Mini-Courses at Waseda University, Tokyo, March 2010

Maximal L_p-Regularity, Quasilinear Parabolic Problems, and the Two-Phase Stokes Flow with Surface Tension

Jan Prüss

Martin-Luther-Universität Halle-Wittenberg, Halle, Germany

Contents

Part I

References

ClPr01 Ph. Clément, J. Prüss. An operator-valued transference principle and maximal regularity on vector-valued L_p-spaces. *Evolution equations and their applications in physical and life sciences*. Lecture Notes in Pure and Appl. Math., 215, Marcel Dekker, New York (2001), 67–87.

DHP03 R. Denk, M. Hieber, J. Prüss, *$\mathcal{R}$-boundedness, Fourier multipliers, and problems of elliptic and parabolic type*, AMS Memoirs 788, Providence, R.I. (2003).

DHP07 R. Denk, M. Hieber, J. Prüss, Optimal L^p-L^q-estimates for parabolic boundary value problems with inhomogeneous data. *Math. Z.* **257** (2007), no. 1, 193–224.

KaWe01 N.J. Kalton, L. Weis. The H^∞-calculus and sums of closed operators. *Math. Ann.* **321** (2001), 319–345.

PrSi04 J. Prüss, G. Simonett, Maximal regularity for evolution equations in weighted L_p-spaces. *Archiv Math.* **82**, 415–431 (2004)

Wei01 L. Weis, A new approach to maximal L_p-regularity. In *Evolution Equ. and Appl. Physical Life Sci.*, volume 215 of *Lect. Notes Pure and Applied Math.*, pages 195–214, New York, 2001. Marcel Dekker.

I.1. Sectorial and R-Sectorial Operators

Definition. *Let X_0 be a Banach space, and A a closed linear operator in X_0. A is called pseudo-sectorial ($A \in \Psi\mathcal{S}(X_0)$), if $(-\infty, 0) \subset \rho(A)$ and there is a constant $M > 0$ such that*

$$|t(t+A)^{-1}| \leq M \qquad \text{for all } t > 0.$$

A is called sectorial ($A \in \mathcal{S}(X_0)$) if in addition $\overline{\mathcal{D}(A)} = \overline{\mathcal{R}(A)} = X_0$.

Let Σ_ϕ denote the open sector with angle $\phi \in (0, \pi)$:

$$\Sigma_\phi = \{z \in \mathbb{C} : \ z \neq 0, \ |\arg(z)| < \phi\}.$$

The spectral angle ϕ_A of A is then defined by

$$\phi_A = \inf\{\phi : \ \rho(-A) \supset \Sigma_{\pi-\phi}, \ \sup_{\lambda \in \Sigma_{\pi-\phi}} |\lambda(\lambda + A)^{-1}| < \infty\}. \tag{1}$$

Evidently, we have

$$\phi_A \in [0, \pi) \ \text{ and } \ \phi_A \geq \sup\{|\arg \lambda| : \lambda \in \sigma(A)\}.$$

A (pseudo-)sectorial operator A is called R-(pseudo)-sectorial, $A \in \mathcal{RS}(X_0)$ (resp. $A \in \Psi\mathcal{RS}(X_0)$) for short, if the set

$$\{t(t+A)^{-1} : t > 0\} \subset \mathcal{B}(X)$$

is R-bounded. The R-angle of A is defined as

$$\phi_A^R := \inf\{\phi : \mathcal{R}\{\lambda(\lambda+A)^{-1} : \lambda \in \Sigma_{\pi-\phi}\} < \infty\}. \tag{2}$$

Evidently $\phi_A^R \geq \phi_A$.

Sectorial operators A allow for the Dunford calculus on sectors Σ_ϕ. For this purpose let

$$\mathcal{H}_0(\Sigma_\phi) := \{h : \Sigma_\phi \to \mathbb{C} \ \ \textit{holomorhic:} \ \sup_{z \in \Sigma_\phi} |(z^{-\alpha} + z^{\alpha})h(z)| < \infty\},$$

where $\alpha \in (0,1)$ is an exponent which may depend on h. For $h \in \mathcal{H}_0(\Sigma_\phi)$ we define

$$h(A) := \frac{1}{2\pi i} \int_{\Gamma_\theta} h(z)(z-A)^{-1} dz. \tag{3}$$

Here $\phi_A < \theta < \phi$ *is arbitrary, thanks to Cauchy's theorem.*

The map $\Psi : h \mapsto h(A)$ *is an algebra-homomorphism from the algebra* $\mathcal{H}_0(\Sigma_\phi)$ *to the algebra* $\mathcal{B}(X_0)$.

The (pseudo)-sectorial operator A *is said to admit an* $\mathcal{H}^\infty$*-calculus if the map* Ψ *is bounded for some* $\phi \in (\phi_A, 2\pi)$. *The infimum of such* $\phi > \phi_A$ *is called the* $\mathcal{H}^\infty$*-angle of* A*, it is denoted by* ϕ_A^∞.

In this case the Dunford calculus extends to an algebra-homomorphism $\Psi : \mathcal{H}^\infty(\Sigma_\phi) \to \mathcal{B}(X)$, *for each* $\phi > \phi_A^\infty$, *and there is a constant* $c_A^\phi > 0$ *such that*

$$|h(A)|_{\mathcal{B}(X_0)} \leq c_A^\phi |h|_{\mathcal{H}^\infty(\Sigma_\phi)}.$$

Such estimates are a powerful tool. The concept H^∞*-calculus is due to McIntosh [McI86].*

Suppose that $A \in \mathcal{H}^\infty(X_0)$. *For* $s \in \mathbb{R}$, *set* $h_s(z) = z^{is}$; *then* $h_s \in \mathcal{H}^\infty(\Sigma)$ *hence* $A^{is} := h_s(A)$ *is well-defined and forms a bounded* C_0*-group in* X_0, *since*

$$|A^{is}|_{\mathcal{B}(X)} \leq c_A^\phi |h_s|_{\mathcal{H}^\infty(\Sigma_\phi)} = c_A^\phi e^{|s|\phi}, \quad s \in \mathbb{R}.$$

Such operators are said to admit bounded imaginary powers, this class is denoted by $\mathcal{BIP}(X_0)$. *The type of the group* A^{is} *is called power angle of* A, *we denote it by* Θ_A. *Obviously* $\Theta_A \leq \phi_A^\infty$.

