^2 *and let* $\boldsymbol{f} \in L^q(\Omega)$*, for some* $q \in (1, 6/5)$*. Then there exists* $\lambda_1 > 0$ *and* $C = C(\Omega, q, \lambda_1) > 0$ *such that if, for some* $\lambda \in (0, \lambda_1]$*,*

$$|\log\lambda|^{-1} + \lambda^{2(1/q-1)}\|\boldsymbol{f}\|_q < C\,,$$

problem (45) *has at least one weak solution that, in addition, satisfies* $(45)_4$ *uniformly pointwise.*

It can be further shown that the above solutions meet all the basic physical requirements. In particular, as in the three-dimensional case (see Section IV.2), they exhibit a "wake" in the region $x_1 < 0$. Moreover, the solutions are unique in a ball of a suitable Banach space, centered at the origin and of "small" radius. For the proof of these two statements, we refer to [32, §§X.4 and X.5].

Though significant, these results leave open several fundamental questions. The most important is, of course, that of whether problem (44) is solvable for *all* $\lambda > 0$ and *all* $\boldsymbol{f}$ in a suitable space. As we already noticed, this solvability would be secured if we can show that the weak solution *does* satisfy $(44)_4$, even in a generalized sense. It should be emphasized that, since, as shown previously, there are functions in $\mathcal{D}_0^{1,2}(\Omega)$ that become unbounded at infinity, the proof of this asymptotic property must be restricted to functions satisfying $(44)_{1,2,3}$. This question has been taken up in a series of remarkable papers by Gilbarg and Weinberger [44], [45] and by Amick [2] in the case when $\boldsymbol{f} \equiv \boldsymbol{0}$. Some of their results lead to the following one due to Galdi [35, Theorem 3.4].

Theorem 10 *Let* $\boldsymbol{w}$ *be a weak solution to* (44) *with* $\boldsymbol{f} \equiv \boldsymbol{0}$*. Then, there exists* $\boldsymbol{\xi} \in \mathbb{R}^2$ *such that*

$$\lim_{|\boldsymbol{x}|\to\infty} \boldsymbol{w}(\boldsymbol{x}) = \boldsymbol{\xi} \quad \textit{uniformly.}$$

Open Question. It is not known if $\boldsymbol{\xi} = \mathbf{0}$, and so it is not known if $\boldsymbol{w}$ satisfies $(44)_4$. Thus, *the question of the solvability of* (44) *for arbitrary* $\lambda > 0$*, even when* $\boldsymbol{f} \equiv \mathbf{0}$*, remains open.*

When Ω is symmetric around the direction of $\boldsymbol{e}_1$, in [33] Galdi has suggested a different approach to the solvability of (44) (with $\boldsymbol{f} \equiv \mathbf{0}$) for arbitrary large $\lambda > 0$. Let $\mathcal{C}$ be the class of vector fields, $\boldsymbol{v} = (v_1, v_2)$, and scalar fields τ such that (i) v_1 and τ are even in x_2 and v_2 is odd in x_2, and (ii) $|\boldsymbol{v}|_{1,2} < \infty$. The following result holds.

Theorem 11 *Let* Ω *be symmetric around the* x_1*-axis. Assume that the homogeneous problem:*

$$\left.\begin{aligned} &\Delta \boldsymbol{u} = \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \nabla \phi \\ &\operatorname{div} \boldsymbol{u} = 0 \end{aligned}\right\} \ \text{in } \Omega$$
$$\boldsymbol{u}|_{\partial\Omega} = \mathbf{0}\,, \quad \lim_{|\boldsymbol{x}| \to \infty} \boldsymbol{u}(\boldsymbol{x}) = \mathbf{0} \ \ \text{uniformly}\,, \tag{56}$$

has only the zero solution, $\boldsymbol{u} \equiv \mathbf{0}$*,* $p = \mathrm{const.}$*, in the class* $\mathcal{C}$ *. Then, there is a set* M *with the following properties:*

(i) $M \subset (0, \infty)$*;*

(ii) $M \supset (0, c)$ *for some* $c = c(\Omega) > 0$*;*

(iii) M *is unbounded;*

(iv) *For any* $\lambda \in M$*, problem* (44) *has at least one solution in the class* $\mathcal{C}$*.*

Open Question. The difficulty with the above theorem relies in establishing the validity of its hypothesis, namely, *whether or not* (56) *has only the zero solution in the class* $\mathcal{C}$. Moreover, supposing that the hypothesis holds true, the other fundamental problem is the study of the properties of the set M. For a detailed discussion of these issues, we refer to [35, §4.3].

V Mathematical Analysis of the Initial-Boundary Value Problem

Objective of this section is to present the main results and open questions regarding the unique solvability of the initial-boundary value problem (7)–(8), and significant related properties.

V.1 Preliminary Considerations

In order to present the basic problems for (7)–(8) and the related results, we shall, for the sake of simplicity, restrict ourselves to the case when $\boldsymbol{f} \equiv \boldsymbol{v}_1 \equiv \mathbf{0}$.

In what follows, we shall denote by $(7)–(8)_{\mathrm{hom}}$ *this homogeneous problem.*

The first, fundamental problem that should be naturally set for $(7)–(8)_{\mathrm{hom}}$ is the classical one of (global) well-posedness in the sense of Hadamard.

Problem 1. *Find a Banach space,* X*, such that for any initial data* $\boldsymbol{v}_0$ *in* X *there is a corresponding solution* $(\boldsymbol{v}, p)$ *to* (7)-$(8)_{\mathrm{hom}}$ *satisfying the following conditions:* (i) *it exists for all* $T > 0$, (ii) *it is unique and* (iii) *it depends continuously on* $\boldsymbol{v}_0$.

In different words, the resolution of Problem 1 will ensure that the Navier-Stokes equations furnish, at all times, a *deterministic description* of the dynamics of the liquid, provided the initial data are given in a "sufficiently rich" class. It is immediately seen that the class X should meet some necessary requirements for Problem 1 to be solvable. For instance, if we take $\boldsymbol{v}_0$ *only bounded*, we find that problem (7)–(8), with $\Omega = \mathbb{R}^n$ and $\boldsymbol{f} = 0$, admits the following two distinct solutions

$$\boldsymbol{v}_1(\boldsymbol{x}, t) = \boldsymbol{0}\,, \quad p_1(\boldsymbol{x}, t) = 0\,; \quad \boldsymbol{v}_2(\boldsymbol{x}, t) = \sin t\, \boldsymbol{e}_1\,, \quad p_2(\boldsymbol{x}, t) = -x_1 \cos t\,;$$

corresponding to the same initial data $\boldsymbol{v}_0 = \boldsymbol{0}$.

Furthermore, we observe that the resolution of Problem 1, does *not* exclude the possibility of the formation of a "singularity", that is, the existence of points in the space-time region where the solution may become unboundedly large in certain norms. This possibility depends, of course, on the regularity of the functional class where well-posedness is established.

One is thus lead to consider the next fundamental (and most popular) problem.

Problem 2. *Given an initial distribution of velocity $\boldsymbol{v}_0$, no matter how smooth, with*

$$\int_\Omega |\boldsymbol{v}_0(\boldsymbol{x})|^2 < \infty\,, \tag{57}$$

determine a corresponding regular *solution $\boldsymbol{v}(x, t), p(x, t)$ to* (7)–(8)$_{\rm hom}$ *for all times $t \in (0, T)$ and all $T > 0$.*

By "regular" here we mean that $\boldsymbol{v}$ and p are both of class C^∞ in the *open* cylinder $\Omega \times (0, T)$, for all $T > 0$. When $\Omega \equiv \mathbb{R}^3$, Problem 1 is, basically, the third Millennium Prize Problem posted by the Clay Mathematical Institute in May 2000.

The requirement (57) on the initial data is meaningful from the physical point of view, in that it ensures that the kinetic energy of the liquid is initially finite. Moreover, it is also necessary from a mathematical viewpoint because, if we relax (57) to the requirement that the initial distribution of velocity is, for example, *only bounded*, then Problem 2 has a simple negative answer. In fact, the following pair

$$\boldsymbol{v}(\boldsymbol{x}, t) = \frac{1}{\tau - t}\boldsymbol{e}_1 \quad p(x, t) = -\frac{x_1}{(\tau - t)^2}\,, \quad t \in [0, \tau)\,,\ \tau > 0$$

is a solution to (7)–(8)$_{\rm hom}$ with $\boldsymbol{f} \equiv \boldsymbol{0}$ and $\Omega = \mathbb{R}^3$, that becomes singular at $t = \tau$, for any given positive τ.

An alternative way of formulating Problem 2 in "more physical" terms is as follows.

Problem 2′. *Can a* spontaneous *singularity arise in a* finite time *in a viscous liquid that is initially in an arbitrarily smooth state?*

Though, perhaps, the gut answer to this question could be in the negative, one can bring very simple examples of dissipative nonlinear evolution equations where spontaneous singularities do occur, if the initial data are sufficiently large. For instance, the initial-value problem $v' + \sigma\, v = v^2$, $v(0) = v_0$, $\sigma > 0$, has the explicit solution

$$v(t) = \frac{\sigma\, v_0}{v_0 - e^{\sigma\, t}(v_0 - \sigma)}$$

which shows that, if $v_0 \le \sigma$, then v is smooth for all $t > 0$, while if $v_0 > \sigma$, then v becomes unbounded in a finite time:

$$v(t) \ge \frac{1}{\tau - t}\,,\ t \in [0, \tau)\,, \quad \tau := \frac{1}{\sigma}\log\left(\frac{v_0}{v_0 - \sigma}\right)\,.$$

If the occurence of singularities for problem (7)–(8)$_{\rm hom}$ can not be at all excluded, one can still theorize that singularities are unstable and, therefore, "undetectable". Another plausible explanation could be that singularity may appear in the Navier-Stokes equations due to the possible break-down of the continuum model at very small scales.

It turns out that, in the case of two-dimensional (2D) flow, both problems 1 and 2 are completely resolved, while they are both open in the three-dimensional (3D) case. The following section will be dedicated to these issues.

V.2 On the Solvability of Problems 1 and 2

As in the steady-state case, a basic tool for the resolution of both Problems 1 and 2 is the accomplishment of "good" *a priori* estimates. By "good", we mean that (i) they have to be *global*, namely, they should hold for all positive times, and (ii) they have to be valid in a sufficiently regular function class. These estimates can then be used along suitable "approximate solutions" which eventually will converge, by an appropriate limit procedure, to a solution to (7)–$(8)_{\rm hom}$. To date, "good" estimates for 3D flow are not known.

Unless explicitly stated, throughout this section we assume that Ω is a bounded, smooth (of class C^2, for example) domain of $\mathbb{R}^n$, $n = 2, 3$.

