New Helmholtz-Weyl decomposition in L^r and its applications to the mathematical fluid mechanics

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Preface

We show that every L^r-vector field on Ω can be uniquely decomposed into two spaces with scalar and vector potentials and the harmonic vector space via rot and div, where Ω is a bounded domain in $\mathbb{R}^3$. This may be regarded as generalization of de Rham-Hodge decomposition for smooth k-forms on compact Riemannian manifolds. Our result holds not only smooth but also general L^r-vector fields. Basically, construction of harmonic vector fields is established by means of the theory of elliptic PDE system of boundary value problems due to Agmon-Douglis-Nirenberg. Since we deal with L^r-vector fields, such a general theory is not directly available. To get around this difficulty, we make use of certain variational inequalities associated with the quadratic forms defined by rot and div. Various kinds of boundary conditions which are compatible to rot and div and which determine the harmonic parts are fully discussed.

As applications, we first consider the stationary problem of the Navier-Stokes equations in multi-connected domains under the inhomogeneous boundary condition. Up to the present, it is an open question whether there exists a solution if the given boundary data satisfies the general flux condition. It will be clarified that if the harmonic extension of the boundary data into Ω is small in $L^3(\Omega)$ compared with the viscosity constant, then there is at least one weak solution.

The second application is on the global Div-Curl lemma. The classical Div-Curl lemma is stated in such a way that the convergence holds in the sense of distributions. Under the boundary condition determining the harmonic vector fields in the L^r-Helmholtz-Weyl decomposition in Ω, we show that the convergence holds in the whole domain Ω.

Introduction

In this article, we first give a survey on our new Helmholtz-Weyl decomposition of L^r-vector fields in bounded domains Ω in $\mathbb{R}^3$ with the smooth boundary $\partial\Omega$. It is known that every vector field $u \in L^r(\Omega)$ with $1 < r < \infty$ can be decomposed as

$$u = v + \nabla p, \tag{0.1}$$

with $v \in L^r_\sigma(\Omega)$ and $p \in W^{1,r}(\Omega)$, where $L^r_\sigma(\Omega)$ denotes the closure in L^r-norm of the space of C^∞-solenoidal vector functions with the compact support in Ω. More precisely, every $v \in L^r_\sigma(\Omega)$ is characterized as div $v = 0$ in the sense of distributions in Ω and $u \cdot \nu = 0$ on $\partial\Omega$, where ν denotes the unit outer normal to $\partial\Omega$. We refer to Fujiwara-Morimoto [20], Solonnikov [44] and Simader-Sohr [41]. Our first purpose is to show a more precise decomposition for v like

$$v = h + \text{rot } w, \tag{0.2}$$

where h is harmonic, i.e., div $h = 0$, rot $h = 0$ in Ω with $h \cdot \nu = 0$ on $\partial\Omega$, while $w \in W^{1,r}(\Omega)$ is called a vector potential of u with the boundary condition as $w \times \nu = 0$ on $\partial\Omega$. Such a representation of u as in (0.1) and (0.2) may be regarded as a special case of the well-known de Rham-Hodge-Kodaira decomposition for general smooth p-forms on compact Riemannian manifolds. Our decomposition does not require any smoothness for u. Indeed, we can deal with all vector fields u in $L^r(\Omega)$. The proof of classical de Rham-Hodge-Kodaira decomposition can be reduced to solving the elliptic boundary problem on the compact Riemannian manifolds. For instance, the vector potential w of u in (0.2) can be derived from the solution of the following equations

$$\begin{cases} \text{rot rot } w = \text{rot } u \quad \text{in } \Omega, \\ \text{div } w = 0 \quad \text{in } \Omega, \\ w \times \nu = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{0.3}$$

Since $u \in L^r(\Omega)$, we need to deal with rot u in the sense of distributions in Ω, and hence the well-known theory due to Agmon-Douglis-Nirenberg [1] on solvability and regularity of solutions to the boundary-value problem of the elliptic system is unavailable to (0.3). To get around such difficulty, we make use of the following variational inequality such that

$$\begin{aligned} &\|w\|_{W^{1,r}(\Omega)} \\ \leqq\ & C \sup\left\{ \frac{|\int_\Omega \text{rot } w \cdot \text{rot } \Phi dx|}{\|\Phi\|_{W^{1,r'}(\Omega)}}; \quad \Phi \in W^{1,r'}(\Omega), \text{div } \Phi = 0 \text{ in } \Omega,\ \Phi \times \nu = 0 \text{ on } \partial\Omega \right\} \\ & + \sum_{i=1}^{L} \left| \int_\Omega w \cdot \psi_i dx \right| \end{aligned} \tag{0.4}$$

holds for all $w \in W^{1,r}(\Omega)$ with div $w = 0$ in Ω, $w \times \nu = 0$ on $\partial\Omega$, where $\{\psi_1, \cdots, \psi_L\}$ is a basis of the finite dimensional space $V_{har}(\Omega) = \{\psi \in C^\infty(\bar{\Omega}); \text{rot } \psi = 0, \text{div } \psi = 0 \text{ in } \Omega, \psi \times \nu|_{\partial\Omega} = 0\}$. Here and in what follows, we denote $r' = r/(r-1)$ so that $1/r + 1/r' = 1$. Based on the variational inequality (0.4), we shall construct a weak solution w of (0.3) for every $u \in L^r(\Omega)$ in terms of a generalization to the reflexive Banach space of the Lax-Milligram theorem which holds for positive definite quadratic forms in the Hilbert space. We also prove a similar variational inequality to (0.4) which states

$$\tag{0.5} \begin{aligned} &\|w\|_{W^{1,r}(\Omega)} \\ \leqq\ & C\sup\left\{\frac{|\int_\Omega \mathrm{rot}\ w\cdot \mathrm{rot}\ \Psi dx|}{\|\Psi\|_{W^{1,r'}(\Omega)}};\quad \Psi\in W^{1,r'}(\Omega), \mathrm{div}\ \Psi=0 \text{ in } \Omega,\ \Psi\cdot\nu=0 \text{ on } \partial\Omega\right\} \\ &+\sum_{i=1}^{N}\left|\int_\Omega w\cdot\varphi_i dx\right| \end{aligned}$$