Clément and Prüss [ClPr01] have shown that if X_0 *is of class* $\mathcal{HT}$, *then*

$$\mathcal{BIP}(X_0) \subset \mathcal{RS}(X_0), \quad \phi_A^R \leq \Theta_A.$$

Here a Banach space X_0 *is said to be of class* $\mathcal{HT}$ *if the Hilbert-transform*

$$Hu(t) := \int_{\mathbb{R}} u(t-\tau)d\tau/\pi\tau, \quad t \in \mathbb{R},$$

is bounded in $L_2(\mathbb{R}; X_0)$. *This class of Banach spaces coincides with the class of UMD-spaces.*

I.2. The Operator $G = d/dt$ in $L_p(J;E)$.

Let $1 < p < \infty$, E *a Banach space, and define the derivation operator* G *in* $X_0 = L_p(\mathbb{R};E)$ *by means of*

$$Gu(t) = \frac{d}{dt}u(t), \quad t \in \mathbb{R}, \quad \mathcal{D}(G) = H_p^1(\mathbb{R};E).$$

This is a sectorial operator and its resolvent is given by

$$(\lambda + G)^{-1}u(t) = \int_0^\infty e^{-\lambda\tau}u(t-\tau)d\tau, \quad t \in \mathbb{R},\ \mathrm{Re}\,\lambda > 0,$$

the spectral angle is $\Phi_G = \pi/2$. *Note that* G *is a causal operator, it is the negative generator of the translation group. When does* G *admit an* $\mathcal{H}^\infty$*-calculus?*

Let $h \in \mathcal{H}_0(\Sigma_\phi)$ *for some* $\phi > \pi/2$. *Then*

$$h(G)u = \frac{1}{2\pi i}\int_{\Gamma_\theta} h(z)(z-G)^{-1}u\,dz = k_h * u, \quad k_h(t) = \frac{1}{2\pi i}\int_{\Gamma_\theta} h(z)e^{zt}u\,dz.$$

Thus k_h is the inverse Laplace-transform of h, or $\mathcal{L}k_h(z) = h(z)$. In other words, for $h \in \mathcal{H}^\infty(\Sigma_\phi)$ the operator $h(G)$ is bounded if the function $m(\xi) = h(i\xi)$ is a Fourier-multiplier for $L_p(\mathbb{R}; E)$.

Now, such functions satisfy the Mikhlin condition, and the Mikhlin theorem is valid in $L_p(\mathbb{R}; E)$ provided E is of class $\mathcal{HT}$. On the other hand, the simplest nontrivial multiplier satisfying the Mikhlin condition is $m(\xi) = -i\,\mathrm{sgn}\,\xi$. But this is the symbol of the Hilbert transform!

Thus we have

Theorem. Let $1 < p < \infty$, E b a Banach space of class $\mathcal{HT}$ and let $G = d/dt$ be defined as above.
Then $G \in \mathcal{H}^\infty(L_p(\mathbb{R}; E))$ with $\phi_G^\infty = \pi/2$. Conversely, if $G \in \mathcal{BIP}(L_p(\mathbb{R}; E))$ for some $p \in (1, \infty)$ then E is necessarily of class $\mathcal{HT}$.

By causality, this result is also valid in $L_p(J; E)$ where $J = \mathbb{R}_+$ or $J = [0, a]$, with $\mathcal{D}(A) := {}_0H_p^1(J; E) := \{u \in H_p^1(J; E) : u(0) = 0\}$.

I.3. An Operator-Valued $\mathcal{H}^\infty$-Calculus

A powerful tool is the following extension of the scalar $\mathcal{H}^\infty$-calculus of a sectorial operator to the operator-valued case.

Theorem. Let $A \in \mathcal{H}^\infty(X_0)$ and $F \in \mathcal{H}^\infty(\Sigma_\phi; \mathcal{B}(X_0))$ such that

$$F(\lambda)(\mu - A)^{-1} = (\mu - A)^{-1}F(\lambda), \quad \mu \in \rho(A),\ \lambda \in \Sigma_\phi.$$

Suppose $\phi > \phi_A^\infty$ and that $F(\Sigma_\phi)$ is R-bounded. Then $F(A) \in \mathcal{B}(X_0)$.

This result is known as the Kalton-Weis theorem; cf. [KaWe01]. It yields a so-called joint functional calculus.

Corollary. *Suppose $A \in \mathcal{H}^\infty(X_0)$ and $B \in \mathcal{S}(X_0)$ are commuting, $f \in \mathcal{H}^\infty(\Sigma_\phi \times \Sigma_\psi)$ with $\phi > \phi_A^\infty$, $\psi > \phi_B$, and assume $\mathcal{R}(f(\Sigma_\phi, B)) < \infty$. Then $f(A, B) \in \mathcal{B}(X_0)$. In particular, this assertion holds if the functional calculus for B is R-bounded.*

Another consequence of the Kalton-Weis theorem is a variant of the Dore-Venni theorem for operator sums.

Corollary. *Suppose* $A \in \mathcal{H}^\infty(X_0)$ *and* $B \in \mathcal{RS}(X_0)$ *are commuting, and such that* $\phi_A^\infty + \phi_B^R < \pi$.
Then $A + B$ *with domain* $\mathcal{D}(A+B) = \mathcal{D}(A) \cap \mathcal{D}(B)$ *is closed,* $A + B \in \mathcal{S}(X_0)$ *with* $\phi_{A+B} \leq \max\{\phi_A^\infty, \phi_B^R\}$, *and*

$$|Ax| + |Bx| \leq C|(A+B)x|, \quad x \in \mathcal{D}(A) \cap \mathcal{D}(B).$$

If in addition $B \in \mathcal{RH}^\infty(X_0)$ *then* $A + B \in \mathcal{H}^\infty(X_0)$ *and* $\phi_{A+B}^\infty \leq \max\{\phi_A^\infty, \phi_B^{R\infty}\}$.

The following corollary deals with products of sectorial operators.

Corollary. *Suppose* $A \in \mathcal{H}^\infty(X_0)$ *is invertible, and* $B \in \mathcal{RS}(X_0)$ *are commuting, and such that* $\phi_A^\infty + \phi_B^R < \pi$.
Then AB *with domain* $\mathcal{D}(AB) = \{x \in \mathcal{D}(B) : Bx \in \mathcal{D}(A)\}$ *is closed and sectorial, with* $\phi_{AB} \leq \phi_A^\infty + \phi_B^R$. *If in addition* $B \in \mathcal{RH}^\infty(X_0)$, *then* $AB \in \mathcal{H}^\infty(X_0)$ *and* $\phi_{AB}^\infty \leq \phi_A^\infty + \phi_B^{R\infty}$.

I.4. Maximal L_p-Regularity

Let X_0 be a Banach space with norm $|\cdot|_0$, and let A be a linear, closed, densely defined operator in X_0.
Let $J = [0, \infty)$ or $[0, a]$ for some $a > 0$ and let $f : J \to X_0$ be given. Consider the inhomogeneous initial value problem

$$\begin{aligned} \dot{u}(t) + Au(t) &= f(t) \quad t \in J, \\ u(0) &= u_0, \end{aligned} \tag{1}$$

in $L_p(J; X_0)$ for $p \in (1, \infty)$.