V.2.1 Derivation of Some Fundamental *A Priori* Estimates

We recall that, for simplicity, we are assuming that the boundary data, $\boldsymbol{v}_1$, in (8) is vanishing. Thus, if we formally dot-multiply through both sides of $(7)_1$ (with $\boldsymbol{f} \equiv \boldsymbol{0}$) by $\boldsymbol{v}$, integrate by parts over Ω and take into account (12) and Lemma 1, we obtain the following equation

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{v}(t)\|_2^2 + \nu\,|\boldsymbol{v}(t)|_{1,2}^2 = 0\,. \tag{58}$$

The physical interpretation of (58) is straightforward. Actually, if we dot-multiply both sides of the identity $\operatorname{div}\boldsymbol{D}(\boldsymbol{v}) = \Delta\boldsymbol{v}$ by $\boldsymbol{v}$, where $\boldsymbol{D} = \boldsymbol{D}(\boldsymbol{v})$ is the stretching tensor (see (3)), and integrate by parts over Ω, we find that $|\boldsymbol{v}|_{1,2} = \|\boldsymbol{D}(\boldsymbol{v})\|_2$. Since, as we observed in Section III, $\boldsymbol{D}$ takes into account the deformation of the parts of the liquid, equation (58) simply relates the rate of decreasing of the kinetic energy to the dissipation inside the liquid, due to the combined effect of viscosity and deformation. If we integrate (58) from $s \geq 0$ to $t \geq s$, we obtain the so-called *energy equality*

$$\|\boldsymbol{v}(t)\|_2^2 + 2\nu\int_s^t |\boldsymbol{v}(\rho)|_{1,2}^2 d\rho = \|\boldsymbol{v}(s)\|_2^2 \quad 0 \leq s \leq t\,. \tag{59}$$

Notice that the nonlinear term $\boldsymbol{v}\cdot\nabla\boldsymbol{v}$ does *not* give any contribution to equation (58) (and, consequently, to equation (59)), in virtue of the fact that $(\boldsymbol{v}\cdot\nabla\boldsymbol{v}, \boldsymbol{v}) = 0$; see Lemma 1. By taking $s = 0$ in (59), we find a *bound on the kinetic energy and on the total dissipation for all times $t \geq 0$, in terms of the initial data only*, provided the latter satisfy (57). In concise words, the energy equality (59) is a *global a priori* estimate. It should be emphasized that the energy equality is, basically, *the only known global* a priori *estimate for 3D flow.*

From (59) it follows, in particular,

$$\boldsymbol{v} \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;\mathcal{D}_0^{1,2}(\Omega))\,, \quad \text{all } T > 0\,. \tag{60}$$

A second estimate can be obtained by dot-multiplying through both sides of $(7)_1$ by $P\Delta\boldsymbol{v}$ and by integrating by parts over Ω. Taking into account that

$$\left(\frac{\partial\boldsymbol{v}}{\partial t}, P\Delta\boldsymbol{v}\right) = \left(\frac{\partial\boldsymbol{v}}{\partial t}, \Delta\boldsymbol{v}\right) = -\frac{1}{2}\frac{d}{dt}|\boldsymbol{v}|_{1,2}^2\,, \quad (\nabla p,\, P\Delta\boldsymbol{v}) = 0\,,$$

we deduce

$$\frac{1}{2}\frac{d}{dt}|\boldsymbol{v}|_{1,2}^2+\nu\|P\Delta\boldsymbol{v}\|_2^2=(\boldsymbol{v}\cdot\nabla\boldsymbol{v},P\Delta\boldsymbol{v})\,. \tag{61}$$

Since the right-hand side of this equation need not be zero, we get that, unlike (58), the nonlinear term *does* contribute to (61). In addition, since the sign of this contribution is basically unknown, in order to obtain useful estimates we have to increase it appropriately. To this end, we recall the validity of the following inequalities

$$\|\boldsymbol{u}\|_\infty\le\begin{cases} c_1\,\|\boldsymbol{u}\|_2^{\frac12}\|P\Delta\boldsymbol{u}\|_2^{\frac12} & \text{if } n=2,\ \text{ for all } \boldsymbol{u}\in L^2_\sigma(\Omega), \text{ with } P\Delta\boldsymbol{u}\in L^2(\Omega) \text{ and } \boldsymbol{u}|_{\partial\Omega}=\boldsymbol{0},\\ c_2\,|\boldsymbol{u}|_{1,2}^{\frac12}\|P\Delta\boldsymbol{u}\|_2^{\frac12} & \text{if } n=3,\ \text{ for all } \boldsymbol{u}\in\mathcal{D}_0^{1,2}(\Omega), \text{ with } P\Delta\boldsymbol{u}\in L^2(\Omega)\,,\end{cases} \tag{62}$$

where $c_i=c_i(\Omega)>0$, $i=1,2$. These relations follow from the Sobolev embedding theorems along with Lemma 2. We shall sketch a proof in the case $n=3$. By the property of the projection operator P, $\boldsymbol{u}$ satisfies the assumptions of Lemma 2 with $\boldsymbol{g}:=P\Delta\boldsymbol{u}$, and, consequently, we have, on the one hand, that $\boldsymbol{u}\in W^{2,2}(\Omega)$, and, on the other hand,

$$\|\boldsymbol{u}\|_{2,2}\le c\,\|P\Delta\boldsymbol{u}\|_2\,, \tag{63}$$

with $c=c(\Omega)>0$. We now recall that there exists an extension operator $E:\boldsymbol{u}\in W^{2,2}(\Omega)\mapsto E(\boldsymbol{u})\in W^{2,2}(\mathbb{R}^3)$ such that

$$\|E(\boldsymbol{u})\|_{k,2}\le C_k\|\boldsymbol{u}\|_{k,2}\,,\quad k=0,1,2\,; \tag{64}$$

see [88, Chapter VI, Theorem 5]. Next, take $\boldsymbol{\varphi}\in C_0^\infty(\mathbb{R}^3)$. From the identity $\Delta(|\boldsymbol{\varphi}|^2)=2(\boldsymbol{\varphi}\cdot\Delta\boldsymbol{\varphi}+|\nabla\boldsymbol{\varphi}|^2)$ we have the representation

$$|\boldsymbol{\varphi}(\boldsymbol{x})|^2=-\frac{1}{2\pi}\int_{\mathbb{R}^3}\frac{\boldsymbol{\varphi}(\boldsymbol{y})\cdot\Delta\boldsymbol{\varphi}(\boldsymbol{y})+|\nabla\boldsymbol{\varphi}(\boldsymbol{y})|^2}{|\boldsymbol{x}-\boldsymbol{y}|}\,d\boldsymbol{y}\,. \tag{65}$$

Using Schwarz inequality on the right-hand side of (65) along with the classical Hardy inequality [31, §II.5]:

$$\int_{\mathbb{R}^3}\frac{|\boldsymbol{\varphi}(\boldsymbol{y})|^2}{|\boldsymbol{x}-\boldsymbol{y}|^2}d\boldsymbol{y}\le 4|\boldsymbol{\varphi}|_{1,2}^2\,,$$

we recover $\|\boldsymbol{\varphi}\|_\infty\le(2/\pi)^{1/2}|\boldsymbol{\varphi}|_{1,2}^{\frac12}|\boldsymbol{\varphi}|_{2,2}^{\frac12}$. Since $C_0^\infty(\mathbb{R}^3)$ is dense in $W^{2,2}(\mathbb{R}^3)$, from this latter inequality we deduce, in particular,

$$\|\boldsymbol{u}\|_\infty\le(2/\pi)^{1/2}|E(\boldsymbol{u})|_{1,2}^{\frac12}|E(\boldsymbol{u})|_{2,2}^{\frac12}\,,$$

which, in turn, by (64) and Poincaré's inequality (18) implies that

$$\|\boldsymbol{u}\|_\infty\le c_5|\boldsymbol{u}|_{1,2}^{\frac12}\|\boldsymbol{u}\|_{2,2}^{\frac12}\,,$$

with $c_5=c_5(\Omega)>0$. Equation $(62)_2$ then follows from this latter inequality and from (63). (For the proof of (62) in more general domains, as well as in domains with less regularity, we refer to [9, 67, 98]. I am not aware of the validity of (62) in an arbitrary (smooth) domain.) We now employ (62) and (29) into (61) to obtain

$$\frac{1}{2}\frac{d}{dt}|\boldsymbol{v}|_{1,2}^2+\frac{\nu}{2}\|P\Delta\boldsymbol{v}\|_2^2\le\begin{cases} c_3\,\|\boldsymbol{v}\|_2^2|\boldsymbol{v}|_{1,2}^4 & \text{if } n=2,\\ c_4\,|\boldsymbol{v}|_{1,2}^6 & \text{if } n=3,\end{cases} \tag{66}$$

where $c_i = c_i(\Omega, \nu) > 0$, $i = 3, 4$. Thus, observing that, from (59), $\|\boldsymbol{v}(t)\|_2 \le \|\boldsymbol{v}_0\|_2$, if we assume, further, that $\boldsymbol{v}_0 \in \mathcal{D}_0^{1,2}(\Omega)$, from the previous differential inequality, we obtain the following uniform bound

$$|\boldsymbol{v}(t)|_{1,2} \le M(\Omega, \nu, t, \|\boldsymbol{v}_0\|_{1,2}), \quad \text{for all } t \in [0, \tau), \text{ and some } \tau \ge 1/(K\, |\boldsymbol{v}_0|_{1,2}^{\alpha}), \tag{67}$$

where M is a continuous function in t and $K = 8c_3$, $\alpha = 2$ if $n = 2$, while $K = 4c_4$, $\alpha = 4$ if $n = 3$. Equation (67) provides the second *a priori* estimate. Notice that, unlike (59), we do not know if in (67) we can take t arbitrary large, namely, we do not know if $\tau = \infty$. Integrating both sides of (66) from 0 to $t < \tau$, and taking into account (67) we find that

$$\int_0^t \|P\Delta \boldsymbol{v}(s)\|_2^2 ds \le M_1(\Omega, \nu, t, \|\boldsymbol{v}_0\|_{1,2}), \quad \text{for all } t \in [0, \tau), \tag{68}$$

with M_1 continuous in t. From (68), (67), (60) and (63) one can then show that

$$\boldsymbol{v} \in L^\infty(0, t; W_0^{1,2}(\Omega)) \cap L^2(0, t; W^{2,2}(\Omega)), \quad t \in [0, \tau). \tag{69}$$

A third *a priori* estimate, on the time derivative of $\boldsymbol{v}$, can be formally obtained by dot-multiplying both sides of $(7)_1$ and by integrating by parts over Ω. By using arguments similar to those leading to (61) we find

$$\frac{\nu}{2}\frac{d}{dt}|\boldsymbol{v}|_{1,2}^2 + \left\|\frac{\partial \boldsymbol{v}}{\partial t}\right\|_2^2 = -(\boldsymbol{v} \cdot \nabla \boldsymbol{v}, \frac{\partial \boldsymbol{v}}{\partial t}), \tag{70}$$

and so, employing Hölder inequality on the right-hand side of (62) along with (29), (67) and (68) we show the following estimate

$$\int_0^t \left\|\frac{\partial \boldsymbol{v}(s)}{\partial s}\right\|_2^2 ds \le M_2(\Omega, \nu, t, \|\boldsymbol{v}_0\|_{1,2}), \quad \text{for all } t \in [0, \tau), \tag{71}$$

with M_2 continuous in t. This latter inequality implies that

$$\frac{\partial \boldsymbol{v}}{\partial t} \in L^2(0, t; L^2(\Omega)), \quad t \in [0, \tau). \tag{72}$$

V.2.2 Existence, Uniqueness, Continuous Dependence and Regularity Results

We shall now use estimates (59), (67), (68), (71) along a suitable approximate solution constructed by the finite-dimensional (Galerkin) method to show existence to (7)–$(8)_{\rm hom}$ in an appropriate function class. We shall briefly sketch the argument. Similarly to the steady-state case, we look for an approximate solution to (7)–$(8)_{\rm hom}$ of the form $\boldsymbol{v}_N(\boldsymbol{x}, t) = \sum_{i=0}^N c_{iN}(t)\boldsymbol{\psi}_i$, where $\{\boldsymbol{\psi}_i\}$ is a base of $L^2_\sigma(\Omega)$ constituted by the eigenvectors of the operator $-P\Delta$, namely,

$$-\nu\Delta\boldsymbol{\psi}_i = \lambda_i\boldsymbol{\psi}_i + \nabla\Phi_i\,,\ \operatorname{div}\boldsymbol{\psi}_i = 0 \ \text{ in } \Omega\,, \quad \boldsymbol{\psi}_i|_{\partial\Omega} = \boldsymbol{0}\,, \quad i \in \mathbb{N}\,, \tag{73}$$

where λ_i are the corresponding eigenvalues. The coefficients $c_{iN}(t)$ are requested to satisfy the following system of ordinary differential equations

$$(\frac{\partial \boldsymbol{v}_N}{\partial t}, \boldsymbol{\psi}_k) + (\boldsymbol{v}_N \cdot \nabla \boldsymbol{v}_N, \boldsymbol{\psi}_k) = \nu(\Delta \boldsymbol{v}_N, \boldsymbol{\psi}_k)\,, \quad k = 1, \cdots, N\,, \tag{74}$$

with initial conditions $c_{iN}(0) = (\boldsymbol{v}_0, \boldsymbol{\psi}_i)$, $i = 1, \cdots, N$. Multiplying both sides of (74), in the order, by c_{kN}, by $\lambda_k c_{kN}$ and by dc_{kN}/dt and summing over k from 1 to N, we at once obtain, with the help of

(73) and of Lemma 1, that $\boldsymbol{v}_N$ satisfies (59), (66) and (70). Consequently, $\boldsymbol{v}_N$ satisfies the uniform (in N) bounds (58) (evaluated at $s = 0$) (67), (68) and (71). Employing these bounds together with, more or less, standard limiting procedures, we can show the existence of a field $\boldsymbol{v}$ in the classes defined by (69) and (72) satisfying the relation

$$\left(\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v}\cdot\nabla\boldsymbol{v} - \nu\Delta\boldsymbol{v}, \boldsymbol{\varphi}\right) = 0\,, \quad \text{for all } \boldsymbol{\varphi}\in\mathcal{D}(\Omega) \text{ and a.a. } t\in[0,\tau). \tag{75}$$

Because of (69) and (72), the function involving $\boldsymbol{v}$ in (75) belongs to $L^2(\Omega)$, a.e. in $[0,\tau)$, and therefore, in view of the orthogonal decomposition $L^2(\Omega) = L^2_\sigma(\Omega)\oplus G(\Omega)$ and of the density of $\mathcal{D}(\Omega)$ in $L^2_\sigma(\Omega)$, we find $p\in L^2(0,t;W^{1,2}(\Omega))$, $t\in[0,\tau)$, such that $(\boldsymbol{v},p)$ satisfies $(7)_1$ for a.a. $(\boldsymbol{x},t)\in\Omega\times[0,\tau)$. We thus find the following result, basically due to G. Prodi, to whose paper [72] we refer for all missing details; see also [86, Chapter V.4].