for all $w\in W^{1,r}(\Omega)$ with div $w=0$ in Ω, $w\cdot\nu=0$ on $\partial\Omega$, where $\{\varphi_1,\cdots,\varphi_N\}$ is a basis of the finite dimensional space $X_{har}(\Omega)=\{\varphi\in C^\infty(\bar\Omega); \mathrm{rot}\ \varphi=0, \mathrm{div}\ \varphi=0 \text{ in } \Omega,\ \varphi\cdot\nu|_{\partial\Omega}=0\}$. This yields an existence theorem of weak solutions to the elliptic system of the boundary value problem

$$\tag{0.6} \begin{cases} \mathrm{rot}\ \mathrm{rot}\ w=\mathrm{rot}\ u & \text{in } \Omega,\\ \mathrm{div}\ w=0 & \text{in } \Omega,\\ \mathrm{rot}\ w\times\nu=u\times\nu & \text{on } \partial\Omega,\\ w\cdot\nu=0 & \text{on } \partial\Omega. \end{cases}$$

As a consequence, we obtain a similar decomposition theorem to (0.1) and (0.2) such as for very $u\in L^r(\Omega)$ it holds

$$\tag{0.7} u=h+\mathrm{rot}\ w+\nabla p,$$

where $h\in V_{har}(\Omega)$, $w\in W^{1,r}(\Omega)$ with div $w=0$ in Ω and $w\cdot\nu=0$ on $\partial\Omega$, and $p\in W^{1,r}(\Omega)$. The spaces $X_{har}(\Omega)$ and $V_{har}(\Omega)$ are called harmonic vector fields on Ω with different boundary conditions on $\partial\Omega$ which are of finite dimension. We shall show that dimensions of $X_{har}(\Omega)$ and $V_{har}(\Omega)$ are closely related to topological invariance of the domain Ω, which is so-called the Betti number.

As an application of our decomposition theorem, we shall establish a new existence result on the inhomogeneous boundary value problem of the stationary Navier-Stokes equations in multi-connected domains Ω in $\mathbb{R}^3$. Let us assume that the boundary $\partial\Omega$ consists of $L+1$ disjoint smooth closed surfaces $\Gamma_0,\Gamma_1,\cdots,\Gamma_L$, where $\Gamma_1,\cdots,\Gamma_L$ lie inside of Γ_0. We consider the stationary Navier-Stokes equations

$$\tag{N-S} \begin{cases} -\mu\Delta v+v\cdot\nabla v+\nabla p=0 & \text{in } \Omega,\\ \mathrm{div}\ v=0 & \text{in } \Omega,\\ v=\beta & \text{on } \partial\Omega, \end{cases}$$

where $\mu>0$ and $\beta\in H^{\frac{1}{2}}(\partial\Omega)$ are the given viscosity constant and the given function on $\partial\Omega$, respectively. Since the unknown vector function v must satisfy div $v=0$, for solvability of (N-S) we need to impose on β *the general flux condition* which means

$$\tag{G.F.} \sum_{j=0}^{L}\int_{\Gamma_j}\beta\cdot\nu dS=0.$$

The solvability of (N-S) under the inhomogeneous boundary data β satisfying (G.F.) has been a famous open problem proposed by Leray [33]. Unfortunately, we have not yet given a complete answer to Leray's problem. Indeed, to solve (N-S), we need to extend the boundary data β on $\partial\Omega$ to the solenoidal vector field b in Ω, i.e., div $b = 0$ in Ω with $b = \beta$ on $\partial\Omega$. Redefining a new unknown function $u \equiv v - b$, we may rewrite (N-S) to the following equations with the *homogeneous* boundary condition on $\partial\Omega$

$$\text{(N-S')} \qquad \begin{cases} -\mu\Delta u + b\cdot\nabla u + u\cdot\nabla b + u\cdot\nabla u + \nabla p = \mu\Delta b - b\cdot\nabla b \quad \text{in } \Omega, \\ \text{div } u = 0 \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega. \end{cases}$$

The existence of solutions u of (N-S') is closely related to the following question; for every $\varepsilon > 0$ does there exist a solenoidal extension $b_\varepsilon \in H^1(\Omega)$ of β such that the inequality

$$\text{(L.I.)} \qquad \left|\int_\Omega u\cdot\nabla b_\varepsilon\cdot u dx\right| \le \varepsilon\int_\Omega |\nabla u|^2 dx$$

holds for all $u \in H^1_0(\Omega)$ with div $u = 0$. We call (L.I.) Leray's inequality which yields, by taking $\varepsilon = \mu/2$, an a priori estimate such as

$$\text{(0.8)} \qquad \int_\Omega |\nabla u|^2 dx \le C\int_\Omega (|\nabla b_{\frac{\mu}{2}}|^2 + |b_{\frac{\mu}{2}}|^2)dx$$

for all possible solutions u of (N-S'). Based on (0.8), from the well-known Leray-Schauder fixed point theorem we obtain at least one solution u in $H^1_0(\Omega)$ of (N-S') whence a solution v in $H^1(\Omega)$ of (N-S).