The definition of maximal L_p-regularity for (1) is as follows.

Definition. *A is said to belong to the class $\mathcal{MR}_p(J; X_0)$ - and we say that there is maximal L^p-regularity for (1) - if for each $f \in L_p(J; X_0)$ there exists a unique $u \in H^1_p(J; X_0) \cap L_p(J; X_1)$ satisfying (1) in the $L_p(J; X)$ sense, with $u_0 = 0$.*

The closed graph theorem implies then that there exists a constant $C > 0$ such that

$$||u||_{H_p^1(J;X_0)} + ||Au||_{L_p(J;X_0)} \leq C||f||_{L_p(J;X_0)}. \tag{2}$$

Theorem. *Let $A \in \mathcal{MR}_p(J;X_0)$ for some $p \in (1,\infty)$. Then the following assertions are valid.*

(i) *If $J = [0,a]$ then there is $\omega \geq 0$ and $M \geq 1$ such that $\{z \in \mathbb{C} : \mathrm{Re} z \leq -\omega\} \subset \rho(A)$ and the estimate*

$$|z(z+A)^{-1}|_{\mathcal{B}(X_0)} \leq M, \quad \mathrm{Re}\, z \geq \omega,$$

is valid. In particular, $\omega + A$ is sectorial with spectral angle $< \pi/2$.

(ii) *If $J = \mathbb{R}_+$ then $\mathbb{C}_- := \{z \in \mathbb{C} : \mathrm{Re} z < 0\} \subset \rho(A)$ and there is a constant $M \geq 1$ such that*

$$|(z+A)^{-1}|_{\mathcal{B}(X_0)} \leq \frac{M}{1+|z|}, \quad \mathrm{Re}\, z > 0.$$

In particular, A is sectorial with spectral angle $< \pi/2$ and $0 \in \rho(A)$.

If one requires for the solution of (1) only $u \in C(\mathbb{R}_+; X_0)$ and $\dot{u}, Au \in L_p(\mathbb{R}_+; X_0)$, we call the class of such operators ${}_0\mathcal{MR}_p(\mathbb{R}_+; X_0)$.

Corollary. *Suppose $A \in {}_0\mathcal{MR}_p(\mathbb{R}_+; X_0)$.*
Then A is pseudo-sectorial in X_0 with spectral angle $< \pi/2$.
Moreover, $A \in \mathcal{MR}_p(\mathbb{R}_+; X_0)$ if and only if $A \in {}_0\mathcal{MR}_p(\mathbb{R}_+; X_0)$ and $0 \in \rho(A)$.

Thus, for a finite interval $J = [0, a]$ its length $a > 0$ plays no role for maximal L_p-regularity, and up to a shift of A, without loss of generality, we may consider $J = [0, \infty)$ and may assume that $-A$ is the generator of an analytic semigroup of negative exponential type.

Assuming the latter, it is well-known that there exists a solution $u \in H^1_p(\mathbb{R}_+; X_0) \cap L_p(\mathbb{R}_+; \mathcal{D}(A))$ satisfying (1) with $f = 0$ in the $L_p(\mathbb{R}_+; X_0)$ sense if and only if $u_0 \in X_\gamma := (X_0, X_1)_{1-\frac{1}{p},p}$, where $X_1 = \mathcal{D}(A)$ equipped with the graph norm of A. In fact, this follows easily from the well-known basic characterization of the real interpolation spaces X_γ in terms of A:

$$X_\gamma = \mathcal{D}_A(1-1/p,p) := \{x \in X_0 : Ae^{-At}x \in L_p(\mathbb{R}_+; X_0)\}.$$

In the sequel, we denote by $|\cdot|_\gamma$ a norm on X_γ. Now we can state

Corollary. Let $A \in \mathcal{MR}(J; X_0)$. Then the map $u \mapsto (\dot{u} + Au, u(0))$ is an isomorphism from $H^1_p(J; X_0) \cap L_p(J; X_1)$ onto $L_p(J; X_0) \times X_\gamma$.

This result is very useful for quasilinear problems since it allows for the use of the implicit function theorem.

The class ${}_0\mathcal{MR}_p(\mathbb{R}_+; X_0)$ does not depend on $p \in (1,\infty)$. This result is due to Sobolevskii [Sob64].

Theorem. *Suppose* $A \in {}_0\mathcal{MR}_{p_0}(\mathbb{R}_+; X_0)$ *for some* $p_0 \in (1,\infty)$. *Then* $A \in {}_0\mathcal{MR}_p(\mathbb{R}_+; X_0)$ *for all* $p \in (1,\infty)$

Another nice property of maximal L_p-regularity is its invariance under perturbations.

Theorem. *Suppose* $A \in \mathcal{MR}_p(\mathbb{R}_+; X_0)$ *and let* B *be linear operator in* X_0 *with* $\mathcal{D}(B) \supset \mathcal{D}(A)$ *and assume there are constants* $\alpha, \beta \geq 0$ *such that*

$$|Bx|_0 \leq \alpha|x|_0 + \beta|Ax|_0, \quad x \in \mathcal{D}(A).$$

Then there are constants $\beta_0 > 0$ *and* $\omega \geq 0$ *such that* $\beta < \beta_0$ *implies* $\omega + A + B \in \mathcal{MR}_p(\mathbb{R}_+; X_0)$.

Note that such a perturbation result is false for the class $\mathcal{H}^\infty(X_0)$!!

As usual, in Hilbert spaces life is easy. The next theorem is an old result due to de Simon [dSi64].

Theorem. *Let* $1 < p < \infty$ *and* X_0 *be a Hilbert space.*
Then $A \in {}_0\mathcal{MR}_p(\mathbb{R}_+; X_0)$ *if and only if* A *is pseudo-sectorial with spectral angle* $\phi_A < \pi/2$.

Recently, Weis [Wei01] obtained the following characterization of maximal L_p-regularity.

Theorem. *Let X_0 be a Banach space of class $\mathcal{HT}$, and let A be pseudo-sectorial with spectral angle $\phi_A < \pi/2$.*
Then $A \in {}_0\mathcal{MR}_p(X)$ if and only if
the set $\{i\rho(i\rho + A)^{-1} : \rho \in \mathbb{R}\}$ is $\mathcal{R}$-bounded,
i.e. if and only if A is R-sectorial with R-angle $\phi_A^R < \pi/2$.

The result of de Simon follows from this one since in Hilbert spaces families of operators are R-bounded if and only if they are uniformly bounded.

The sufficiency part in the result of Weis can be obtained as a consequence of the operator sum corollary to the Kalton-Weis theorem.