Theorem 12 (Existence) *For every $\boldsymbol{v}_0\in\mathcal{D}^{1,2}_0(\Omega)$, there exist $\boldsymbol{v}=\boldsymbol{v}(\boldsymbol{x},t)$ and $p=p(\boldsymbol{x},t)$ such that*

$$\left.\begin{aligned}
&\boldsymbol{v}\in L^\infty(0,\tau;L^2(\Omega))\cap L^2(0,\tau;\mathcal{D}^{1,2}_0(\Omega))\,,\\
&\boldsymbol{v}\in C([0,t];\mathcal{D}^{1,2}_0(\Omega))\cap L^2(0,t;W^{2,2}(\Omega))\,,\\
&\frac{\partial\boldsymbol{v}}{\partial t}\in L^2(0,t;L^2(\Omega))\,,\quad p\in L^2(0,t;W^{1,2}(\Omega))\,,
\end{aligned}\right\}\ \text{for all } t\in[0,\tau)\,, \tag{76}$$

with τ given in (67)*, satisfying* (7) *for a.a. $(\boldsymbol{x},t)\in\Omega\times[0,\tau)$, and* $(8)_2$ *(with $\boldsymbol{v}_1\equiv\boldsymbol{0}$) for a.a. $(\boldsymbol{x},t)\in\partial\Omega\times[0,\tau)$. Moreover, the initial condition* $(8)_1$ *is attained in the following sense:*

$$\lim_{t\to0^+}\|\boldsymbol{v}(t)-\boldsymbol{v}_0\|_{1,2} = 0\,.$$

We also have.

Theorem 13 (Uniqueness and Continuous Dependence on the Initial Data) *Let $\boldsymbol{v}_0$ be as in* Theorem 12. *Then the corresponding solution is unique in the class* (76)*. Moreover, it depends continuously on $\boldsymbol{v}_0$ in the norm of $L^2(\Omega)$, in the time interval $[0,\tau)$.*

Proof. Let $(\boldsymbol{v},p)$ and $(\boldsymbol{v}+\boldsymbol{u},p+p_1)$ be two solutions corresponding to data $\boldsymbol{v}_0$ and $\boldsymbol{v}_0+\boldsymbol{u}_0$, respectively. From (7)–$(8)_{\rm hom}$ we then find

$$\frac{\partial\boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} + \boldsymbol{v}\cdot\nabla\boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{v} = -\nabla(p_1/\rho) + \nu\Delta\boldsymbol{u}\,,\quad \operatorname{div}\boldsymbol{u}=0\,,\quad \text{a.e. in } \Omega\times(0,\tau)\,. \tag{77}$$

Employing the properties of the function $\boldsymbol{v}$ and $\boldsymbol{u}$, it is not hard to show the following equation

$$\frac12\frac{d}{dt}\|\boldsymbol{u}\|_2^2 + \nu|\boldsymbol{u}|_{1,2}^2 = -(\boldsymbol{u}\cdot\nabla\boldsymbol{v},\boldsymbol{u})\,, \tag{78}$$

that is formally obtained by dot-multiplying both sides of $(77)_1$ by $\boldsymbol{u}$, and by using $(77)_2$ and Lemma 1 along with the fact that $\boldsymbol{u}$ has zero trace at $\partial\Omega$. By Hölder inequality and inequalities (14), (18) and (29), we find

$$|(\boldsymbol{u}\cdot\nabla\boldsymbol{v},\boldsymbol{u})| \le \|\boldsymbol{u}\|_2\|\boldsymbol{u}\|_4|\boldsymbol{v}|_{1,4} \le c_1\|\boldsymbol{u}\|_2|\boldsymbol{u}|_{1,2}\|\boldsymbol{v}\|_{2,2} \le c_2\,\|\boldsymbol{v}\|_{2,2}^2\|\boldsymbol{u}\|_2^2 + \frac{\nu}{2}|\boldsymbol{u}|_{1,2}^2\,,$$

where $c_1 = c_1(\Omega) > 0$ and $c_2 = c_2(\Omega,\nu) > 0$. If we replace this inequality back into (78) we deduce

$$\frac{d}{dt}\|\boldsymbol{u}\|_2^2 \le 2c_2\|\boldsymbol{v}\|_{2,2}^2\|\boldsymbol{u}\|_2^2\,,$$

and so, by Gronwall's lemma and by the fact that $\int_0^t \|\boldsymbol{v}(s)\|_{2,2}^2 ds < \infty$, $t \in [0, \tau)$ (see Theorem 12), we prove the desired result. ■

Finally, we have the following result concerning the regularity of solutions determined in Theorem 12, for whose proof we refer to [39] and [34, Theorem 5.2].

Theorem 14 (Regularity) *Let Ω be a bounded domain of class C^∞. Then, the solution $(\boldsymbol{v}, p)$ constructed in* Theorem 12 *is of class $C^\infty(\overline{\Omega} \times (0, \tau))$.*

Remark 7 The results of Theorem 12–Theorem 14 can be extended to *arbitrary* domains of $\mathbb{R}^n$, provided their boundary is sufficiently smooth; see [50, 34]. Moreover, the continuous dependence result in Theorem 13 can be proved in in the stronger norm of $\mathcal{D}_0^{1,2}(\Omega)$. □

V.2.3 Times of Irregularity and Resolution of Problems 1 and 2 in 2D

Theorem 12–Theorem 14 furnish a complete and positive answer to both Problems 1 and 2, *provided we show that $\tau = \infty$.* Our next task is to give necessary and sufficient conditions for this latter situation to occur. To this end, we give the following.

Definition 6 *Let $(\boldsymbol{v}, p)$ be a solution to* (7)–(8)$_{\rm hom}$ *in the class* (76). *We say that τ is a* time of irregularity *if and only if* (i) $\tau < \infty$, *and* (ii) $(\boldsymbol{v}, p)$ *can not be continued, in the class* (76)*, to an interval $[0, \tau_1)$ with $\tau_1 > \tau$.*

If τ is a time of irregularity, we expect that some norms of the solution may become infinite at $t = \tau$, while being bounded for all $t \in [0, \tau)$. In order to show this rigorously, we premise a simple but useful result.

Lemma 7 *Assume that $\boldsymbol{v}$ is a solution to* (7)–(8)$_{\rm hom}$ *in the class* (76) *for some $\tau > 0$. Then, $|\boldsymbol{v}(t)|_{1,2} < \infty$, for all $t \in [0, \tau)$. Furthermore, for all $q \in (n, \infty]$, and all $t \in [0, \tau)$*

$$\int_0^t \|\boldsymbol{v}(s)\|_q^r \, ds < \infty \,, \quad \frac{2}{r} + \frac{n}{q} = 1 \,.$$

Proof. The proof of the first statement is obvious. Moreover, by the Sobolev embedding theorem (see [69, Theorem at p. 125]) we find, for $n = 2$,

$$\|\boldsymbol{v}\|_q \le c_1 \, \|\boldsymbol{v}\|_2^{2/q} |\boldsymbol{v}|_{1,2}^{1-2/q} \,, \; q \in (2, \infty) \tag{79}$$

and

$$\|\boldsymbol{v}\|_\infty \le c_2 \|\boldsymbol{v}\|_{2,2} \,,$$

with $c_1 = c_1(q) > 0$, $c_2 = c_2(\Omega) > 0$ while, if $n = 3$,

$$\begin{aligned} &\|\boldsymbol{v}\|_q \le c_3, \|\boldsymbol{v}\|_2^{(6-q)/2q} |\boldsymbol{v}|_{1,2}^{3(q-2)/2q} \,, \quad \text{if } q \in [2, 6] \,, \\ &\|\boldsymbol{v}\|_q^{2q/(q-3)} \le c_4 \|\boldsymbol{v}\|_{2,2}^{(q-6)/(q-3)} |\boldsymbol{v}|_{1,2}^{(6+q)/(q-3)} \,, \quad \text{if } q \in (6, \infty] \,, \end{aligned} \tag{80}$$

where $c_i = c_i(\Omega, q) > 0$, $i = 3, 4$. Since, by (76),

$$\sup_{t \in [0,\tau]} \|\boldsymbol{v}(t)\|_2 + |\boldsymbol{v}(t)|_{1,2} + \int_0^t \|\boldsymbol{v}(s)\|_{2,2}^2 ds < \infty \,, \quad t \in [0, \tau) \,,$$

the lemma follows by noting that $(q - 6)/(q - 3) < 2$ for $q > 3$. ■

We shall now furnish some characterization of the possible times of irregularity in terms of the behavior, around them, of the norms of the solution considered in Lemma 7.

Lemma 8 (Criteria for the Existence of a Time of Irregularity) *Let* $(\boldsymbol{v}, p)$ *be a solution to* (7)–(8)$_{\rm hom}$ *in the class* (76) *for some* $\tau \in (0, \infty]$. *Then, the following properties hold:*

(i) *If* τ *is a time of irregularity, then*

$$\lim_{t\to\tau^-} |\boldsymbol{v}(t)|_{1,2} = \infty\,. \tag{81}$$

Conversely, if $\tau < \infty$ *and* (81) *holds, then* τ *is a time of irregularity. Moreover, if* τ *is a time of irregularity, for all* $t \in (0, \tau)$ *the following growth estimates hold*

$$|\boldsymbol{v}(t)|_{1,2}^2 \geq \begin{cases} \dfrac{C}{\tau - t} & \text{if } n = 2\,, \\ \dfrac{C}{\sqrt{\tau - t}} & \text{if } n = 3\,, \end{cases} \tag{82}$$

with $C = C(\Omega, \nu) > 0$.

(ii) *If* τ *is a time of irregularity, then, for all* $q \in (n, \infty]$,

$$\int_0^\tau \|\boldsymbol{v}(s)\|_q^r\, ds = \infty\,, \quad \frac{2}{r} + \frac{n}{q} = 1\,. \tag{83}$$

Conversely, if $\tau < \infty$ *and* (83) *holds for some* $q = \overline{q} \in (n, \infty]$, *then,* τ *is a time of irregularity.*

(iii) *If* $n = 3$, *there exists* $K = K(\Omega, \nu) > 0$ *such that, if* $\|\boldsymbol{v}_0\|_2 |\boldsymbol{v}_0|_{1,2} < K$, *then* $\tau = \infty$.

Proof. (i) Clearly, if $\tau < \infty$ and (81) holds, then τ is a time of irregularity. Conversely, suppose τ is a time of irregularity and assume, by contradiction, that there exists a sequence $\{t_k\}$ in $[0, \tau)$ and $M > 0$, independent of k, such that

$$t_k \to \tau\,, \quad |\boldsymbol{v}(t_k)|_{1,2} \leq M\,.$$

Since $\boldsymbol{v}(t_k) \in \mathcal{D}_0^{1,2}(\Omega)$, by Theorem 12 we may construct a solution $(\overline{\boldsymbol{v}}, \overline{p})$ with initial data $\boldsymbol{v}(t_k)$, in a time interval $[t_k, t_k + \tau^*)$ where (see (67))

$$\tau^* \geq A/|\boldsymbol{v}(t_k)|_{1,2}^\alpha \geq A M^\alpha \equiv \tau_*\,, \quad \alpha = 2(n-1)\,,$$

and A depends only on Ω and ν. By Theorem 12, $\overline{\boldsymbol{v}}$ belongs to the class (76) in the time interval $[t_k, t_k + \tau_*]$, with τ_* independent of k and, by Theorem 13, $\overline{\boldsymbol{v}}$ must coincide with $\boldsymbol{v}$ in the time interval $[t_k, \tau)$. We may now choose $t_k \equiv \tau_0$ such that $\tau_0 + \tau_* > \tau$, contradicting the assumption that τ is time of irregularity. We next show (82) when $n = 3$, the proof for $n = 2$ being completely analogous. Integrating (66) (with $n = 3$) we find

$$\frac{1}{|\boldsymbol{v}(t)|_{1,2}^4} - \frac{1}{|\boldsymbol{v}(s)|_{1,2}^4} \leq c_4 (s - t), \quad 0 < t < s < \tau.$$

Letting $s \to \tau$ and recalling (81), we prove (82).