It is known that under the restricted flux condition

$$\text{(R.F.)} \qquad \int_{\Gamma_j} \beta\cdot\nu dS = 0, \quad \text{for all } j = 0, 1, \cdots, L,$$

Leray's inequality holds. Therefore, if $\beta \in H^{\frac{1}{2}}(\partial\Omega)$ satisfies (R.F.), then there is a solution $v \in H^1(\Omega)$ of (N-S), which is a partial answer to Leray's problem. See, e.g., Leray [33], Fujita [16] and Ladyzehnskaya [32]. The relation between (L.I.) and (R.F.) is well understood in terms of our new decomposition (0.7). Indeed, it is shown that for every solenoidal extension b in Ω of β satisfying (R.F.) we have $p = 0$ and $h = 0$ in (0.7) with u replaced by b, which yields an expression $b = \text{rot } w$. Taking a family $\{\theta_\varepsilon\}$ of cut-off functions with their support near the ε-neighborhood of the boundary $\partial\Omega$, we see that $b_\varepsilon \equiv \text{rot }(\theta_\varepsilon w)$ satisfies Leray's inequality (L.I.). Now, a natural question arises whether the boundary data β satisfying (G.F.), but not (R.F.) fulfills Leray's inequality (L.I.). Unfortunately, Takeshita [47] gave a negative answer to this question. He treaded the case when Ω is an annular region, i.e., $\Omega = \{x \in \mathbb{R}^3; R_1 < |x| < R_0\}$, and proved that Leray's inequality (L.I.) holds if and only if

$$\text{(0.9)} \qquad \int_{|x|=R_0} \beta\cdot\nu dS = \int_{|x|=R_1} \beta\cdot\nu dS = 0.$$

Takeshita's result implies that it is impossible to solve (N-S) under the general flux condition (G.F.) provided we rely on Leray's inequality (L.I.).

To overcome such an obstruction given by Takeshita [47], we shall investigate possible decomposition as in (0.7) of the solenoidal extension b in Ω of the boundary data β. We shall first show that although there are infinitely many solenoidal extensions b of β, their harmonic part $h \in V_{har}(\Omega)$ of b as in (0.7) is determined only by means of the flux $\int_{\Gamma_j} \beta \cdot \nu dS$ for $j = 0, 1, \cdots, L$. Next, we shall prove that if h is small enough in the L^3-norm on Ω in comparison with the viscosity constant μ, then there exists a solution v of (N-S). Our theorem includes the previous existence theorem on (N-S) under the restricted flux condition (R.F.). We also consider the relation between Leray's inequality (L.I.) and the restricted flux condition (R.F.). Indeed, we shall generalize Takeshita's results and prove that in more domains Ω, Leray's inequality (L.I.) holds if and only if β satisfies the restricted flux condition (R.F.).

Our second application is to derive the global Div-Curl lemma in which the convergence holds not only in the sense of distributions in Ω but also in the sense of integral over the whole domain Ω. It is well-known that if $u_j \to u$, $v_j \to v$ weakly in $L^2(\Omega)$ and if $\{\text{div } u_j\}_{j=1}^{\infty}$ and $\{\text{rot } v_j\}_{j=1}^{\infty}$ are both bounded in $L^2(\Omega)$, then it holds $u_j \cdot v_j \to u \cdot v$ in the sense of distributions in Ω. It seems an interesting question whether it does hold

$$\int_{\Omega} u_j \cdot v_j dx \to \int_{\Omega} u \cdot v dx \quad \text{as } j \to \infty. \tag{0.10}$$

We shall give a positive answer to this question under the additional assumption that either $\{u_j \cdot \nu\}_{j=1}^{\infty}$ or $\{v_j \times \nu\}_{j=1}^{\infty}$ is bounded in $H^{\frac{1}{2}}(\partial\Omega)$. For the proof, we make use of our decompositions (0.1)-(0.2) and (0.7). The essential difference of proofs between usual Div-Curl lemma and our convergence like (0.10) stems from the precise investigation into the harmonic part h according to the boundary condition $h \cdot \nu = 0$ or $h \times \nu = 0$ on $\partial\Omega$. Since our decompositions are both direct sum, for validity of (0.10) it suffices to show that

$$\int_{\Omega} h_j \cdot \tilde{h}_j dx \to \int_{\Omega} h \cdot \tilde{h} dx \tag{0.11}$$

$$\int_{\Omega} \text{rot } w_j \cdot \text{rot } \tilde{w}_j dx \to \int_{\Omega} \text{rot } w \cdot \text{rot } \tilde{w} dx, \quad \int_{\Omega} \nabla p_j \cdot \nabla \tilde{p}_j dx \to \int_{\Omega} \nabla p \cdot \nabla \tilde{p} \, dx, \tag{0.12}$$

where $u_j = h_j + \text{rot } w_j + \nabla p_j$, $v_j = \tilde{h}_j + \text{rot } \tilde{w}_j + \nabla \tilde{p}_j$, $j = 1, 2, \cdots$, and $u = h + \text{rot } w + \nabla p$, $v = \tilde{h} + \text{rot } \tilde{w} + \nabla \tilde{p}$ are the expressions according to (0.1)-(0.2) and (0.7). The convergence (0.12) follows from the bound of $\{u_j \cdot \nu\}_{j=1}^{\infty}$ or $\{v_j \times \nu\}_{j=1}^{\infty}$ in $H^{\frac{1}{2}}(\partial\Omega)$ with the aid of the a priori estimates in $W^{2,2}(\Omega)$ for the elliptic systems (0.3) and (0.6). The advantage of our decomposition is that the harmonic spaces $X_{har}(\Omega)$ and $V_{har}(\Omega)$ are of finite dimensions so that the convergence (0.11) is an easy consequence of equivalence of weak and strong topologies in finite dimensional vector spaces. It should be noted that the convergence (0.10) is discussed in the couple of vector functions between $L^r(\Omega)$ and $L^{r'}(\Omega)$.

1 Helmholtz-Weyl decomposition in L^r

Throughout this article, we impose the following assumption on the domain Ω:

Assumption. Ω is a bounded domain in $\mathbb{R}^3$ with the C^∞-boundary $\partial\Omega$.