I.5. Time-Weighted L_p-Spaces

Let as before $1 < p < \infty$ and $J = \mathbb{R}_+$. We now consider the time-weighted spaces

$$L_{p,\mu}(J, X_0) := \{u : J \to X_0 : t^{1-\mu}u \in L_p(J; X_0)\}, \quad 1/p < \mu < 1.$$

The time weight allows for weak singularities at time $t = 0$.
We define similarly

$$H^1_{p,\mu}(J; X_0) := \{u \in L_{p,\mu}(J; X_0) : t^{1-\mu}\dot{u} \in L_p(J; X_0)\}.$$

Then the trace space $X_{\gamma,\mu}$ of $\mathbb{E}_\mu(J) := H^1_{p,\mu}(J; X_0) \cap L_{p,\mu}(J, X_1)$ is given by $X_{\gamma,\mu} = (X_0, X_1)_{\mu-1/p,p}$.

We are again interested in the Cauchy problem (1). The following result has been proved in Prüss-Simonett [PrSi04].

Theorem. *Let* $1 < p < \infty$, $1/p < \mu < 1$ *and assume* $A \in \mathcal{MR}(J; X_0)$. *Then the map* $u \mapsto (\dot{u} + Au, u(0))$ *is an isomorphism from* $H^1_{p,\mu}(J; X_0) \cap L_{p,\mu}(J; X_1)$ *onto* $L_{p,\mu}(J; X_0) \times X_{\gamma,\mu}$.

This result shows parabolic regularization since

$$\mathbb{E}_\mu(J) \hookrightarrow \mathbb{E}_1([\delta,\infty)) \hookrightarrow C_0([\delta,\infty); X_\gamma),$$

for each $\delta > 0$.

This is of particular importance for compactness of orbits of quasilinear problems, as we will see below.

Moreover, in view of the Kalton-Weis theorem, the following result, also proved in Prüss-Simonett [PrSi04], is important.

Theorem. *Let $1 < p < \infty$, $1/p < \mu < 1$, E b a Banach space of class $\mathcal{HT}$, and let $G = d/dt$ with domain ${}_0H^1_{p,\mu}(J;E)$.*
Then $G \in \mathcal{H}^\infty(L_{p,\mu}(\mathbb{R}_+;E))$ with $\phi^\infty_G = \pi/2$.

Note that for $1/p < \mu < 1$, the translation semigroup generated by $-G$ is not bounded in $L_{p,\mu}(\mathbb{R}_+;E)$, in contrast to the case $\mu = 1$!

Part II

References

Ama05 H. Amann, Quasilinear parabolic problems via maximal regularity. *Adv. Differential Equation* **10**, 1081–1110 (2005)

CleLi9 Ph. Clément, S. Li, Abstract parabolic quasilinear equations and application to a groundwater flow problem. *Adv. Math. Sci. Appl.* **3**, (1993)

KPW09 M. Köhne, J. Prüss, M. Wilke, On quasilinear parabolic evolution equations in weighted L_p-spaces. *J. Evolu. Eqns* (to appear 2010)

Pru03 J. Prüss, Maximal regularity for evolution equations in L_p-spaces. *Conf. Sem. Mat. Univ. Bari* **285**, 1–39 (2003)

PrSi04 J. Prüss, G. Simonett, Maximal regularity for evolution equations in weighted L_p-spaces. *Archiv Math.* **82**, 415–431 (2004)

PSZ09 J. Prüss, G. Simonett, R. Zacher, Convergence of solutions to equilibria for nonlinear parabolic problems. *J. Diff. Equations* **246**, 3902–3931 (2009)

II.1. Quasilinear Evolution Equations

Let $X_1 \hookrightarrow X_0$ densely, $J_0 = [0, a_0]$, and let $1 < p < \infty$. Consider the abstract quasilinear problem

$$\begin{aligned} &\dot{u}(t) + A(t, u(t))u(t) = F(t, u(t)), \quad t \in J, \\ &u(0) = u_0. \end{aligned} \tag{1}$$

Here $u_0 \in X_\gamma := (X_0, X_1)_{1-1/p,p}$, $A : J_0 \times X_\gamma \to \mathcal{B}(X_1, X_0)$ is continuous, and $F : J_0 \times X_\gamma \to X_0$ is Caratheodory, i.e. such that $F(\cdot, u)$ is measurable for each $u \in X_\gamma$, $F(t, \cdot)$ continuous for a.a. $t \in J_0$. Moreover, we assume the following Lipschitz continuity of A_1 and F_1.

(A_1) For each $R > 0$ there is a constant $L(R) > 0$ such that

$$\begin{aligned} &|A(t, u)v - A(t, \bar{u})v|_0 \le L(R)|u - \bar{u}|_\gamma |v|_1, \\ &t \in J_0,\ u, \bar{u} \in X_\gamma, |u|_\gamma, |\bar{u}|_\gamma \le R,\ v \in X_1. \end{aligned}$$

(F_1) $f(\cdot) := F(\cdot, 0) \in L_p(J_0; X)$; for each $R > 0$ there is a function $\phi_R \in L_p(J_0)$ such that

$$|F(t,u) - F(t,\bar{u})|_0 \leq \phi_R(t)|u - \bar{u}|_\gamma,$$
$$\text{for a.a. } t \in J_0,\ u, \bar{u} \in X_\gamma, |u|_\gamma, |\bar{u}|_\gamma \leq R.$$

The following result is essentially due to Clément and Li [ClLi94]; see also Prüss [Mon03].

Theorem. *Suppose assumptions* **(A_1)** *and* **(F_1)** *are satisfied, and assume that $A_0 = A(0, u_0)$ has the property of maximal L_p-regularity.*

Then there is $a \in (0, a_0]$ such that (1) admits a unique solution u on $J = [0, a]$ in the maximal regularity class $u \in H^1_p(J; X_0) \cap L_p(J; X_1)$. The solution depends continuously on u_0.

Concerning continuation of the solution u, observe that $u \in C(J; X_\gamma)$ holds. Therefore the natural phase space for the problem is the space X_γ, and by uniqueness of the solutions, in the autonomous case the map $u_0 \mapsto u(t)$ defines a local semiflow on X_γ.

Corollary. *Suppose assumptions* (A_1) *and* (F_1) *are satisfied, and assume that* $A(t,v)$ *has maximal* L_p*-regularity for each* $t \in J_0$*,* $v \in X_\gamma$*. Then the solution* $u(t)$ *of (1) has a maximal interval of existence* $J(u_0) = [0, t_+(u_0))$*, which is characterized by the equivalent conditions*

$$\int_{J(u_0)} [|u(t)|_1^p + |\dot{u}(t)|_0^p] dt = \infty,$$

and

$$\lim_{t \to t_+(u_0)} u(t) \quad \text{does not exist in } X_\gamma.$$

In the autonomous case, the map $u_0 \mapsto u(t)$ *defines a local semiflow on the natural phase space* X_γ*.*

We now give two abstract criteria for global existence.