(ii) Assume that τ is a time of irregularity. Then, (82)$_2$ holds. Now, by the Sobolev embedding theorems, one can show that (see [34, proof of Lemma 5.4])

$$(\boldsymbol{v}\cdot\nabla\boldsymbol{v}, P\Delta\boldsymbol{v}) \leq C\,\|\boldsymbol{v}\|_q^{2q/(q-n)} |\boldsymbol{v}|_{1,2}^2 + \frac{\nu}{2}\|P\Delta\boldsymbol{v}\|_2^2\,, \quad \text{for all } q \in (n, \infty]\,,$$

where $C = C(\Omega, \nu) > 0$. If we replace this relation into (61) and integrate the resulting differential inequality from 0 to $t < \tau$, we find

$$|\boldsymbol{v}(t)|_{1,2}^2 \leq |\boldsymbol{v}_0|_{1,2}^2 \exp\{2C \int_0^t \|\boldsymbol{v}(s)\|_q^r ds\}\,, \quad \text{for all } t \in [0, \tau)\,. \tag{84}$$

If condition (83) is not true for some $q \in (n, \infty]$, then (84) evaluated for that particular q, would contradict (82). Conversely, assume (83) holds for some $q = \overline{q} \in (n, \infty]$, but that, by contradiction, the solution of Theorem 12 can be extended to $[0, \tau_1)$ with $\tau_1 > \tau$. Then, by Lemma 7 we would get the invalidity of condition (83) with $q = \overline{q}$, and the proof of (ii) is completed.

(iii) By integrating the differential inequality $(66)_2$, we find

$$|\boldsymbol{v}(t)|_{1,2}^2 \leq \frac{|\boldsymbol{v}_0|_{1,2}^2}{1 - 2c_4|\boldsymbol{v}_0|_{1,2}^2 \int_0^t |\boldsymbol{v}(s)|_{1,2}^2 ds}\,, \quad t \in [0, \tau)\,.$$

Thus, by (59) with $s = 0$ in this latter inequality we find

$$|\boldsymbol{v}(t)|_{1,2}^2 \leq \frac{|\boldsymbol{v}_0|_{1,2}^2}{1 - c_5|\boldsymbol{v}_0|_{1,2}^2 \|\boldsymbol{v}_0\|_2^2}\,, \quad t \in [0, \tau)\,,$$

with $c_5 = c_5(\Omega, \nu) > 0$, which shows that $(82)_2$ can not occur if the initial data satisfy the imposed "smallness" restriction. ■

A fundamental consequence of Lemma 8 is that, in the case $n = 2$, a time of irregularity can not occur. In fact, for example, (82) is incompatible with the fact that $\boldsymbol{v} \in L^2(0, \tau; \mathcal{D}_0^{1,2}(\Omega))$ (see $(75)_1$). We thus have the following theorem, which answers positively both Problems 1 and 2 in the 2D case.

Theorem 15 (Resolution of Problems 1 and 2 in 2D) *Let $\Omega \subset \mathbb{R}^2$. Then, in* Theorem 12–Theorem 14 *we can take $\tau = \infty$.*

The conclusion of Theorem 15 also follows from Lemma 8(ii). In fact, in the case $n = 2$, by (79) we at once find that

$$L^{2q/(q-2)}(0, T : L^q(\Omega)) \subset L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; \mathcal{D}_0^{1,2}(\Omega))\,, \quad \text{for all } T > 0 \text{ and all } q \in (2, \infty)\,. \tag{85}$$

so that, from $(76)_1$, we deduce

$$\int_0^\tau \|\boldsymbol{v}(t)\|_q^{2q/(2-q)} dt < \infty\,, \quad \text{for all } q \in (2, \infty)\,.$$

which, by Lemma 8(ii), excludes the occurence of time of irregularity.

Unfortunately, from all we know, in the case $n = 3$, we can not draw the same conclusion. Actually, in such a case, $(82)_2$ and $(77)_1$ are no longer incompatible. Moreover, from Lemma 8(ii) it follows that a sufficient condition for τ *not* to be a time of irregularity is that

$$\int_0^\tau \|\boldsymbol{v}(t)\|_q^r dt < \infty\,, \quad \frac{2}{r} + \frac{3}{q} = 1\,, \quad \text{some } q \in (3, \infty]\,. \tag{86}$$

However, from $(80)_1$ and from $(76)_1$, it is immediately verified that, in the case $n = 3$, the solutions constructed in Theorem 12 satisfy the following condition

$$\int_0^\tau \|\boldsymbol{v}(t)\|_q^r dt < \infty\,, \quad \frac{2}{r} + \frac{3}{q} = 1 + \frac{1}{2}\,, \quad \text{all } q \in [2, 6]\,. \tag{87}$$

Therefore, in view of Lemma 8(iii), the best conclusion we can draw is that for 3D flow Problems 1 and 2 can be positively answered *if the size of the initial data is suitably restricted.*

Remark 8 In the case $n = 3$, besides (86), one may furnish other sufficient conditions for the absence of a time of irregularity. We refer, among others, to the papers [6, 15, 56, 18, 80, 8, 57, 70]. In particular, we would like to direct attention to the work [18, 80], where the difficult borderline case $q = n = 3$ in condition (86) is worked out by completely different methods than those used here. □

Open Question. *In the case $n = 3$, it is not known whether or not condition* (86) *(or any of the other conditions referred to in* Remark 8*) holds along solutions of* Theorem 12.

V.3 Less Regular Solutions and Partial Regularity Results in 3D

As shown in the previous section, we do not know if, for 3D flow, the solutions of Theorem 12 exist in an arbitrarily large time interval, without restricting the magnitude of the initial data: they are *local solutions*. However, following the line of thought introduced by J. Leray [62], we may extend them to solutions defined for all times, for initial data of arbitrary magnitude, namely, to *global solutions*, but belonging to a functional class, $\mathcal{C}$, *a priori* less regular than that given in (76) (*weak solutions*). Thus, if, besides existence, we could prove in $\mathcal{C}$ also uniqueness and continuous dependence, Problem 1 would receive a positive answer. Unfortunately, to date, the class where global solutions are proved to exist is, in principle, too large to secure the validity of these latter two properties and some extra assumptions are needed. Alternatively, in relation to Problem 2, one may investigate the "size" of the space-time regions where these generalized solutions may (possibly) become irregular. As a matter of fact, singularities, if they at all occur, have to be concentrated within "small" sets of space-time. Our objective in this section is to discuss the above issues and to present the main results.

For future purposes, we shall present some of these results also in space dimension $n = 2$, even though, as shown in Theorem 15, in this case, both Problems 1 and 2 are answered in the affirmative.

V.3.1 Weak Solutions and Related Properties

We begin to introduce the corresponding definition of weak solutions in the sense of Leray-Hopf [62, 54]. By formally dot-multiplying through both sides of (7) by $\boldsymbol{\varphi} \in \mathcal{D}(\Omega)$ and by integrating by parts over Ω, with the help of (12) we find

$$\frac{d}{dt}(\boldsymbol{v}(t), \boldsymbol{\varphi}) + \nu(\nabla \boldsymbol{v}(t), \nabla \boldsymbol{\varphi}) + (\boldsymbol{v}(t) \cdot \nabla \boldsymbol{v}(t), \boldsymbol{\varphi}) = 0\,, \quad \text{for all } \boldsymbol{\varphi} \in \mathcal{D}(\Omega)\,. \tag{88}$$

Definition 7 *Let* $\Omega \subset \mathbb{R}^n$, $n = 2, 3$. *A field* $\boldsymbol{v} : \Omega \times (0, \infty) \mapsto \mathbb{R}^n$ *is a* weak solution *to* (7)–(8)$_{\rm hom}$ *if and only if:* (i) $\boldsymbol{v} \in L^\infty(0, T; L^2_\sigma(\Omega)) \cap L^2(0, T; \mathcal{D}^{1,2}_0(\Omega))$, *for all* $T > 0$; (ii) $\boldsymbol{v}$ *satisfies* (88) *for a.a.* $t \geq 0$, *and* (iii) $\displaystyle\lim_{t \to 0^+} (\boldsymbol{v}(t) - \boldsymbol{v}_0, \boldsymbol{\varphi}) = 0$, *for all* $\boldsymbol{\varphi} \in \mathcal{D}(\Omega)$.

A. Existence. The proof of existence of weak solutions is easily carried out, for example, by the finite-dimensional (Galerkin) method indicated in Subsection V.2.2. This time, however, along the "approximate" solutions $\boldsymbol{v}_N$ to (74), we only use the estimate corresponding to the energy equality (59). We thus obtain

$$\|\boldsymbol{v}_N(t)\|_2^2 + 2\nu \int_s^t |\boldsymbol{v}_N(\rho)|_{1,2}^2 d\rho = \|\boldsymbol{v}_N(s)\|_2^2\,, \quad 0 \leq s \leq t\,. \tag{89}$$

As we already emphasized, the important feature of this estimate is that it holds for *all* $t \geq 0$ *and all data* $v_0 \in L^2_\sigma(\Omega)$. From (89) it follows, in particular, that the sequence $\{\boldsymbol{v}_N\}$ is uniformly bounded in the class of function specified in (i) of Definition 7. Using this fact together with classical weak and strong compactness arguments in (74), we can show the existence of at least one subsequence converging, in suitable topologies, to a weak solution $\boldsymbol{v}$. We then have the following result for whose complete proof we refer, *e.g.*, to [34, Theorem 3.1].

Theorem 16 *For any* $\boldsymbol{v}_0 \in L^2_\sigma(\Omega)$ *there exists at least one weak solution to* (7)–(8)$_{\rm hom}$. *This solution verifies, in addition, the following properties.*

(i) *The energy inequality:*

$$\|\boldsymbol{v}(t)\|_2^2 + 2\nu \int_s^t |\boldsymbol{v}(\rho)|_{1,2}^2 d\rho \leq \|\boldsymbol{v}(s)\|_2^2\,, \quad \textit{for a.a. } s \in [0, \infty),\ 0 \textit{ included, and all } t \geq s. \tag{90}$$

(ii) $\displaystyle\lim_{t \to 0^+} \|\boldsymbol{v}(t) - \boldsymbol{v}_0\|_2 = 0$.

B. On the Energy Equality. In the case $n = 3$, weak solutions in Theorem 16 only satisfy the energy *inequality* (90) instead of the energy *equality* (see (59)). (For the case $n = 2$, see Remark 9.) This is an undesired feature that is questionable from the physical viewpoint. As a matter of fact, for fixed $s \geq 0$, in time intervals $[s, t]$ where (90) were to hold as a *strict* inequality, the kinetic energy would decrease by an amount which is not only due to the dissipation. From a strictly technical viewpoint, this happens because the convergence of (a subsequence of) the sequence $\{\boldsymbol{v}_N\}$ to the weak solution $\boldsymbol{v}$ can be proved only in the weak topology of $L^2(0, T; \mathcal{D}_0^{1,2}(\Omega))$ and this only ensures that, as $N \to \infty$, the second term on the left-hand side of (89) tends to a quantity not less than the one given by the second term on the left-hand side of (90). One may think that this circumstance is due to the special method used for constructing the weak solutions. Actually, this is not the case because, in fact, we have the following.