Before stating our results, we introduce some function spaces. Let $C_{0,\sigma}^\infty(\Omega)$ denote the set of all C^∞-vector functions $\varphi = (\varphi^1, \varphi^2, \varphi^3)$ with compact support in Ω, such that div $\varphi = 0$. $L^r_\sigma(\Omega)$ is the closure of $C_{0,\sigma}^\infty(\Omega)$ with respect to the L^r-norm $\|\cdot\|_r$; $(\cdot,\cdot)$ denotes the duality pairing between $L^r(\Omega)$ and $L^{r'}(\Omega)$, where $1/r + 1/r' = 1$. $L^r(\Omega)$ stands for the usual (vector-valued) L^r-space over Ω, $1 < r < \infty$. Let us recall the generalized trace theorem for $u \cdot \nu$ and $u \times \nu$ on $\partial\Omega$ defined on the spaces $E^r_{div}(\Omega)$ and $E^r_{rot}(\Omega)$, respectively.

$$
\begin{aligned}
E^r_{div}(\Omega) &\equiv \{u \in L^r(\Omega); \text{div } u \in L^r(\Omega)\} \text{ with the norm } \|u\|_{E^r_{div}} = \|u\|_r + \|\text{div } u\|_r, \\
E^r_{rot}(\Omega) &\equiv \{u \in L^r(\Omega); \text{rot } u \in L^r(\Omega)\} \text{ with the norm } \|u\|_{E^r_{rot}} = \|u\|_r + \|\text{rot } u\|_r.
\end{aligned}
$$

It is known that there are bounded operators γ_ν and τ_ν on $E^r_{div}(\Omega)$ and $E^r_{rot}(\Omega)$ with properties that

$$
\begin{aligned}
&\gamma_\nu : u \in E^r_{div}(\Omega) \mapsto \gamma_\nu u \in W^{1-1/r',r'}(\partial\Omega)^*, \qquad \gamma_\nu u = u \cdot \nu|_{\partial\Omega} \text{ if } u \in C^1(\bar{\Omega}), \\
&\tau_\nu : u \in E^r_{rot}(\Omega) \mapsto \tau_\nu u \in W^{1-1/r',r'}(\partial\Omega)^*, \qquad \tau_\nu u = u \times \nu|_{\partial\Omega} \text{ if } u \in C^1(\bar{\Omega}),
\end{aligned}
$$

respectively. We have the following generalized Stokes formula

$$(u, \nabla p) + (\text{div } u, p) = \langle \gamma_\nu u, \gamma_0 p \rangle_{\partial\Omega} \quad \text{for all } u \in E^r_{div}(\Omega) \text{ and all } p \in W^{1,r'}(\Omega), \tag{1.1}$$

$$(u, \text{rot } \phi) = (\text{rot } u, \phi) + \langle \tau_\nu u, \gamma_0 \phi \rangle_{\partial\Omega} \quad \text{for all } u \in E^r_{rot}(\Omega) \text{ and all } \phi \in W^{1,r'}(\Omega), \tag{1.2}$$

where γ_0 denotes the usual trace operator from $W^{1,r'}(\Omega)$ onto $W^{1-1/r',r'}(\partial\Omega)$, and $\langle \cdot,\cdot \rangle_{\partial\Omega}$ is the duality paring between $W^{1-1/r',r'}(\partial\Omega)^*$ and $W^{1-1/r',r'}(\partial\Omega)$. Notice that $L^r_\sigma(\Omega) = \{u \in L^r(\Omega); \text{div } u = 0 \text{ in } \Omega \text{ with } \gamma_\nu u = 0\}$.

Let us define two spaces $X^r(\Omega)$ and $V^r(\Omega)$ for $1 < r < \infty$ by

$$X^r(\Omega) \equiv \{u \in L^r(\Omega); \text{div } u \in L^r(\Omega), \text{rot } u \in L^r(\Omega), \gamma_\nu u = 0\}, \tag{1.3}$$

$$V^r(\Omega) \equiv \{u \in L^r(\Omega); \text{div } u \in L^r(\Omega), \text{rot } u \in L^r(\Omega), \tau_\nu u = 0\}. \tag{1.4}$$

Equipped with the norms $\|u\|_{X^r}$ and $\|u\|_{V^r}$

$$\|u\|_{X^r}, \|u\|_{V^r} \equiv \|\text{div } u\|_r + \|\text{rot } u\|_r + \|u\|_r, \tag{1.5}$$

we may regard $X^r(\Omega)$ and $V^r(\Omega)$ as Banach spaces. Indeed, in Theorem 1.2 below, we shall see that both $X^r(\Omega)$ and $V^r(\Omega)$ are closed subspaces in $W^{1,r}(\Omega)$ since it holds that

$$\|\nabla u\|_r \leqq C\|u\|_{X^r} \quad \text{for all } u \in X^r(\Omega) \quad \text{and} \quad \|\nabla u\|_r \leqq C\|u\|_{V^r} \quad \text{for all } u \in V^r(\Omega), \tag{1.6}$$

respectively, where $C = C(r)$ is a constant depending only on r. Furthermore, we define two spaces $X^r_\sigma(\Omega)$ and $V^r_\sigma(\Omega)$ by

$$X^r_\sigma(\Omega) \equiv \{u \in X^r(\Omega); \text{div } u = 0 \quad \text{in } \Omega\}, \quad V^r_\sigma(\Omega) \equiv \{u \in V^r(\Omega); \text{div } u = 0 \quad \text{in } \Omega\}. \tag{1.7}$$

Finally, we denote by $X^r_{har}(\Omega)$ and $V^r_{har}(\Omega)$ the L^r-spaces of harmonic vector fields on Ω as

$$X^r_{har}(\Omega) \equiv \{u \in X^r_\sigma(\Omega); \text{rot } u = 0\}, \quad V^r_{har}(\Omega) \equiv \{u \in V^r_\sigma(\Omega); \text{rot } u = 0\}. \tag{1.8}$$

Our main result in this section now reads

Theorem 1.1 *Let Ω be as in the Assumption. Suppose that $1 < r < \infty$.*

(1) It holds that

$$\begin{aligned} X^r_{har}(\Omega) &= \{h \in C^\infty(\bar{\Omega}); \mathrm{div}\ h = 0, \mathrm{rot}\ h = 0 \ \textit{in}\ \Omega \ \textit{with}\ h\cdot\nu = 0 \ \textit{on}\ \partial\Omega\}(\equiv X_{har}(\Omega)), \\ V^r_{har}(\Omega) &= \{h \in C^\infty(\bar{\Omega}); \mathrm{div}\ h = 0, \mathrm{rot}\ h = 0 \ \textit{in}\ \Omega \ \textit{with}\ h\times\nu = 0 \ \textit{on}\ \partial\Omega\}(\equiv V_{har}(\Omega)). \end{aligned}$$

Both $X_{har}(\Omega)$ and $V_{har}(\Omega)$ are of finite dimensional vector spaces.