Proposition. *Let the assumptions of the previous Corollary hold. Suppose that the solution* u *of (1) with maximal interval* $J(u_0)$ *satisfies* *one of the following conditions.*

(i) *u is uniformly continuous in X_γ on $J(u_0)$;*

(ii) *$u(J(u_0)) \subset X_\gamma$ is relatively compact.*

Then the solution $u(t)$ of (1) exists globally on J_0.

Specializing to the case $A(t,v) = A(t)$ and $F(t,v) = B(t)v + f(t)$, where $A : J_0 \to \mathcal{B}(X_1, X_0)$ is continuous, $B \in L_p(J_0; \mathcal{B}(X_\gamma, X_0))$, and $f \in L_p(J_0; X_0)$, we obtain a result for the nonautonomous linear problem

$$\begin{aligned} &\dot{u}(t) + A(t)u(t) = B(t)u(t) + f(t), \quad t \in J_0, \\ &u(0) = u_0, \end{aligned} \tag{2}$$

Corollary *Let $A \in C(J_0; \mathcal{B}(X_1, X_0))$ be such that $A(t)$ has maximal L_p-regularity for each $t \in J_0$, and let $B \in L_p(J_0; \mathcal{B}(X_\gamma, X_0))$.*
Then (2) admits a unique solution

$$u \in H^1_p(J_0; X_0) \cap L_p(J_0; X_1),$$

if and only if $f \in L_p(J_0; X_0)$ and $u_0 \in X_\gamma$.

II.2. Weighted L_p-Spaces

We want to extend the existence result from the previous section to the case of weighted L_p-spaces to obtain parabolic smoothing also for quasilinear equations. So we now assume

$(\mathbf{A_\mu})$ For each $R > 0$ there is a constant $L(R) > 0$ such that

$$|A(t,u)v - A(t,\bar{u})v|_0 \le L(R)|u-\bar{u}|_{\gamma,\mu}|v|_1,$$
$$t \in J_0,\ u,\bar{u} \in X_{\gamma,\mu}, |u|_{\gamma,\mu}, |\bar{u}|_{\gamma,\mu} \le R,\ v \in X_1.$$

$(\mathbf{F_\mu})$ $f(\cdot) := F(\cdot,0) \in L_{p,\mu}(J_0;X)$; for each $R > 0$ there is a function $\phi_R \in L_{p,\mu}(J_0)$ such that

$$|F(t,u) - F(t,\bar{u})|_0 \le \phi_R(t)|u-\bar{u}|_{\gamma,\mu},$$
$$\text{for a.a. } t \in J_0,\ u,\bar{u} \in X_{\gamma,\mu}, |u|_{\gamma,\mu}, |\bar{u}|_{\gamma,\mu} \le R.$$

The following result is due to Köhne, Prüss and Wilke [KPW09].

Theorem. *Suppose assumptions* (A_μ) *and* (F_μ) *are satisfied for some* $\mu \in (1/p, 1)$*, and assume that* $A_0 = A(0, u_0)$ *has the property of maximal* L_p*-regularity.*

Then there is $a \in (0, a_0]$ *such that for* $u_0 \in X_{\gamma,\mu}$*, problem (1) admits a unique solution* u *on* $J = [0, a]$ *in the maximal regularity class* $u \in H^1_{p,\mu}(J; X_0) \cap L_{p,\mu}(J; X_1)$*. The solution map* $u_0 \mapsto u$ *is continuous.*

Note that conditions (A_μ) and (F_μ) become stronger for decreasing μ, in particular these conditions imply (A_1) and (F_1). Therefore once we have the solution on some time-interval $[0, a]$, we may continue it in the natural phase space X_γ rather than in $X_{\gamma,\mu}$.

We use this result to obtain compactness of orbits which are bounded in the natural phase space X_γ. For simplicity we restrict here to the autonomous case on $\mathbb{R}_+$. Thus we assume

(H_μ) For each $R > 0$ there is a constant $L(R) > 0$ such that

$$|A(u)v - A(\bar{u})v|_0 \le L(R)|u - \bar{u}|_{\gamma,\mu}|v|_1,$$
$$|F(u) - F(\bar{u})|_0 \le L(R)|u - \bar{u}|_{\gamma,\mu},$$
$$\text{for all } u, \bar{u} \in X_{\gamma,\mu}, |u|_{\gamma,\mu}, |\bar{u}|_{\gamma,\mu} \le R.$$

Now we can prove the following result which is also due to Köhne, Prüss and Wilke [KPW09].

Theorem. *Assume (H_μ) for some $\mu \in (1/p, 1)$, suppose $A(u) \in \mathcal{MR}_p(\mathbb{R}_+; X_0)$ for each $u \in X_{\gamma,\mu}$, and that the embedding $X_\gamma \hookrightarrow X_{\gamma,\mu}$ is compact. Let u be a solution of $\dot{u} + A(u)u = F(u)$ on its maximal interval of existence $[0, t_+)$ and assume that u is bounded in X_γ.*

Then $t_+ = \infty$, i.e. the solution is global, and its orbit $u(\mathbb{R}_+) \subset X_\gamma$ is relatively compact in X_γ. In particular the limit set $\omega(u) \subset X_\gamma$ is nonempty.

II.3. The Generalized Principle of Linear Stability

In this section we consider the autonomous quasilinear problem

$$\dot{u}(t) + A(u(t))u(t) = F(u(t)), \quad t > 0, \quad u(0) = u_0. \tag{3}$$

Here we assume

$$(A, F) \in C^1(V, \mathcal{B}(X_1, X_0) \times X_0), \tag{4}$$

where $V \subset X_\gamma$ is open. Let $\mathcal{E} \subset V \cap X_1$ denote the set of equilibrium solutions of (3), which means that

$$u \in \mathcal{E} \quad \text{if and only if} \quad u \in V \cap X_1,\ A(u)u = F(u).$$

Given an element $u_* \in \mathcal{E}$, we assume that u_* is contained in an m-dimensional manifold of equilibria. This means that there is an open subset $U \subset \mathbb{R}^m$, $0 \in U$, and a C^1-function $\psi : U \to X_1$, such that

$$\psi(U) \subset \mathcal{E}, \quad \psi(0) = u_*, \quad A(\psi(\zeta))\psi(\zeta) = F(\psi(\zeta)), \quad \zeta \in U. \tag{5}$$

and the rank of $\psi'(0)$ equals m.