Open Question. *If $n = 3$, it is not known if there are solutions satisfying* (90) *with the equality sign and corresponding to initial data $\boldsymbol{v}_0 \in L^2_\sigma(\Omega)$ of unrestricted magnitude.*

Remark 9 A sufficient condition for a weak solution, $\boldsymbol{v}$, to satisfy the energy equality (59) is that $\boldsymbol{v} \in L^4(0, t; L^4(\Omega))$, for all $t > 0$ [34, Theorem 4.1]. Consequently, from (85) it follows that, if $n = 2$, weak solutions satisfy (59) for all $t > 0$. Moreover, from $(80)_1$, we find that the solutions of Theorem 12, for $n = 3$, satisfy the energy equality (59), at least for all $t \in [0, \tau)$. □

Remark 10 For future purposes, we observe that the definition of weak solution and the results of Theorem 16 can be extended easily to the case when $\boldsymbol{f} \not\equiv \mathbf{0}$. In fact, it is enough to change Definition 7 by requiring that $\boldsymbol{v}$ satisfies the modification of (88) obtained by adding to its right-hand side the term $(\boldsymbol{f}, \boldsymbol{\varphi})$. Then, if $\boldsymbol{f} \in L^2(0, T; \mathcal{D}_0^{-1,2}(\Omega))$, for all $T > 0$, one can show the existence of a weak solution satisfying condition (ii) of Theorem 16 and the variant of (90) obtained by adding the term $\int_0^t (\boldsymbol{f}, \boldsymbol{v})\, ds$ on its right-hand side. □

C. Uniqueness and Continuous Dependence. The following result, due to Serrin [83, Theorem 6] and Sather [75, Theorem 5.1], is based on ideas of Leray [62, §32] and Prodi [71]. A detailed proof is given in [34, Theorem 4.2].

Theorem 17 *Let $\boldsymbol{v}$, $\boldsymbol{u}$ be two weak solutions corresponding to data $\boldsymbol{v}_0$ and $\boldsymbol{u}_0$. Assume that $\boldsymbol{u}$ satisfies the energy inequality* (90) *with $s = 0$, and that*

$$\boldsymbol{v} \in L^r(0, T; L^q(\Omega)), \quad \textit{for some } q \in (n, \infty] \textit{ such that } \frac{2}{r} + \frac{n}{q} = 1. \tag{91}$$

Then,

$$\|\boldsymbol{v}(t) - \boldsymbol{u}(t)\|_2^2 \leq C\, \|\boldsymbol{v}_0 - \boldsymbol{u}_0\|_2^2 \exp\{\int_0^t \|\boldsymbol{v}(\rho)\|_q^r d\rho\}, \quad \textit{for all } t \in [0, T]$$

where $C = C(\Omega, \nu) > 0$. Thus, in particular, if $\boldsymbol{v}_0 = \boldsymbol{u}_0$, then $\boldsymbol{v} = \boldsymbol{u}$ a.e. in $\Omega \times [0, T]$.

Remark 11 If $n = 2$, from (85) and Remark 9 we find that every weak solution satisfies the assumptions of Theorem 17. Therefore, in such a case, every weak solution is unique in the class of weak solutions and depends continuously upon the data. Furthermore, if $n = 3$, the uniqueness result continues to hold if in condition (91), we take $q = n = 3$; see [87, 55]. □

Open Question. While the existence of weak solutions $\boldsymbol{u}$ satisfying the hypothesis Theorem 17 is secured by Theorem 16, in the case $n = 3$ it is not known if there exist weak solutions having the property stated for $\boldsymbol{v}$. (In principle, as a consequence of Definition 7(i) and $(80)_1$, $\boldsymbol{v}$ only satisfies (87).) Consequently *in the case $n = 3$ uniqueness and continuous dependence in the class of weak solution remains open, and so does the resolution of* Problem 1.

Remark 12 As a matter of fact, weak solutions possess more regularity than that implied by their very definition. Actually, if $n = 2$, they are indeed smooth (see Remark 13). If $n = 3$, by means of sharp estimates for solutions to the linear problem –obtained from (7)–(8) by neglecting the nonlinear term $\boldsymbol{v} \cdot \nabla \boldsymbol{v}$ (*Stokes problem*)– one can show that every corresponding weak solution satisfies the following additional properties (see [43, Theorem 3.1], [87, 3.4 Theorem])

$$\frac{\partial \boldsymbol{v}}{\partial t} \in L^l(\delta, T; L^s(\Omega)) \,, \quad \boldsymbol{v} \in L^l(\delta, T; W^{2,s}(\Omega)) \,, \quad \text{for all } T > 0 \text{ and all } \delta \in (0, T) \,,$$

where the numbers l, s obey the following conditions

$$\frac{2}{l} + \frac{3}{s} \geq 4 \,, \quad l \in [1, 2] \,, \quad s \in [1, \frac{3}{2}] \,.$$

Moreover, there exists $p \in L^l(\delta, T; W^{1,s}(\Omega)) \cap L^l(\delta, T; L^{3s/(3-s)}(\Omega))$ such that the pair $(\boldsymbol{v}, p)$ satisfies (7) for a.a. $(\boldsymbol{x}, t) \in \Omega \times (0, \infty)$. If, in addition, $\boldsymbol{v}_0$ lies in a sufficiently regular subspace of $L^2_\sigma(\Omega)$, we can take $\delta = 0$. However, the above properties are still not enough to ensure the validity of condition (91). Weak solutions enjoying further regularity properties are constructed in [21, Theorem 3.1 and Corollary 3.2], [14] and [65]. □

D. Partial Regularity and "Suitable" Weak Solutions. A problem of fundamental importance is to investigate the set of space-time where weak solutions may possibly become irregular, and to give an estimate of how "big" this set can be.

To this end, we recall that, for a given $S \subset \mathbb{R}^{d+1}$, $d \in \mathbb{N} \cup \{0\}$, and $\kappa \in (0, \infty)$, the *κ-dimensional (spherical) Hausdorff measure $\mathcal{H}^\kappa$* of S is defined as

$$\mathcal{H}^\kappa(S) = \lim_{\delta \to 0} \mathcal{H}^\kappa_\delta(S),$$

where $\mathcal{H}^\kappa_\delta(S) = \inf \sum_i r_i^\kappa$, the infimum being taken over all at most countable coverings $\{B_i\}$ of S of closed balls $B_i \subset \mathbb{R}^d$ of radius r_i with $r_i < \delta$; see, *e.g.*, [20]. If $d \in \mathbb{N}$, the κ-dimensional *parabolic* Hausdorff measure $\mathcal{P}^\kappa$ of S is defined as above, by replacing the ball B_i with a *parabolic cylinder* of radius r_i:

$$Q_{r_i}(x, t) = \{(y, s) \in \mathbb{R}^d \times \mathbb{R} : \ |y - x| < r_i \,, \ |s - t| < r_i^2\} \,. \tag{92}$$

In general, it is $\mathcal{H}^\kappa(S) \leq C \mathcal{P}^\kappa(S)$, $C > 0$; see [20, §2.10.1] for details.

The following lemma is a direct consequence of the preceding definition.

Lemma 9 *For any $S \subset \mathbb{R}^{d+1}$, $d \in \mathbb{N} \cup \{0\}$ (respectively, $d \in \mathbb{N}$), we have $\mathcal{H}^\kappa(S) = 0$ (respectively, $\mathcal{P}^\kappa(S) = 0$) if and only if, for each $\delta > 0$, S can be covered by closed balls $\{B_i\}$ (respectively, parabolic cylinders $\{Q_i\}$) of radii r_i, $i \in \mathbb{N}$, such that $\sum_{i=1}^\infty r_i^\kappa < \delta$.*

We begin to consider the collection of times where weak solutions are (possibly) not smooth and show that they constitute a very "small" region of $(0, \infty)$. Specifically, we have the following result, basically, due to Leray [62, pp. 244-245] and completed by Scheffer [76].

Theorem 18 *Let Ω be a bounded domain of class C^∞. Assume $\boldsymbol{v}$ is a weak solution determined in* Theorem 16. *Then, there exists a union of disjoint and, at most, countable open time intervals $\mathcal{T} = \mathcal{T}(\boldsymbol{v}) \subset (0, \infty)$ such that:*

(i) *$\boldsymbol{v}$ is of class C^∞ in $\overline{\Omega} \times \mathcal{T}$,*

(ii) *There exists $T^* \in (0, \infty)$ such that $\mathcal{T} \supset (T^*, \infty)$;*

(iii) *If* $\boldsymbol{v}_0 \in \mathcal{D}_0^{1,2}(\Omega)$ *then* $\mathcal{T} \supset (0, T_1)$ *for some* $T_1 > 0$;

(iv) *Let* (s, τ) *be a generic bounded interval in* $\mathcal{T}(\boldsymbol{v})$ *and suppose* $\boldsymbol{v} \notin C^\infty(\overline{\Omega} \times (s, \tau_1))$, $\tau_1 > \tau$. *Then, both following conditions must hold*

$$|\boldsymbol{v}(t)|_{1,2}^2 \geq \frac{C}{(\tau - t)^{1/2}}\,, \quad t \in (s, \tau) \quad \textit{and} \quad \lim_{t \to \tau^-} \int_s^t \|\boldsymbol{v}(s)\|_q^{2q/(q-3)} ds = \infty\,, \quad \textit{for all } q > 3\,, \tag{93}$$

where $C = C(\Omega, \nu) > 0$;

(v) *The* $\frac{1}{2}$*-dimensional Hausdorff measure of* $\mathcal{I}(\boldsymbol{v}) := (0, \infty) - \mathcal{T}$ *is zero;*

Proof. By (90) we may select $T^* > 0$ with the following properties: (a) $\|\boldsymbol{v}(T^*)\|_2 |\boldsymbol{v}(T^*)|_{1,2} < K$, and (b) the energy inequality (90) holds with $s = T^*$, where K the constant introduced in Lemma 8(iii). Let us denote by $\widetilde{\boldsymbol{v}}$ the solution of Theorem 12 corresponding to the data $\boldsymbol{v}(T^*)$. By Lemma 8(iii), $\widetilde{\boldsymbol{v}}$ exists for all times $t \geq T^*$ and, by Theorem 14, it is of class C^∞ in $\Omega \times (T^*, \infty)$. By Lemma 7 and by Theorem 17 we must have $\boldsymbol{v} \equiv \widetilde{\boldsymbol{v}}$ in $\Omega \times (T^*, \infty)$, and part (iii) is proved. Next, denote by I the set of those $t \in [0, T^*)$ such that (a) $\|\boldsymbol{v}(t)\|_{1,2} < \infty$, and (b) the energy inequality (90) holds with $s \in I$. Clearly, $[0, T^*] - I$ is of zero Lebesgue measure. Moreover, for every $t_0 \in I$ we can construct in $(t_0, t_0 + T(t_0))$ a solution $\widetilde{\boldsymbol{v}}$ assuming at t_0 the initial data $\boldsymbol{v}(t_0)$ $(\in \mathcal{D}_0^{1,2}(\Omega))$; see Theorem 12. From Theorem 14, Lemma 7 and Theorem 17, we know that $\widetilde{\boldsymbol{v}}$ is of class C^∞ in $\Omega \times (t_0, t_0 + T(t_0))$ and that it coincides with $\boldsymbol{v}$, since this latter satisfies the energy inequality with $s = t_0$. Furthermore, if $\boldsymbol{v}_0 \in \mathcal{D}_0^{1,2}(\Omega)$, then $0 \in I$. Properties (ii)–(iv) thus follow with $\mathcal{T} \equiv \bigcup_{i \in \mathfrak{I}} (s_i, \tau_i) \cup (T^*, \infty)$, where (s_i, τ_i) are the connected components in I. Notice that

$$(s_i, \tau_i) \subset [0, T^*], \quad \text{for all } i \in \mathfrak{I}; \quad (s_i, \tau_i) \cap (s_j, \tau_j) = \emptyset, \quad i \neq j\,, \tag{94}$$

and that, moreover, the (1-dimensional) Lebesgue measure of $\mathcal{I} := \mathcal{T} - (0, \infty)$ is 0. Finally, property (iv) is an immediate consequence of Lemma 8 and Theorem 14. It remains to show (v). From (iv) and (90) we find

$$\sum_{i \in \mathfrak{I}} (\tau_i - s_i)^{1/2} \leq 1/(2C) \sum_{i \in \mathfrak{I}} \int_{\tau_i}^{s_i} \|\nabla \boldsymbol{v}(\tau)\|_2^2 dt \leq \|\boldsymbol{v}_0\|_2^2/(4C).$$