(2) For every $u \in L^r(\Omega)$, there are $p \in W^{1,r}(\Omega)$, $w \in V^r_\sigma(\Omega)$ and $h \in X_{har}(\Omega)$ such that u can be represented as

$$u = h + \mathrm{rot}\ w + \nabla p. \tag{1.9}$$

Such a triplet $\{p, w, h\}$ is subordinate to the estimate

$$\|\nabla p\|_r + \|w\|_{V^r} + \|h\|_r \leqq C\|u\|_r \tag{1.10}$$

with the constant $C = C(r)$ independent of u. The above decomposition (1.9) is unique. In fact, if u has another expression

$$u = \tilde{h} + \mathrm{rot}\ \tilde{w} + \nabla\tilde{p}$$

for $\tilde{p} \in W^{1,r}(\Omega)$, $\tilde{w} \in V^r_\sigma(\Omega)$ and $\tilde{h} \in X_{har}(\Omega)$, then we have

$$h = \tilde{h}, \quad \mathrm{rot}\ w = \mathrm{rot}\ \tilde{w}, \quad \nabla p = \nabla\tilde{p}. \tag{1.11}$$

(3) For every $u \in L^r(\Omega)$, there are $p \in W^{1,r}_0(\Omega)$, $w \in X^r_\sigma(\Omega)$ and $h \in V_{har}(\Omega)$ such that u can be represented as

$$u = h + \mathrm{rot}\ w + \nabla p. \tag{1.12}$$

Such a triplet $\{p, w, h\}$ is subordinate to the estimate

$$\|\nabla p\|_r + \|w\|_{X^r} + \|h\|_r \leqq C\|u\|_r \tag{1.13}$$

with the constant $C = C(r)$ independent of u. The above decomposition (1.12) is unique. In fact, if u has another expression

$$u = \tilde{h} + \mathrm{rot}\ \tilde{w} + \nabla\tilde{p}$$

for $\tilde{p} \in W^{1,r}_0(\Omega)$, $\tilde{w} \in X^r_\sigma(\Omega)$ and $\tilde{h} \in V_{har}(\Omega)$, then we have

$$h = \tilde{h}, \quad \mathrm{rot}\ w = \mathrm{rot}\ \tilde{w}, \quad \nabla p = \nabla\tilde{p}. \tag{1.14}$$

An immediate consequence of the above theorem is

Corollary 1.1 *Let Ω be as in the Assumption.*

(1) By the unique decomposition (1.9) and (1.12) we have two kinds of direct sums

$$L^r(\Omega) = X_{har}(\Omega) \oplus \mathrm{rot}\ V^r_\sigma(\Omega) \oplus \nabla\ W^{1,r}(\Omega), \tag{1.15}$$

$$L^r(\Omega) = V_{har}(\Omega) \oplus \mathrm{rot}\ X^r_\sigma(\Omega) \oplus \nabla\ W^{1,r}_0(\Omega) \tag{1.16}$$

for $1 < r < \infty$.

(2) Let S_r, R_r and Q_r be projection operators associated to both (1.9) and (1.12) from $L^r(\Omega)$ onto $X_{har}(\Omega)$, rot $V^r_\sigma(\Omega)$ and $\nabla\ W^{1,r}(\Omega)$, and from $L^r(\Omega)$ onto $V_{har}(\Omega)$, rot $X^r_\sigma(\Omega)$ and $\nabla\ W^{1,r}_0(\Omega)$, respectively, i.e.,

(1.17)
$$S_r u \equiv h, \quad R_r u \equiv \text{rot } w, \quad Q_r u \equiv \nabla p.$$

Then we have

(1.18)
$$\|S_r u\|_r \leqq C\|u\|_r, \quad \|R_r u\|_r \leqq C\|u\|_r, \quad \|Q_r u\|_r \leqq C\|u\|_r$$

for all $u \in L^r(\Omega)$, where $C = C(r)$ is the constant depending only on $1 < r < \infty$. Moreover, there holds

(1.19)
$$\begin{cases} S_r^2 = S_r, & S_r^* = S_{r'}, \\ R_r^2 = R_r, & R_r^* = R_{r'} \\ Q_r^2 = Q_r, & Q_r^* = Q_{r'}, \end{cases}$$

where S_r^*, R_r^* and Q_r^* denote the adjoint operators on $L^{r'}(\Omega)$ of S_r, R_r and Q_r, respectively.

Remark 1.1 (1) It is known that

(1.20)
$$L^r(\Omega) = L^r_\sigma(\Omega) \oplus \nabla\ W^{1,r}(\Omega), \quad 1 < r < \infty, \quad \text{(direct sum)}.$$

See Fujiwara-Morimoto [20], Solonnikov [44] and Simader-Sohr [41]. Our decomposition (1.15) gives a more precise direct sum of $L^r_\sigma(\Omega)$ such as

(1.21)
$$L^r_\sigma(\Omega) = X_{har}(\Omega) \oplus \text{rot } V^r_\sigma(\Omega), \quad 1 < r < \infty. \quad \text{(direct sum)}$$

On the other hand, our new decomposition (1.16) imposes on p the homogeneous boundary condition on $\partial\Omega$. Compared with $L^r_\sigma(\Omega)$ in (1.21), any boundary condition on $\partial\Omega$ cannot be prescribed on the vector filed $v \equiv h + \text{rot } w$ in (1.12). If $u \in W^{1,r}(\Omega)$, then we have $u \times \nu = v \times \nu$ on $\partial\Omega$.