Let A_0 denote the linearization of $A(u) - F(u)$ at u_*. We call $u_* \in \mathcal{E}$ normally stable if the following conditions hold.

(i) near u_* the set $\mathcal{E}$ is a C^1-manifold in X_1, $\dim \mathcal{E} = m \in \mathbb{N}_0$,
(ii) the tangent space for $\mathcal{E}$ at u_* is isomorphic to $N(A_0)$,
(iii) 0 is a semi-simple eigenvalue of A_0, i.e. $N(A_0) \oplus R(A_0) = X_0$,
(iv) $\sigma(A_0) \setminus \{0\} \subset \mathbb{C}_+ = \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$.

The following result is due to Prüss, Simonett and Zacher [PSZ09].

Theorem. *Let $1 < p < \infty$. Suppose $u_* \in V \cap X_1$ is an equilibrium of (3), (A, F) satisfy (4), and that $A(u_*)$ has the property of maximal L_p-regularity. Assume that u_* is normally stable.*

Then u_* is stable in X_γ, and there exists $\delta > 0$ such that the unique solution $u(t)$ of (3) with initial value $u_0 \in X_\gamma$ satisfying $|u_0 - u_*|_\gamma < \delta$ exists on $\mathbb{R}_+$ and converges at an exponential rate in X_γ to some $u_\infty \in \mathcal{E}$ as $t \to \infty$.

In the special case $m = 0$ we have $\mathcal{E} = \{u_*\}$. This case is the classical principle of linear stability.

The conditions (ii)∼(iii) cannot be relaxed as the following two-dimensional examples show.

Examples. Consider the following system in $G = \mathbb{R}^2 \setminus \{(0,0)\}$.

$$\dot{r} = -r(r-1)^3, \quad \dot{\theta} = (r-1)^k \tag{6}$$

Equilibria are precisely the points on the unit sphere S_1.

(i) Set $k = 1$. Then the eigenvalue zero at any point of the unit sphere $\mathcal{E} = S_1$ is not semisimple.

(ii) Set $k = 2$. Then the eigenvalue zero at any point of the unit sphere $\mathcal{E} = S_1$ is semisimple, but has multiplicity $2 > \dim \mathcal{E} = 1$.

In both cases the function $\Phi(r,\theta) = (r-1)^2$ is a strict Ljapunov function for the system, and the solutions will spiral towards S_1 but do not converge.

We call $u_* \in \mathcal{E}$ normally hyperbolic if

(i) near u_* the set $\mathcal{E}$ is a C^1-manifold in X_1, $\dim \mathcal{E} = m \in \mathbb{N}_0$,
(ii) the tangent space for $\mathcal{E}$ at u_* is given by $N(A_0)$,
(iii) 0 is a semi-simple eigenvalue of A_0, i.e. $N(A_0) \oplus R(A_0) = X_0$,
(iv) $\sigma(A_0) \cap i\mathbb{R} = \{0\}$, $\sigma_u := \sigma(A_0) \cap \mathbb{C}_- \neq \emptyset$.

The following result is also due to Prüss, Simonett and Zacher [PSZ09].

Theorem. *Let* $1 < p < \infty$. *Suppose* $u_* \in V \cap X_1$ *is an equilibrium of (3), the functions* (A, F) *satisfy (4), and that* $A(u_*)$ *has the property of maximal* L_p*-regularity. Assume that* u_* *is normally hyperbolic.*

Then the equilibrium u_* *is unstable in* X_γ *and even in* X_0. *There exists* $\rho > \delta > 0$ *such that the unique solution* $u(t)$ *of (3) with* $|u_0 - u_*| < \delta$ *either satisfies* $\operatorname{dist}(u(t_0), \mathcal{E}) > \rho$ *for some time* $t_0 > 0$, *or it exists globally and* $u(t) \in B_{X_\gamma}(u_*, \rho)$ *for all* $t \geq 0$. *In the latter case* $u(t)$ *converges at an exponential rate to some* $u_\infty \in \mathcal{E}$ *in* X_γ *as* $t \to \infty$.

These local results become global if combined with a strict Ljapunv functional and compactness.

So let $\Phi : X_\gamma \to \mathbb{R}$ be continuous and strictly decreasing along non-constant solutions, and consider a global solution with relatively compact orbit. Then

$$\emptyset \neq \omega(u) \subset \mathcal{E}.$$

Suppose that there exists $u_* \in \omega(u)$ which is normally stable or normally hyperbolic.
The solution then comes arbitrarily close to u_* and stays in a neighbourhood of $\mathcal{E}$.
By the generalized principle of linear stability it converges to u_*.

Example: Consider the 2-D-system

$$\dot{r} = -r(r-1), \quad \dot{\theta} = r - 1.$$

Here this argument yields convergence of all solutions.

III.1. The Two-Phase Stokes Flow with Surface Tension

The Stokes equations read

$$\begin{aligned} -\mathrm{div}\, T &= 0, & x \in \Omega \setminus \Gamma(t),\ t > 0, \\ \nabla \cdot u &= 0, & x \in \Omega \setminus \Gamma(t),\ t > 0, \\ \mu(\nabla u + [\nabla u]^T) - \pi I &= T, & x \in \Omega \setminus \Gamma(t),\ t > 0. \end{aligned}$$

At the interface we have the conditions

$$\begin{aligned} [\![u]\!] &= 0, & x \in \Gamma(t),\ t > 0, \\ (u|\nu_\Gamma) &= V_\Gamma, & x \in \Gamma(t),\ t > 0, \\ -[\![T]\!]\,\nu_\Gamma &= \sigma H_\Gamma \nu_\Gamma, & x \in \Gamma(t),\ t > 0. \end{aligned}$$

The initial condition reads

$$\Gamma(0) = \Gamma_0,$$

and we require no-slip at $\partial\Omega$.

Define the energy functional by means of

$$\Phi(\Gamma) := \sigma \operatorname{mes} \Gamma.$$

Then

$$\partial_t \Phi(\Gamma) + 2\|\mu^{1/2} E\|_{\Omega}^2 = 0, \tag{1}$$

hence the energy functional is a Ljapunov functional, even a strict one. We have

Theorem. *Let $\mu_i, \sigma > 0$ be constants. Then*

(a) The energy equality is valid for smooth solutions.
(b) The equilibria are zero velocities, constant pressures in the phase-components, the dispersed phase is a union of nonintersecting balls.
(c) The energy functional is a strict Ljapunov-functional.
(d) The critical points of the energy functional for constant phase volumes are precisely the equilibria.