Thus, for every $\delta > 0$ we can find a finite part $\mathfrak{I}_\delta$ of $\mathfrak{I}$ such that

$$\sum_{i \notin \mathfrak{I}_\delta} (\tau_i - s_i) < \delta, \qquad \sum_{i \notin \mathfrak{I}_\delta} (\tau_i - s_i)^{1/2} < \delta. \tag{95}$$

By $(94)_1$, $\cup_{i \in \mathfrak{I}} (s_i, \tau_i) \subset [0, T^*]$ and so the set $[0, T^*] - \cup_{i \in \mathfrak{I}_\delta} (s_i, \tau_i)$ consists of a finite number of disjoint closed intervals B_j, $j = 1, \ldots, N$. Clearly,

$$\bigcup_{j=1}^N B_j \supset \mathcal{I}(\boldsymbol{v}). \tag{96}$$

By $(94)_2$, each interval (s_i, τ_i), $i \notin \mathfrak{I}_\delta$, is included in one and only one B_j. Denote by $\mathfrak{I}_j$ the set of all indeces i satisfying $B_j \supset (s_i, \tau_i)$. We thus have

$$\mathfrak{I} = \mathfrak{I}_\delta \cup \left(\bigcup_{j=1}^N \mathfrak{I}_j \right)\,, \quad B_j = \left(\bigcup_{i \in \mathfrak{I}_j} (s_i, \tau_i) \right) \cup (B_j \cap \mathcal{I}(\boldsymbol{v}))\,. \tag{97}$$

Since $\mathcal{I}$ has zero Lebesgue measure, from $(97)_2$ we have $\operatorname{diam} B_j = \sum_{i\in\Im_j}(\tau_i - s_i)$. Thus, by (95) and $(97)_1$,

$$\operatorname{diam} B_j \le \sum_{i\notin\Im_\delta}(\tau_i - s_i) < \delta \tag{98}$$

and, again by (95) and $(97)_1$,

$$\sum_{j=1}^{N}(\operatorname{diam} B_j)^{1/2} \le \sum_{j=1}^{N}\left(\sum_{i\in\Im_j}(\tau_i - s_i)\right)^{1/2} \le \sum_{i\notin\Im_\delta}(\tau_i - s_i)^{1/2} < \delta. \tag{99}$$

Therefore, property (v) follows from (96), (98), (99) and Lemma 9. ■

Remark 13 If Ω is of class C^∞, from (85), Remark 11 and from Theorem 18(v) we at once obtain that, for $n = 2$, every weak solution is of class $C^\infty(\overline{\Omega}\times(0,t))$, for all $t > 0$. □

We shall next analyze, in more details, the set of points where weak solutions may possibly lose regularity. In view of Remark 13, *we shall restrict ourselves to consider the case $n = 3$ only.*

Let (s,τ) be a bounded interval in $\mathcal{T}(\boldsymbol{v})$ and assume that, at $t = \tau$, the weak solution $\boldsymbol{v}$ becomes irregular. We may wish to estimate the spatial set $\Sigma = \Sigma(\tau) \subseteq \Omega$ where $\boldsymbol{v}(\tau)$ becomes irregular. By defining Σ as the set of $\boldsymbol{x}\in\Omega$ where $\boldsymbol{v}(\boldsymbol{x},\tau)$ is not continuous, in the case $\Omega = \mathbb{R}^3$, Scheffer has shown that $\mathcal{H}^1(\Sigma) < \infty$ [77].

More generally, one may wish to estimate the "size" of the region of space-time where points of irregularity (appropriately defined, see Definition 9 below) may occur. This study, initiated by Scheffer [77, 78] and continued and deepened by Caffarelli, Kohn and Nirenberg [11], can be performed in a class of solutions called *suitable weak solutions* which, in principle, due to the lack of an adequate uniqueness theory, is more restricted than that of weak solutions.

Definition 8 *A pair $(\boldsymbol{v},p)$ is called a* suitable weak solution *to* (7)–$(8)_{\rm hom}$ *if and only if:* (i) $\boldsymbol{v}$ *satisfies* Definition 7(i) *and* $p\in L^{3/2}((0,T);L^{3/2}(\Omega))$*, for all* $T>0$*;* (ii) $(\boldsymbol{v},p)$ *satisfies*

$$\frac{d}{dt}(\boldsymbol{v}(t),\boldsymbol{\psi}) + \nu(\nabla\boldsymbol{v}(t),\nabla\boldsymbol{\psi}) + (\boldsymbol{v}(t)\cdot\nabla\boldsymbol{v}(t),\boldsymbol{\psi}) - (p(t),\operatorname{div}\boldsymbol{\psi}) = 0\,,\quad \textit{for all } \boldsymbol{\psi}\in C_0^\infty(\Omega)\,,$$

and (iii) $(\boldsymbol{v},p)$ *obeys the following* localized energy inequality

$$2\int_0^T\int_\Omega |\nabla\boldsymbol{v}|^2\boldsymbol{\phi}\,d\boldsymbol{x}dt \le \int_0^T\int_\Omega\{|\boldsymbol{v}|^2(\frac{\partial\boldsymbol{\phi}}{\partial t}+\Delta\boldsymbol{\phi}) + (|\boldsymbol{v}|^2+2p)\boldsymbol{v}\cdot\nabla\boldsymbol{\phi}\}d\boldsymbol{x}dt\,,$$

for all non-negative $\boldsymbol{\phi}\in C_0^\infty(\Omega\times(0,T))$.

Remark 14 By taking $l = 3/2$, $s = 9/8$ in Remark 12, it follows that *every* weak solution, corresponding to sufficiently regular initial data, matches requirements (i) and (ii) of Definition 8 (recall that Ω is bounded). However, it is not known, to date, if it satisfies also condition (iii). Moreover, it is not clear if the finite-dimensional (Galerkin) method used for Theorem 16 is appropriate to construct solutions obeying such a condition (see, [47] for a partial answer). Nevertheless, by using different methods, one can show the existence of at least one weak solution satisfying the properties stated in Theorem 16, and which, in addition, is a suitable weak solution; see [11, Theorem A.1], [5], [64, Theorem 2.2]. □

Definition 9 *A point $P := (\boldsymbol{x}, t) \in \Omega_T := \Omega \times (0, T)$ is called* regular *for a suitable weak solution $(\boldsymbol{v}, p)$, if and only if there exists a neighborhood, I, of P such that $\boldsymbol{v}$ is in $L^\infty(I)$. A point which is not regular will be called* irregular.

Remark 15 The above definition of regular point is reinforced by a result of Serrin [82], from which we deduce that, *in the neighborhood of every regular point, a suitable weak solution is, in fact, of class C^∞ in the space variables*.

The next result is crucial in assessing the "size" of the set of possible irregular points in the space-time domain. For its proof, we refer to [64], [60, Theorem 2.2], [11, Proposition 2]. We recall that $Q_r(\boldsymbol{x}, t)$ is defined in (92).

Lemma 10 *Let $(\boldsymbol{v}, p)$ be a suitable weak solution and let $(\boldsymbol{x}, t) \in \Omega_T$. There exists $K > 0$ such that, if*

$$\limsup_{r \to 0} r^{-1} \int_{Q_r(\boldsymbol{x},t)} |\nabla \boldsymbol{v}(\boldsymbol{y}, s)|^2 d\boldsymbol{y}\, ds < K\,, \tag{100}$$

then $(\boldsymbol{x}, t)$ is regular.

Now, let let $\mathcal{S} = \mathcal{S}(\boldsymbol{v}, p) \subseteq \Omega \times (0, T)$ be the set of possible irregular points for a suitable weak solution $(\boldsymbol{v}, p)$, and let V be a neighborhood of $\mathcal{S}$. By Lemma 10 we then have that, for each $\delta > 0$, there is $Q_r(\boldsymbol{x}, t) \subset V$ with $r < \delta$, such that

$$K^{-1} \int_{Q_r(\boldsymbol{x},t)} |\nabla \boldsymbol{v}(\boldsymbol{y}, s)|^2 d\boldsymbol{y}\, ds > r\,. \tag{101}$$

Let $\mathcal{Q} = \{Q_r(\boldsymbol{x}, t)\}$ be the collection of all Q satisfying this property. Since $\Omega \times (0, T)$ is bounded, from [11, Lemma 6.1] we can find an at most countable, disjoint subfamily of $\mathcal{Q}$, $\{Q_{r_i}(\boldsymbol{x}_i, t_i)\}$, such that $S \subset \cup_{i \in \mathfrak{I}} Q_{5r_i}(\boldsymbol{x}_i, t_i)$. From (101) it follows, in particular, that

$$\sum_{i \in \mathfrak{I}} r_i \leq K^{-1} \sum_{i \in \mathfrak{I}} \int_{Q_{r_i}(\boldsymbol{x}_i, t_i)} |\nabla \boldsymbol{v}(\boldsymbol{y}, s)|^2 d\boldsymbol{y}\, ds \leq K^{-1} \int_V |\nabla \boldsymbol{v}|^2\,. \tag{102}$$

Since δ is arbitrary, (102) implies, on the one hand, that $\mathcal{S}$ is of zero Lebesgue measure and, on the other hand, that

$$\mathcal{P}^1(S) \leq \frac{5}{K} \int_V |\nabla \boldsymbol{v}|^2\,,$$

for every neighborhood V of $\mathcal{S}$. Thus, by the absolute continuity of the Lebesgue integral, from this latter inequality and from Lemma 9 we have the following [11, Theorem B].

Theorem 19 *Let $(\boldsymbol{v}, p)$ be a suitable weak solution and let $\mathcal{S} = \mathcal{S}(\boldsymbol{v}, p) \subseteq \Omega \times (0, T)$ be the corresponding set of possible irregular points. Then $\mathcal{P}^1(\mathcal{S}) = 0$.*

Remark 16 Let

$$\boldsymbol{\mathcal{Z}} = \mathcal{Z}(\boldsymbol{v}, p) := \{t \in (0, T) :\ (\boldsymbol{x}, t) \in \mathcal{S}(\boldsymbol{v}, p)\,,\ \text{for some } \boldsymbol{x} \in \Omega\}\,.$$

Clearly, $t \in \mathcal{Z}$ if and only if $\boldsymbol{v}$ becomes essentially unbounded around $(\boldsymbol{x}, t)$, for some $\boldsymbol{x} \in \Omega$. Namely, for any $M > 0$ there is a neighborhood of $(\boldsymbol{x}, t)$, I_M, such that $|\boldsymbol{v}(\boldsymbol{y}, s)| > M$ for a.a. $(\boldsymbol{y}, s) \in I_M$. From Theorem 18(ii) we deduce at once that $\mathcal{Z}(\boldsymbol{v}, p) \subset \mathcal{I}(\boldsymbol{v})$, where $\mathcal{I}(\boldsymbol{v})$ is the set of all possible times of irregularity; see Definition 6. Thus, from Theorem 18(i) we find $\mathcal{H}^{1/2}(\mathcal{Z}) = 0$. □

Remark 17 There is a number of papers dedicated to the formulation of sufficient conditions for the absence of irregular points, $(\boldsymbol{x}, t)$, for a suitable weak solution, when $\boldsymbol{x}$ is either an interior or a boundary point. In this latter case, the definition of regular point as well as that of suitable weak solution must be, of course, appropriately modified. Among others, we refer to [11, 64, 81, 49, 48]. □