(2) Let us characterize Ω by topological invariance which is called a Betti number. To be more precise, we make the following definition.

Definition 1.1 Let Ω be as in the Assumption and let N and L be two positive integers.

(i) We say that Ω has the first Betti number N if there are N C^∞-surfaces $\Sigma_1, \cdots, \Sigma_N$ transversal to $\partial\Omega$ such that $\Sigma_i \cap \Sigma_j = \phi$ for $i \neq j$, and such that

(1.22)
$$\dot{\Omega} \equiv \Omega \setminus \Sigma \quad \text{is a simply connected domain, where } \Sigma \equiv \bigcup_{j=1}^N \Sigma_j.$$

(ii) We say that Ω has the second Betti number L if the boundary $\partial\Omega$ has $L+1$ connected components $\Gamma_0, \Gamma_1, \cdots, \Gamma_L$ of C^∞-closed surfaces such that $\Gamma_1, \cdots, \Gamma_L$ lie inside of Γ_0 with $\Gamma_i \cap \Gamma_j = \phi$ for $i \neq j$, and such that

(1.23)
$$\partial\Omega = \bigcup_{j=0}^L \Gamma_j.$$

Foias-Temam [15] showed that if Ω has the first Betti number N as in Definition 1.1 (i), then it holds

$$\dim . X_{har}(\Omega) = N. \tag{1.24}$$

They [15] also gave an orthogonal decomposition of $L^2_\sigma(\Omega)$ such as

$$L^2_\sigma(\Omega) = X_{har}(\Omega) \oplus H_1(\Omega) \quad \text{(orthogonal sum in } L^2(\Omega)),$$

where

$$H_1(\Omega) \equiv \left\{ u \in L^2_\sigma(\Omega); \int_{\Sigma_j} u \cdot \nu dS = 0 \quad \text{for all } j = 1, \cdots, N \right\}.$$

Yoshida-Giga [53] investigated the operator rot with its domain $D(\mathrm{rot}) = \{u \in H_1(\Omega); \mathrm{rot}\ u \in H_1(\Omega)\}$ and showed that $H_1(\Omega) = \mathrm{rot}\ V^2_\sigma(\Omega)$. Furthermore, they [53] gave another type of orthogonal L^2-decomposition of vector fields $u \in D(\mathrm{rot})$. From our decomposition (1.21) with $r = 2$, it follows also that $H_1(\Omega) = \mathrm{rot}\ V^2_\sigma(\Omega)$.

(3) In the case when Ω is a star-shaped domain, Griesinger [23] gave a similar decomposition in $L^r(\Omega)$ for $1 < r < \infty$. In her case, it holds $N = 0$. Since she took the smaller space $W^{1,r}_0(\Omega)$ than our space $V^r(\Omega)$, it seems to be an open question whether, in the same way as in (1.15), the annihilator rot $W^{1,r}_0(\Omega)^\perp$ of rot $W^{1,r}_0(\Omega)$ in $L^{r'}(\Omega)$ coincides with $\nabla\ W^{1,r'}(\Omega)$.

(4) If Ω has the second Betti number L as in Definition 1.1 (ii), then we shall show in Subsection 2.3 that

$$\dim . V_{har}(\Omega) = L. \tag{1.25}$$

Moreover, it holds

$$\begin{aligned} &\{\mathrm{rot}\ w; w \in W^{2,r}(\Omega) \cap X^r_\sigma(\Omega)\} \\ = \ &\left\{ v \in W^{1,r}(\Omega);\ \mathrm{div}\ v = 0 \text{ in } \Omega, \int_{\Gamma_j} v \cdot \nu dS = 0 \text{ for all } j = 0, 1, \cdots, L \right\}. \end{aligned} \tag{1.26}$$

(5) If Ω has the first and the second Betti numbers N and L as in Definition 1.1, then von Wahl [50] gave also a representation formula like (1.9) and (1.12) by means of the potential theory. Our theorem does not need any restriction on the topological type of Ω, which seems to be an advantage for the use of L^r-variational inequalities (0.4) and (0.5). In the more general case when Ω is an n-dimensional C^∞-manifold with the boundary, Schwarz [40] established an orthogonal decomposition of p-forms in $L^2(\Omega)$ and in $W^{s,r}(\Omega)$ for $s \geqq 1$ and $2 \leqq r < \infty$. However, his method depends on the theory of pseudo-differential operators, which is different from our L^r-variational approach based on the theory of Agmon-Douglis-Nirenberg.

As an application of our decomposition, we have the following gradient and higher order estimates of vector fields via div and rot .

Theorem 1.2 *Let Ω be as in the Assumption. Suppose that $1 < r < \infty$.*
(1) *(prescribed $\gamma_\nu u$) Let $\dim . X_{har}(\Omega) = N$ and let $\{\varphi_1, \cdots, \varphi_N\}$ be a basis of $X_{har}(\Omega)$.*
(i) *It holds $X^r(\Omega) \subset W^{1,r}(\Omega)$ with the estimate*

$$\|\nabla u\|_r + \|u\|_r \leqq C\left(\|\mathrm{div}\ u\|_r + \|\mathrm{rot}\ u\|_r + \sum_{j=1}^N |(u, \varphi_j)|\right) \quad \text{for all } u \in X^r(\Omega), \tag{1.27}$$

where $C = C(\Omega, r)$.

(ii) *Let* $s \geq 1$. *Suppose that* $u \in L^r(\Omega)$ *with* $\mathrm{div}\, u \in W^{s-1,r}(\Omega)$, $\mathrm{rot}\, u \in W^{s-1,r}(\Omega)$ *and* $\gamma_\nu u \in W^{s-1/r,r}(\partial\Omega)$. *Then we have* $u \in W^{s,r}(\Omega)$ *with the estimate*

$$\begin{aligned} &\|u\|_{W^{s,r}(\Omega)} \\ &\leqq C(\|\mathrm{div}\, u\|_{W^{s-1,r}(\Omega)} + \|\mathrm{rot}\, u\|_{W^{s-1,r}(\Omega)} + \|\gamma_\nu u\|_{W^{s-1/r,r}(\partial\Omega)} + \sum_{j=1}^{N} |(u, \varphi_j)|), \end{aligned} \tag{1.28}$$

where $C = C(\Omega, r)$.