III.2. Transformation to a Fixed Domain

Approximate Γ_0 by a smooth hypersurface Σ (set $\Sigma = \Gamma_*$ near an equilibrium $(0, \Gamma_*)$).
Let $d(x)$ denote the signed distance of $x \in \mathbb{R}^n$ to Σ, and $\Pi(x)$ the projection of $x \in \mathbb{R}^n$ to Σ. Then

$$\Lambda : \Sigma \times (-a, a) \to \mathbb{R}^n$$
$$\Lambda(p, r) := p + r\nu_\Sigma(p), \quad \Lambda^{-1}(x) = (\Pi(x), d(x))$$

is a diffeomorphism from $\Sigma \times (-a, a)$ onto $\mathcal{R}(\Lambda) = \{x \in \mathbb{R}^n : |d(x)| < a\}$, provided

$$0 < a < \min\{r(p), 1/\kappa_j(p) : j = 1, \ldots, n-1,\ p \in \Sigma\},$$

where $\kappa_j(p)$ mean the principal curvatures of Σ at $p \in \Sigma$ and

$$\bar{B}_r(p \pm r\nu_\Sigma(p)) \cap \Sigma = \{p\}, \quad p \in \Sigma.$$

Use this to parametrize $\Gamma(t)$ over Σ:

$$\Gamma(t) : p \mapsto p + h(t, p)\nu_\Sigma(p), \quad p \in \Sigma,\ t \geq 0.$$

Extend this diffeomorphism to all of Ω:

$$\Theta(t,x) = x + \chi(d(x))h(t,\Pi(x))\nu_\Sigma(\Pi(x)).$$

Here χ denotes a suitable cut-off function. This way $\Omega \setminus \Gamma(t)$ is transformed to the fixed domain $\Omega \setminus \Sigma$. Then we define

$$\bar{u} = u \circ \Theta^{-1}, \quad \bar{\pi} = \pi \circ \Theta^{-1}.$$

This gives the following problem for $\bar{u}, \bar{\pi}, h$. (Drop the bars!)

$$\begin{aligned}
&-\mu\mathcal{A}(h)u + \mathcal{G}(h)\pi = 0 \quad \text{in } \Omega \setminus \Sigma,\\
&(\mathcal{G}(h)|u) = 0 \quad \text{in } \Omega \setminus \Sigma,\\
&u = 0 \text{ on } \partial\Omega,\\
&[\![-\mu(\mathcal{G}(h)u + [\mathcal{G}(h)u]^T) + \pi]\!]\nu_\Gamma(h) = \sigma H_\Gamma(h)\nu_\Gamma(h) \text{ on } \Sigma, \qquad (2)\\
&[\![u]\!] = 0 \quad \text{on } \Sigma,\\
&\beta(h)\partial_t h - (u|\nu_\Gamma) = 0, \quad \text{on } \Sigma,\\
&h(0) = h_0, \quad \text{on } \Sigma.
\end{aligned}$$

This is the direct mapping approach also called Hanzawa transform.

Here $\mathcal{A}(h)$ and $\mathcal{G}(h)$ denote the transformed Laplacian, resp. gradient. With the curvature tensor L_Σ and the surface gradient ∇_Σ we have

$$\nu_\Gamma(h) = \beta(h)(\nu_\Sigma - \alpha(h)), \qquad \alpha(h) = M(h)\nabla_\Sigma h,$$
$$M(h) = (I - hL_\Sigma)^{-1}, \qquad \beta(h) = (1 + |\alpha(h)|^2)^{-1/2},$$

and

$$V = (\partial_t \Theta|\nu_\Gamma) = \partial_t h(\nu_\Gamma|\nu_\Sigma) = \beta(h)\partial_t h.$$

The curvature $H_\Gamma(h)$ becomes

$$\begin{aligned} H_\Gamma(h) &= \beta(h)\{\mathrm{tr}[M(h)(L_\Sigma + \nabla_\Sigma \alpha(h))] - \beta^2(h)(M(h)\alpha(h)|[\nabla_\Sigma \alpha(h)]\alpha(h))\}, \\ &= \mathcal{B}(h)h + \mathcal{C}(h) \end{aligned}$$

a differential expression involving second order derivatives of h only linearly. $\mathcal{B}, \mathcal{C}$ depend only on $h, \nabla_\Sigma h$. Its linearization is given by

$$H'_\Gamma(0) = \mathrm{tr}\, L_\Sigma^2 + \Delta_\Sigma.$$

Here Δ_Σ denotes the Laplace-Beltrami operator on Σ.

Rewrite this problem in reduced quasilinear form, employing its principal linear part.

$$
\begin{aligned}
&-\mu\Delta u + \nabla\pi = F_u(h)u + F_\pi(h)\pi \quad \text{in } \Omega\setminus\Sigma,\\
&\nabla\cdot u = G_d(h)u \quad \text{in } \Omega\setminus\Sigma,\\
&u = 0 \quad \text{on } \partial\Omega,\\
&P_\Sigma[\![-\mu(\nabla u + \nabla u^T)]\!]\nu_\Sigma = G_\tau(h)u, \quad \text{on } \Sigma,\\
&([\![-\mu(\nabla u + \nabla u^T)]\!]\nu_\Sigma|\nu_\Sigma) + [\![\pi]\!] = \sigma H_\Gamma(h) + G_\nu(h)u, \quad \text{on } \Sigma,\\
&[\![u]\!] = 0 \quad \text{on } \Sigma,\\
&\partial_t h - (u|\nu_\Sigma) = (M(h)\nabla_\Sigma h|u) \quad \text{on } \Sigma,\\
&h(0) = h_0, \quad \text{on } \Sigma.
\end{aligned} \tag{3}
$$

The right hand sides in this problem consist of lower order terms and of terms of the same order appearing on the left, but carrying a factor $|\nabla_\Sigma h|$, which is small by construction. The operators F_j, G_j are analytic in h and $F_j(0) = G_j(0) = 0$. Observe that the problem is linear in (u, π).

III.3. Reduction to a Quasilinear Evolution Equation

The idea is now simple. Suppose that h is known. Solve the transmission problem for the perturbed Stokes problem to obtain

$$u = \sigma\mathcal{S}(h)H_\Gamma(h) = (\mathcal{S}_0 + \mathcal{S}_1(h))(\mathcal{B}(h)h + \mathcal{C}(h)),$$

where $\mathcal{S}_1$ is analytic in h and $S_1(0) = 0$. Then inserting into the dynamic equation for the height function h we obtain the following quasilinear evolution equation for h on Σ.

$$\dot{h} + A(h)h = F(h),\ t > 0, \quad h(0) = h_0. \tag{4}$$

Here A and F are given by

$$A(h)k = -(\nu_\Sigma - M(h)\nabla_\Sigma h|\mathcal{S}(h)\mathcal{B}(h)k), \quad F(h) = (\nu_\Sigma - M(h)\nabla_\Sigma h|\mathcal{S}(h)\mathcal{C}(h)).$$

Note that F contains only lower order terms, and $A(0)k = -(\nu_\Sigma|\mathcal{S}_0\Delta_\Sigma k)$.