V.4 Long-Time Behavior and Existence of the Global Attractor

Suppose a viscous liquid moves in a fixed spatial bounded domain, Ω, under the action of a given time-independent driving mechanism, m, and denote by $\lambda > 0$ a non-dimensional parameter measuring the "magnitude" of m. To fix the ideas, we shall assume that the velocity field of the liquid, $\boldsymbol{v}_1$, vanishes at $\partial\Omega$, for each time $t \geq 0$, This assumption is made for the sake of simplicity. All main results can be extended to the case $\boldsymbol{v}_1 \not\equiv \boldsymbol{0}$, provided $\boldsymbol{v}_1(\boldsymbol{x}, t) \cdot \boldsymbol{n}|_{\partial\Omega} = 0$, for all $t \geq 0$, where $\boldsymbol{n}$ is the unit normal on $\partial\Omega$. We then take m to be a (non-conservative) time-independent body force, $\boldsymbol{f} \in L^2(\Omega)$ with $\lambda \sim \|\boldsymbol{f}\|_2$. We shall denote by (7)–(8)$_{\mathrm{Hom}}$ the initial-boundary value problem (7)–(8) with $\boldsymbol{v}_1 \equiv \boldsymbol{0}$. As we know from Theorem 3 and Remark 5, if λ is "sufficiently" small, less than λ_c, say, there exists one and only one (steady-state) solution, $(\overline{\boldsymbol{v}}, \overline{p})$ to the boundary-value problem (9)–(10) (with $\boldsymbol{v}_* \equiv \boldsymbol{0}$). Actually, it is easy to show that, with the above restriction on λ, every solution to the initial-boundary value problem (8)–(11)$_{\mathrm{Hom}}$, belonging to a sufficiently regular function class and corresponding to the given $\boldsymbol{f}$ and to arbitrary $\boldsymbol{v}_0 \in L^2_\sigma(\Omega)$, decays exponentially fast in time to $(\overline{\boldsymbol{v}}, \overline{p})$. In fact, by setting $\boldsymbol{u} := \boldsymbol{v} - \overline{\boldsymbol{v}}$, $P := p - \overline{p}$, from (7)–(10) we find

$$
\left.\begin{aligned}
&\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \overline{\boldsymbol{v}} + \overline{\boldsymbol{v}} \cdot \nabla \boldsymbol{u} = -\nabla P + \nu \Delta \boldsymbol{u}\,,\\
&\operatorname{div} \boldsymbol{u} = 0
\end{aligned}\right\} \text{ in } \Omega \times (0, T)\,;
$$
$$
\boldsymbol{u}(\boldsymbol{x}, 0) = \boldsymbol{v}_0(\boldsymbol{x}) - \overline{\boldsymbol{v}}(\boldsymbol{x})\,, \quad \boldsymbol{x} \in \Omega\,; \quad \boldsymbol{u} = \boldsymbol{0} \text{ at } \partial\Omega \times (0, T)\,. \tag{103}
$$

If we formally dot-multiply through both sides of (103)$_1$ by $\boldsymbol{u}$, integrate by parts over Ω and take into account (12) and Lemma 1, we obtain

$$
\frac{1}{2}\frac{d}{dt}\|\boldsymbol{u}\|_2^2 + \nu |\boldsymbol{u}|_{1,2}^2 = -(\boldsymbol{u} \cdot \nabla \overline{\boldsymbol{v}}, \boldsymbol{u})\,. \tag{104}
$$

From Lemma 1, (21) and (18), we find $|(\boldsymbol{u} \cdot \nabla \overline{\boldsymbol{v}}, \boldsymbol{u})| \leq c_1\, |\boldsymbol{u}|_{1,2}^2 |\overline{\boldsymbol{v}}|_{1,2}$, with $c_1 = c_1(\Omega) > 0$. Thus, by Remark 5, it follows that

$$
|(\boldsymbol{u} \cdot \nabla \overline{\boldsymbol{v}}, \boldsymbol{u})| \leq \frac{c_2}{\nu}\, \|\boldsymbol{f}\|_2 |\boldsymbol{u}|_{1,2}^2\,,
$$

with $c_2 = c_2(\Omega) > 0$, which, in turn, once replaced in (104), furnishes

$$
\frac{1}{2}\frac{d}{dt}\|\boldsymbol{u}\|_2^2 + \left(\nu - \frac{c_2}{\nu}\, \|\boldsymbol{f}\|_2\right) |\boldsymbol{u}|_{1,2}^2 \leq 0\,.
$$

Therefore, if $\gamma := \nu - (c_2/\nu)\|\boldsymbol{f}\|_2 > 0$, from (18) and from the latter displayed equation we deduce

$$
\|\boldsymbol{u}(t)\|_2^2 \leq \|\boldsymbol{u}(0)\|_2^2\, e^{-2(\gamma/C_P)\,t}\,, \tag{105}
$$

which gives the desired result. Estimate (105) can be read, in "physical terms" in the following way: after a certain amount of time (depending on how close γ is to 0, namely, on how close λ is to λ_c), the transient motion will die out exponentially fast and the "true" dynamics of the fluid will be described by the unique steady-state flow corresponding to the given force $\boldsymbol{f}$. From a mathematical point of view, the steady-state $(\overline{\boldsymbol{v}}, \overline{p})$ is, in a suitable function space, a one-point set which is invariant under the flow. Now, let us increase λ higher and higher beyond λ_c. Following Eberhard Hopf [53], we expect that, after a while, the transient motion will yet die out, and that the generic flow will approach a certain manifold, $\mathfrak{M} = \mathfrak{M}(\lambda)$ which need not reduce to a single point. (There are, however, explicit examples where $\mathfrak{M}(\lambda)$ remains a single point for any $\lambda > 0$; see [66].) Actually, in principle, the structure of $\mathfrak{M}$ can be very involved. Nevertheless, we envisage that $\mathfrak{M}$ is still invariant under the flow, and that it is $\mathfrak{M}$ where, eventually, the "true" dynamics of the liquid will take place. For obvious reasons, the manifold $\mathfrak{M}$ is called *global attractor*.

The existence of a global attractor and the study of the dynamics of the liquid on it, could be of the utmost importance in the effort of formulating a mathematical theory of *turbulence*. Actually, as is well known, if the magnitude of the driving force becomes sufficiently large, the corresponding flow becomes chaotic and the velocity and pressure of the liquid exhibit large and completely random variation in space and time. According to the ideas proposed by Smale [85] and by Ruelle and Takens [74], this chaotic behavior could be explained by the existence of a very complicated global attractor, where, as mentioned before, the ultimate dynamics of the liquid occurs.

V.4.1 Existence of the Global Attractor for Two-Dimensional Flow, and Related Properties.

Throughout this section we shall consider two-dimensional flow, so that, in particular, $\Omega \subset \mathbb{R}^2$.

Let $\boldsymbol{f} \in L^2(\Omega)$ and $\nu > 0$ be given. Consider the one-parameter family of operators

$$S_t : \boldsymbol{a} \in L^2_\sigma(\Omega) \mapsto S_t(\boldsymbol{a}) := \boldsymbol{v}(t) \in L^2_\sigma(\Omega)\,, \quad t \in [0,\infty)$$

where $\boldsymbol{v}(t)$ is, at each t, the weak solution to (7)–(8)$_{\rm Hom}$ corresponding to the initial data $\boldsymbol{a}$; see Remark 10. From Remark 9 and Remark 11 we deduce that the family $\{S_t\}_{t\geq 0}$ defines a (strongly) *continuous semigroup* in $L^2_\sigma(\Omega)$, namely (i) $S_{t_1}S_{t_2}(\boldsymbol{a}) = S_{t_1+t_2}(\boldsymbol{a})$ for all $\boldsymbol{a} \in L^2_\sigma(\Omega)$ and all $t_1, t_2 \in [0,\infty)$; (ii) $S_0(\boldsymbol{a}) = \boldsymbol{a}$, and (iii) the map $t \mapsto S_t(\boldsymbol{a})$ is continuous for all $a \in L^2_\sigma(\Omega)$.

Definition 10 *For any given $\boldsymbol{f} \in L^2(\Omega)$, the corresponding pair $\{L^2_\sigma(\Omega), S_t\}$ is called* semi-flow *associated to* (7)–(8)$_{\rm Hom}$

Our objective is to study the asymptotic properties (as $t \to \infty$) of the semi-flow $\{L^2_\sigma(\Omega), S_t\}$. To this end, we need to recall some basic facts.

Given $\mathcal{A}_1, \mathcal{A}_2 \subset L^2_\sigma(\Omega)$, we set

$$\mathfrak{d}(\mathcal{A}_1, \mathcal{A}_2) = \sup_{\boldsymbol{u}_1 \in \mathcal{A}_1} \inf_{\boldsymbol{u}_2 \in \mathcal{A}_2} \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_2\,.$$

Notice that $\mathfrak{d}(\mathcal{A}_1, \mathcal{A}_2) = 0 \Rightarrow \mathcal{A}_1 \subseteq \overline{\mathcal{A}}_2$. Moreover, we denote by $\mathfrak{A}$ the class of all bounded subset of $L^2_\sigma(\Omega)$.

Definition 11 *$\mathcal{B} \subset L^2_\sigma(\Omega)$ is called:* (i) absorbing *iff for any $\mathcal{A} \in \mathfrak{A}$ there is $t_0 = t_0(\mathcal{A}) \geq 0$ such that $S_t(\mathcal{A}) \subseteq \mathcal{B}$, for all $t \geq t_0$;* (ii) attracting *iff* $\lim\limits_{t\to\infty} \mathfrak{d}(S_t(\mathcal{A}), \mathcal{B}) = 0$, *for all $\mathcal{A} \in \mathfrak{A}$;* (iii) invariant *iff $S_t(\mathcal{B}) = \mathcal{B}$, for all $t \geq 0$;* (iv) maximal invariant *iff it is invariant and contains every invariant set in $\mathfrak{A}$;* (v) global attractor *iff it is compact, attracting and maximal invariant.*

Clearly, if a global attractor exists, it is unique. Furthermore, roughly speaking, its existence is secured whenever the semiflow admits a bounded absorbing set on which the semiflow becomes, eventually, relatively compact; see [94, Theorem 1.1]. The following result holds.

Theorem 20 *For any $\boldsymbol{f} \in L^2_\sigma(\Omega)$ and $\nu > 0$, the corresponding semi-flow $\{L^2_\sigma(\Omega), S_t\}$ admits a global attractor $\mathfrak{M} = \mathfrak{M}(\boldsymbol{f}, \nu)$ which is also connected.*

Proof. In view of [94, Theorem 1.1] and of Rellich compactness theorem [31, Theorem II.4.2], it suffices to show the following two properties: (a) existence of a bounded absorbing set, and (b) given $M > 0$, there is $t_0 = t_0(M, \boldsymbol{f}, \nu) > 0$ such that $\|\boldsymbol{a}\|_2 \le M$ implies $|S_t(\boldsymbol{a})|_{1,2} \le C$, for all $t \ge t_0$ and for some $C > 0$ independent of t. The starting point is the analog of (58) and (61) which, this time, take the form

$$
\begin{aligned}
&\tfrac{1}{2}\frac{d}{dt}\|\boldsymbol{v}(t)\|_2^2 + \nu\,|\boldsymbol{v}(t)|_{1,2}^2 = (\boldsymbol{f}, \boldsymbol{v})\,,\\
&\tfrac{1}{2}\frac{d}{dt}|\boldsymbol{v}(t)|_{1,2}^2 + \nu\,\|P\boldsymbol{v}(t)\|_2^2 = (\boldsymbol{v}\cdot\nabla\boldsymbol{v}, P\Delta\boldsymbol{v}) - (\boldsymbol{f}, P\Delta\boldsymbol{v})
\end{aligned}
\tag{106}
$$

By using in these equations the Schwarz inequality, and inequalities (18), (29), $(62)_1$ and (63), we deduce

$$
\begin{aligned}
&\frac{d}{dt}\|\boldsymbol{v}(t)\|_2^2 + (\nu/C_P)\,\|\boldsymbol{v}(t)\|_2^2 \le\, F\,,\\
&\frac{d}{dt}|\boldsymbol{v}(t)|_{1,2}^2 - g(t)\,|\boldsymbol{v}(t)|_{1,2}^2 \le F\,,
\end{aligned}
\tag{107}
$$

where $F := \|\boldsymbol{f}\|_2^2/\nu$, $g(t) := c_1\|\boldsymbol{v}(t)\|_2^2|\boldsymbol{v}(t)|_{1,2}^2$, and $c_1 = c_1(\Omega) > 0$. By integrating $(107)_1$, we find

$$
\|S_t(\boldsymbol{a})\|_2^2 := \|\boldsymbol{v}(t)\|_2^2 \le \|\boldsymbol{a}\|_2^2\, e^{-\nu t/C_P} + (1 - e^{-\nu t/C_P})F/C_P\,. \tag{108}
$$

Thus, setting

$$
\mathfrak{B} := \{\boldsymbol{\varphi} \in L^2_\sigma(\Omega) :\ \|\boldsymbol{\varphi}\|_2 \le (2F/C_P)^{1/2} \equiv \rho\}\,, \tag{109}
$$

from (107) we deduce that, whenever $\|\boldsymbol{a}\|_2 < M$, there exists $t_1 = t_1(M, F, \nu) > 0$ such that $S_t(\boldsymbol{a}) \in \mathfrak{B}$, which shows that $\mathfrak{B}$ is absorbing. Next, again from Schwarz inequality and from $(106)_1$, we obtain

$$
\int_t^{t+1} |\boldsymbol{v}(s)|_{1,2}^2 ds \le \frac{1}{\nu}\|\boldsymbol{f}\|_2 \int_t^{t+1} \|\boldsymbol{v}(s)\|_2 ds \le \left(\frac{\rho^4 M^2 C_P}{2\nu^2}\right)^{1/2} \equiv \rho_1 \quad \text{for all } t \ge t_1\,, \tag{110}
$$

which, in particular, implies

$$
|\boldsymbol{v}(\bar{t})|_{1,2}^2 \le \rho_1^2\,, \quad \text{for some } \bar{t} \in (t, t+1), \quad \text{all } t \ge t_1\,. \tag{111}
$$