(2) *(prescribed* $\tau_\nu u$*) Let* $\dim.V_{har}(\Omega) = L$ *and let* $\{\psi_1, \cdots \psi_L\}$ *be a basis of* $V_{har}(\Omega)$.

(i) *It holds* $V^r(\Omega) \subset W^{1,r}(\Omega)$ *with the estimate*

$$\|\nabla u\|_r + \|u\|_r \leqq C(\|\mathrm{div}\, u\|_r + \|\mathrm{rot}\, u\|_r + \sum_{j=1}^{L} |(u, \psi_j)|) \quad \textit{for all } u \in V^r(\Omega), \tag{1.29}$$

where $C = C(\Omega, r)$.

(ii) *Let* $s \geq 1$. *Suppose that* $u \in L^r(\Omega)$ *with* $\mathrm{div}\, u \in W^{s-1,r}(\Omega)$, $\mathrm{rot}\, u \in W^{s-1,r}(\Omega)$ *and* $\tau_\nu u \in W^{s-1/r,r}(\partial\Omega)$. *Then we have* $u \in W^{s,r}(\Omega)$ *with the estimate*

$$\begin{aligned} &\|u\|_{W^{s,r}(\Omega)} \\ &\leqq C(\|\mathrm{div}\, u\|_{W^{s-1,r}(\Omega)} + \|\mathrm{rot}\, u\|_{W^{s-1,r}(\Omega)} + \|\tau_\nu u\|_{W^{s-1/r,r}(\partial\Omega)} + \sum_{j=1}^{L} |(u, \psi_j)|), \end{aligned} \tag{1.30}$$

where $C = C(\Omega, r)$.

Remark 1.2 von Wahl [51] treated the homogeneous gradient bound such as

$$\|\nabla u\|_r \leqq C(\|\mathrm{div}\, u\|_r + \|\mathrm{rot}\, u\|_r)$$

for $u \in W^{1,r}(\Omega)$ with $\gamma_\nu u = 0$ and $\tau_\nu u = 0$. He proved that such a homogeneous estimate holds if and only if $N = 0$, i.e., Ω is simply connected in the case $\gamma_\nu u = 0$, and if and only if $L = 0$, i.e., Ω has only one connected component of the boundary $\partial\Omega$ in the case $\tau_\nu u = 0$, respectively. Our variational inequalities (0.4) and (0.5) make it possible to prove (1.27) and (1.29) for an arbitrary bounded domain Ω. So, von Wahl's estimate [51] may be regarded as a special case of ours since our Assumption on Ω does not require any topological type such as (1.23) or (1.22). His method is based on the representation formula for $u \in W^{1,r}(\Omega)$ via $\mathrm{div}\, u$ and $\mathrm{rot}\, u$ which is different from ours. Similar estimate to (1.28) with $\sum_{j=1}^{N} |(u, \varphi_j)|$ replaced by $\|u\|_r$ was obtained by Temam[48, Proposition 1.4, Appendix I] for $s \geq 1$, $r = 2$ and by Bourguignon-Brezis [9, Lemma 5] for $s \geq 2$, $1 < r < \infty$, respectively. See also Duvaut-Lions [12, Theorem 6.1, Chapter 7].

2 L^r-variational inequality

2.1 Variational inequalities in $X^r(\Omega)$ and $V^r(\Omega)$

In what follows, we shall denote by C the constants which may change from line to line. If we need to specify the constants, we shall denote by $C(*, \cdots, *)$ the constants depending only on the quantities appearing in the parenthesis.

Let us first introduce auxiliary function spaces $\hat{\mathcal{X}}(\Omega)$, $\hat{X}^r(\Omega)$, $\hat{\mathcal{V}}(\Omega)$ and $\hat{V}^r(\Omega)$ defined by

$$\hat{\mathcal{X}}(\Omega) \equiv \{\phi \in C^\infty(\bar{\Omega}); \phi \cdot \nu|_{\partial\Omega} = 0\}, \quad \hat{X}^r(\Omega) \equiv \{u \in W^{1,r}(\Omega); u \cdot \nu|_{\partial\Omega} = 0\}, \tag{2.1}$$

$$\hat{\mathcal{V}}(\Omega) \equiv \{\psi \in C^\infty(\bar{\Omega}); \psi \times \nu|_{\partial\Omega} = 0\}, \quad \hat{V}^r(\Omega) \equiv \{u \in W^{1,r}(\Omega); u \times \nu|_{\partial\Omega} = 0\} \tag{2.2}$$

for $1 < r < \infty$, respectively. Obviously, it holds that $\hat{X}^r(\Omega) \subset X^r(\Omega)$ and $\hat{V}^r(\Omega) \subset V^r(\Omega)$, but in Lemma 2.2 below, we will see that

$$\hat{X}^r(\Omega) = X^r(\Omega), \quad \hat{V}^r(\Omega) = V^r(\Omega), \quad 1 < r < \infty, \tag{2.3}$$

where $X^r(\Omega)$ and $V^r(\Omega)$ are the spaces defined by (1.3) and (1.4), respectively. It should be noted that for every $u \in \hat{X}^r(\Omega)$ and $u \in \hat{V}^r(\Omega)$, we have $u|_{\partial\Omega} \in W^{1-1/r,r}(\partial\Omega)$. On the other hand, for $u \in X^r(\Omega)$ and $u \in V^r(\Omega)$, we have only that $\gamma_\nu u = 0$ and $\tau_\nu u = 0$ in the sense of functionals on $W^{1-1/r',r'}(\partial\Omega)$, respectively.