For the base space X_0 we make the following choice. If we want $u(t,\cdot) \in H_p^2(\Omega \setminus \Sigma)$ at each instant, then $H_\Gamma(h)$ must belong to $W_p^{1-1/p}(\Sigma)$, hence $h \in W_p^{3-1/p}(\Sigma)$ since $\mathcal{B}$ has order 2. The Neumann-to-Dirichlet operator $\mathcal{S}$ has order -1, hence A is of order 1.

Therefore we choose

$$X_0 = W_p^{2-1/p}(\Sigma), \quad X_1 = W_p^{3-1/p}(\Sigma), \quad \text{hence} \;\; X_{\gamma,\mu} = W_p^{\mu+2-2/p}(\Sigma).$$

Then the solutions will satisfy

$$h \in H_p^1(J; W_p^{2-1/p}(\Sigma)) \cap L_p(J; W_p^{3-1/p}(\Sigma)) \hookrightarrow C(J; W_p^{3-2/p}(\Sigma)),$$

and

$$u \in L_p(J; H_p^2(\Omega \setminus \Sigma) \cap H_p^1(\Omega)), \quad \pi \in L_p(J; \dot{H}_p^1(\Omega \setminus \Sigma)).$$

This is in the L_p-setting natural regularity.

We choose $p > n+1, \mu \in ((n+1)/p, 1)$ to obtain the embedding $W_p^{\mu+2-2/p}(\Sigma) \hookrightarrow C^2(\Sigma)$. Therefore the curvatures are well-defined pointwise.

To apply the results of Section II, we have to study the Stokes problem with linear dynamic boundary condition, given by

$$
\begin{aligned}
\omega^2 u - \mu\Delta u + \nabla\pi &= f_u, && x \in \Omega\setminus\Sigma,\ t>0,\\
\nabla\cdot u &= f_d, && x \in \Omega\setminus\Sigma,\ t>0,\\
u &= 0 && x \in \partial\Omega,\\
[\![u]\!] &= 0, && x\in\Sigma,\ t>0,\\
-[\![\mu(\nabla u + (\nabla u)^{\mathsf{T}}) + \pi]\!]\nu_\Sigma + \sigma\mathcal{A}_\Sigma h\nu_\Sigma &= g_u, && x\in\Sigma,\ t>0,\\
\partial_t h - (u|\nu_\Sigma) &= g_h, && x\in\Sigma,\ t>0,\\
h(0) &= h_0, && x\in\Sigma.
\end{aligned}
$$

Here $\mathcal{A}_\Sigma = -(\operatorname{tr} L_\Sigma^2 + \Delta_\Sigma)$ and $\omega \geq 0$.

For this problem we have to prove maximal L_p-regularity and normal stability. Note that this problem lives on the domain Ω with fixed interface Σ!

III.4. The Local Semiflow

We now introduce the phase manifold of the two-phase Stokes problem with surface tension. Let $\mathcal{MH}^2(\Omega)$ denote the set of all closed hypersurfaces contained in Ω. The second normal bundle $\mathcal{N}^2(\Gamma)$ is defined by

$$\mathcal{N}^2(\Gamma) = \{(p, \nu_\Gamma(p), \nabla_\Gamma \nu_\Gamma(p) : p \in \Gamma\}.$$

Next we need the Haussdorff-distance for sets $A, B \subset \mathbb{R}^N$ defined by

$$d_H(A, B) = \max\{\sup_{a \in A} \mathrm{dist}(a, B), \sup_{b \in B} \mathrm{dist}(b, A)\}.$$

We define a metric on $\mathcal{MH}^2(\Omega)$ by

$$d(\Gamma_1, \Gamma_2) = d_H(\mathcal{N}^2(\Gamma_1), \mathcal{N}^2(\Gamma_2)), \quad \Gamma_1, \Gamma_2 \in \mathcal{MH}^2(\Omega).$$

This way $\mathcal{MH}^2(\Omega)$ becomes a Banach manifold; the charts are given by parameterizations over a given hypersurface Σ, and the tangent space consists of the normal vector fields on Σ.

As above, let $d_\Sigma(x)$ denote the signed distance for Σ. We may then define the level function φ_Σ by means of

$$\varphi_\Sigma(x) = \phi(d_\Sigma(x)), \quad x \in \mathbb{R}^n,$$

where

$$\phi(s) = s(1 - \chi(s/a)) + \chi(s/a)\operatorname{sgn} s, \quad s \in \mathbb{R}.$$

Then $\Sigma = \varphi_\Sigma^{-1}(0)$, and $\nabla_\Sigma \varphi(p) = \nu_\Sigma(p)$, for each $p \in \Sigma$.

If we consider the subset $\mathcal{MH}^2(\Omega, r)$ consisting of all closed hypersurfaces $\Gamma \in \mathcal{MH}^2(\Omega)$ such that $\Gamma \subset \Omega$ satisfies the ball condition with fixed radius $r > 0$ then the map $\Phi : \mathcal{MH}^2(\Omega, r) \to C^2(\bar{\Omega})$ defined by $\Phi(\Gamma) = \varphi_\Gamma$ is an isomorphism of the metric space $\mathcal{MH}^2(\Omega, r)$ onto $\Phi(\mathcal{MH}^2(\Omega, r)) \subset C^2(\bar{\Omega})$.

Let $s - (n-1)/p > 2$; for $\Gamma \in \mathcal{MH}^2(\Omega, r)$, we define $\Gamma \in W_p^s(G, r)$ if $\varphi_\Gamma \in W_p^s(\Omega)$. A subset $A \subset W_p^s(\Omega, r)$ is said to be (relatively) compact, if $\Phi(A) \subset W_p^s(\Omega)$ is (relatively) compact.

Applying the theory from Lecture II we proved

Theorem 1. *The two-phase Stokes problem with surface tension has a unique local-in-time L_p-solution, in the sense that the transformed problem has a solution in the class described in Section III.3. These solutions generate a local semiflow in the phase manifold $\mathcal{PM}$.*

Theorem 2. *The equilibria are stable in $\mathcal{PM}$. Solutions starting near an equilibrium in $\mathcal{PM}$ converge in $\mathcal{PM}$ to another equilibrium as $t \to \infty$.*

Theorem 3. *Suppose $\Gamma(t)$ is a solution which satisfies*
(i) *the uniform ball condition*
(ii) $\|\Gamma(t)\|_{W_p^{3-2/p}} \leq C$
on its life time. Then this solution exists globally and converges in $\mathcal{PM}$ to an equilibrium.