We next integrate $(107)_2$ from $\bar{t}$ to $t+1$ and use (110), to get

$$
|\boldsymbol{v}(t+1)|_{1,2}^2 \le (\rho_1^2 + F)\, e^{\int_t^{t+1} g(\zeta)d\zeta} \le (\rho_1^2 + F)\, e^{\rho^2 \int_t^{t+1} |\boldsymbol{v}(\zeta)|_{1,2}^2 d\zeta} \le (\rho_1^2 + F)\, e^{\rho^2\rho_1^2}\,, \quad \text{for all } t \ge t_1\,,
$$

which proves also property (b). ■

Remark 18 In the proof of the previous theorem the assumption of the boundedness of Ω is crucial, in order to ensure the validity of Rellich's compactness theorem. However, a different approach, due to Rosa [73, Theorem 3.2] allows us to draw the same conclusion of Theorem 20 under the more general assumption that in Ω the Poincaré inequality (18) holds. This happens whenever Ω is contained in a strip of finite width (like in a flow in an infinite channel). □

We shall now list some further properties of the global attractor $\mathfrak{M}$, for whose proofs we refer to the monographs [59, 94, 22]

A. Smoothness. The restriction of the semigroup S_t to $\mathfrak{M}$ can be extended to a group, $\widetilde{S}_t$, defined for all $t \in (-\infty, \infty)$. Therefore, the pair $\{\mathfrak{M}, \widetilde{S}_t\}$ constitutes a *flow* (*dynamical system*). This flow is as smooth as allowed by $\boldsymbol{f}$ and Ω. In particular, if $\boldsymbol{f}$ and Ω are of class C^∞, then the solutions to (7)–$(8)_{\rm Hom}$ belonging to $\mathfrak{M}$ are of class C^∞ in space and time as well. Further significant regularity properties can be found in [27].

B. Finite Dimensionality. Let X be a bounded set of a metric space and let $N(X, \varepsilon)$ be the smallest number of balls of radius ε necessary to cover X. The non-negative (possibly infinite) number

$$d_f(X) = \limsup_{\varepsilon \to 0^+} \frac{\ln N(X, \varepsilon)}{\ln(1/\varepsilon)}\,,$$

is called the *fractal dimension* of X. If X is closed with $d_f(X) < \infty$, then there exists a Lipschitz-continuous function, $g : X \mapsto \mathbb{R}^m$, $m > 2d_f(X)$, possessing a Hölder-continuous inverse on $g(X)$ [24, Theorem 1.2].

The fundamental result states that $d_f(\mathfrak{M})$ is finite and that, moreover,

$$d_f(\mathfrak{M}) \leq c\, \|\boldsymbol{f}\|_2/(\nu^2 C_P) := c\, G\,, \tag{112}$$

where c is a positive constant depending only on the "shape" of Ω; see [92]. The quantity G is non-dimensional (often called *Grashof number*). Consequently, $\mathfrak{M}$ is (in particular) homeomorphic to a compact set of $\mathbb{R}^m$, with $m = 2c\,G + 1$. Notice that, in agreement with what conjectured by E. Hopf, (112) gives a rigorous estimate of how the dimension of $\mathfrak{M}$ is expected to increase with the magnitude of the driving force. (Recall, however, that, as remarked previously, there are example where $d_f(\mathfrak{M}(G)) = 0$, for all $G > 0$.)

Open Question. Since $\mathfrak{M}$ can be parameterized by a finite number of parameters, or, equivalently, it can be "smoothly" embedded in a finite-dimensional space, it is a natural question to ask *whether or not one can construct a* finite-dimensional *dynamical system having a global attractor on which the dynamics is "equivalent" to the Navier-Stokes dynamics on* $\mathfrak{M}$. This question, which is still *unresolved*, has lead to the introduction of the idea of *inertial manifold* [25] and of the associated *approximate inertial manifold* [23], for whose definitions and detailed properties we refer to [94, Chapter VIII]; see, however, also [51].

V.4.2 Further Questions Related to the Existence of the Global Attractor.

In this section we shall address two further important aspects of the theory of attractors for the Navier-Stokes equations, namely, the three-dimensional case (in a bounded domain) and the case of a flow past an obstacle.

A. Three-Dimensional Flow in a Bounded Domain. If we go through the first part of the proof of Theorem 20, we see that the assumption of planar flow has not been used. In fact, by the same token, we can still prove for three-dimensional flow the existence of an "absorbing set", in the sense that every solution departing from any bounded set of L^2_σ, $\mathcal{A}$, will end up in the set $\mathfrak{B}$ defined in (109), after a time t_0, dependent on $\mathcal{A}$ and F. However, the difficulty in extending the results of Theorem 20 to three-dimensional flow, resides, fundamentally, in the lack of well-posedness of (7)–(8)$_{\rm Hom}$ in the space $L^2_\sigma(\Omega)$; see Part C in section V.3.1. In order to overcome this situation, several strategies of attack have been proposed. One way is to make an *unproved* assumption on all possible solutions, which guarantees the existence of a semiflow on $L^2_\sigma(\Omega)$; see [16, Chapter I]. In such a case, all the fundamental results proven for the 2D flow, continue to hold in 3D as well [16]. Another way is to weaken the definition of global attractor, by requiring the attractivity property in the weak topology of L^2; see [22, Chapter III.3], and a third way is to generalize the definition of semiflow in such a way that the uniqueness property is no longer required; see [79, 3, 13].

B. Flow Past an Obstacle. In this case, the relevant initial-boundary value to be investigated is (7)–(8) with $\boldsymbol{f} \equiv \boldsymbol{v}_1 \equiv \boldsymbol{0}$, endowed with the condition at infinity $\lim_{|\boldsymbol{x}|\to\infty} \boldsymbol{v}(\boldsymbol{x}, t) = \boldsymbol{U}$, where $\boldsymbol{U} = U\boldsymbol{e}_1$ is a given, non-zero constant vector. For the reader's convenience, we shall rewrite here this set of equations and put them in a suitable non-dimensional form:

$$\left.\begin{aligned}&\frac{\partial \boldsymbol{v}}{\partial t}+\lambda \boldsymbol{v}\cdot\nabla\boldsymbol{v}=\Delta\boldsymbol{v}-\nabla\mathsf{p}\\&\operatorname{div}\boldsymbol{v}=0\end{aligned}\right\}\ \text{ in }\ \Omega\times(0,\infty)$$
$$\boldsymbol{v}(x,t)|_{\partial\Omega}=\mathbf{0},\quad \lim_{|x|\to\infty}\boldsymbol{v}(x,t)=\boldsymbol{e}_1\,,\quad t>0\,;\quad \boldsymbol{v}(x,0)=\boldsymbol{v}_0(x)\,. \tag{113}$$

In (113), $\lambda := |U|d/\nu$, with d a length scale, is the appropriate Reynolds number which furnishes the magnitude of the driving mechanism.

As we observed in Remark 18, the least requirement on the spatial domain for the existence of a global attractor in a two-dimensional flow, is that there holds Poincaré's inequality (18). Since this inequality is no longer valid in an exterior domain, the problem of the existence of an attractor for a flow past an obstacle is, basically, unresolved. Actually, the situation is even more complicated than what we just described. In fact, from Theorem 9 and from the considerations developed after it, we know that there is $\lambda_c > 0$ such that if $\lambda < \lambda_c$, the corresponding boundary-value problem has one and only one solution, $(\overline{\boldsymbol{v}}, \overline{p})$, in a suitable function class. Now, it is not known if this solution is attracting. More precisely, *in the two-dimensional flow, it is not known if, for sufficiently small λ, solutions to* (113)*, defined in an appropriate class, tend, as $t\to\infty$, to the only corresponding steady solution.*

In the three-dimensional case, the situation is slightly better, but still, the question of the existence of an attractor is completely open. We would like to go in more detail about this point. We begin to observe that there is $\lambda_0 > 0$such that if $\lambda < \lambda_0$, for any given $\boldsymbol{v}_0$ with $(\boldsymbol{v}_0 - \boldsymbol{e}_1) \in L^3_\sigma(\Omega)$, problem (113) has one and only one (smooth) solution that tends to the (uniquely determined) corresponding steady-state solution, $(\overline{\boldsymbol{v}}, \overline{p})$. In particular

$$\lim_{t\to\infty}\|\boldsymbol{v}(t)-\overline{\boldsymbol{v}}\|_3 = 0\,; \tag{114}$$

see [38]. The fundamental question that stays open is then that of investigating the behavior of solutions to (113) for large t, when $\lambda > \lambda_0$. As a matter of fact, *it is not known whether there exists a norm with respect to which solutions to* (113)*, in a suitable class, remain bounded uniformly in time, for all $\lambda > 0$.* In this respect, it is readily seen that, unlike the bounded domain situation, solutions to (113), in general, can not be bounded in $L^2(\Omega)$, uniformly in time, *even when $\lambda < \lambda_0$*. This means that the kinetic energy associated to the motion described by (113) has to grow unbounded for large times. To see this, assume $\lambda < \lambda_0$ and that there exists $K > 0$, independent of t, such that

$$\|\boldsymbol{v}(t)-\boldsymbol{e}_1\|_2 \le K\,, \tag{115}$$

where $\boldsymbol{v}$ is a solution to (113). Then, we can find an unbounded sequence, $\{t_m\}$, and an element $\overline{\boldsymbol{w}} \in L^2_\sigma(\Omega)$ (possibly depending on the sequence) such that

$$\lim_{m\to\infty}(\boldsymbol{v}(t_m)-\boldsymbol{e}_1,\boldsymbol{\varphi}) = (\overline{\boldsymbol{w}},\boldsymbol{\varphi})\,,\quad \text{for all } \boldsymbol{\varphi}\in\mathcal{D}(\Omega)\,. \tag{116}$$

By (114) and (116) we thus must have $\overline{\boldsymbol{w}} = \overline{\boldsymbol{v}} - \boldsymbol{e}_1$, which in turn implies $(\overline{\boldsymbol{v}} - \boldsymbol{e}_1) \in L^2_\sigma(\Omega)$, which, from Theorem 6, we know to be impossible. Consequently, (115) can not be true. Thus, the basic open question is whether or not there exists a function space, Y, where the solution $\boldsymbol{v}(t) - \boldsymbol{e}_1$ remains *uniformly* bounded in $t \in (0,\infty)$, for *all* $\lambda > 0$. (The bound, of course, may depend on λ.) The above considerations along with Theorem 6 suggest then that a plausible candidate for Y is $L^q_\sigma(\Omega)$, for *some* $q > 2$. However, the proof of this property for $q \ge 3$ appears to be overwhelmingly challenging because, in view of Theorem 18 and Remark 8, it would be closely related to the existence of a global, regular solution. Nevertheless, one could investigate the validity of the following weaker property

$$\|\boldsymbol{v}(t) - \boldsymbol{e}_1\|_q \leq K_1\,, \quad \text{for some } q \in (2,3)\,, \tag{117}$$

where K_1 is independent of $t \in (0,\infty)$. Of course, the requirement is that (117) holds for all $\lambda > 0$ and for all corresponding solutions. It is worth emphasizing that the proof of (117) would be of "no harm" to the outstanding global regularity Problem 2, since, according to the available regularity criteria for weak solutions that we discussed in Section V.3, the corresponding solutions, while global in time, will still be weak, even though more regular than those described in Theorem 16. However, notwithstanding its plausibility and "harmlessly", the property (117) appears to be very difficult to establish.

VI Future Directions

The fundamental open questions that we have pointed out throughout the chapter constitute as many topics for future investigation. Actually, it is commonly believed that the answer to most of these questions (in the affirmative or in the negative) will probably shed an entirely new light not only on the mathematical theory of the Navier-Stokes equations but also on other disciplines of applied mathematics.

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