The purpose of this subsection is to show the following variational inequalities.

Lemma 2.1 *Let Ω be as in the Assumption.*
(1) For the boundary condition $u \times \nu = 0$ on $\partial\Omega$, we have the following properties (i) *and* (ii)*.*

(i) *For every $1 < r < \infty$, there is a constant $C = C(r)$ such that*

$$\|\nabla u\|_r + \|u\|_r \leqq C \sup_{\psi \in \hat{\mathcal{V}}(\Omega)} \frac{|(\nabla u, \nabla \psi) + (u, \psi)|}{\|\nabla \psi\|_{r'} + \|\psi\|_{r'}} \tag{2.4}$$

holds for all $u \in \hat{V}^r(\Omega)$.

(ii) *Let $u \in \hat{V}^q(\Omega)$ for some $1 < q < \infty$. If u satisfies*

$$\sup_{\psi \in \hat{\mathcal{V}}(\Omega)} \frac{|(\nabla u, \nabla \psi) + (u, \psi)|}{\|\nabla \psi\|_{r'} + \|\psi\|_{r'}} < \infty \tag{2.5}$$

for some $1 < r < \infty$, then we have $u \in \hat{V}^r(\Omega)$, and the estimate (2.4) holds.
(2) For the boundary condition $u \cdot \nu = 0$ on $\partial\Omega$, we have the following properties (iii) *and* (iv)*.*

(iii) *For every $1 < r < \infty$, there is a constant $C = C(r)$ such that*

$$\|\nabla u\|_r + \|u\|_r \leqq C \sup_{\phi \in \hat{\mathcal{X}}(\Omega)} \frac{|(\nabla u, \nabla \phi) + (u, \phi)|}{\|\nabla \phi\|_{r'} + \|\phi\|_{r'}} \tag{2.6}$$

holds for all $u \in \hat{X}^r(\Omega)$.

(iv) *Let $u \in \hat{X}^q(\Omega)$ for some $1 < q < \infty$. If u satisfies*

$$\sup_{\phi \in \hat{\mathcal{X}}(\Omega)} \frac{|(\nabla u, \nabla \phi) + (u, \phi)|}{\|\nabla \phi\|_{r'} + \|\phi\|_{r'}} < \infty \tag{2.7}$$

for some $1 < r < \infty$, then we have $u \in \hat{X}^r(\Omega)$, and the estimate (2.6) holds.

Based on Lemma 2.1, we shall first show (1.6) which guarantees (2.3). Recall the spaces $X^r(\Omega)$ and $V^r(\Omega)$ defined by (1.3) and (1.4), respectively. Both $X^r(\Omega)$ and $V^r(\Omega)$ are Banach spaces with norms $\|\cdot\|_{X^r}$ and $\|\cdot\|_{V^r}$ as in (1.5). It follows from Duvaut-Lions [12, Lemmata 4.2 and 6.1, Chapter 7] that $\hat{\mathcal{X}}(\Omega)$ and $\hat{\mathcal{V}}(\Omega)$ are dense in $X^r(\Omega)$ and $V^r(\Omega)$, respectively.

Lemma 2.2 *Let Ω be as in the Assumption and let $1 < r < \infty$.*

(1)*(in case $\gamma_\nu u = 0$) It holds $X^r(\Omega) = \hat{X}^r(\Omega)$ with the estimate*

$$\|\nabla u\|_r + \|u\|_r \leqq C\|u\|_{X^r} \quad \text{for all } u \in X^r(\Omega). \tag{2.8}$$

(2)*(in case $\tau_\nu u = 0$) It holds $V^r(\Omega) = \hat{V}^r(\Omega)$ with the estimate*

$$\|\nabla u\|_r + \|u\|_r \leqq C\|u\|_{V^r} \quad \text{for all } u \in V^r(\Omega). \tag{2.9}$$

Proof. (1) Since $\hat{\mathcal{X}}(\Omega)$ is dense in $X^r(\Omega)$, it suffices to show (2.8) for $u \in \hat{\mathcal{X}}(\Omega)$. We make use of the following identity

$$(\nabla u, \nabla\phi) = (\text{rot } u, \text{rot } \phi) + (\text{div } u, \text{div } \phi) - \int_{\partial\Omega} u \cdot (\phi \cdot \nabla\nu + \phi \times \text{rot } \nu)dS \tag{2.10}$$

for all $u, \phi \in \hat{\mathcal{X}}^(\Omega)$, where dS denotes the surface element of $\partial\Omega$. Notice that the unit outer normal ν to $\partial\Omega$ can be extended as a smooth vector field in some neighborhood of $\partial\Omega$. Indeed, since $u \cdot \nu = 0$ on $\partial\Omega$, we have by integration by parts

$$\begin{aligned}
& (\nabla u, \nabla\phi) \\
= \ & (u, -\Delta\phi) + \int_{\partial\Omega} u \cdot (\nu \cdot \nabla\phi)dS \\
= \ & (u, \text{rot rot } \phi - \nabla(\text{div } \phi)) + \int_{\partial\Omega} u \cdot (\nu \cdot \nabla\phi)dS \\
= \ & (\text{rot } u, \text{rot } \phi) + (\text{div } u, \text{div } \phi) + \int_{\partial\Omega} u \times \nu \cdot \text{rot } \phi dS + \int_{\partial\Omega} u \cdot (-\nu \text{div } \phi + \nu \cdot \nabla\phi)dS \qquad (2.11) \\
= \ & (\text{rot } u, \text{rot } \phi) + (\text{div } u, \text{div } \phi) + \int_{\partial\Omega} u \cdot (\nu \times \text{rot } \phi + \nu \cdot \nabla\phi)dS \\
= \ & (\text{rot } u, \text{rot } \phi) + (\text{div } u, \text{div } \phi) + \int_{\partial\Omega} u \cdot (\nabla(\phi \cdot \nu) - \phi \cdot \nabla\nu - \phi \times \text{rot } \nu)dS.
\end{aligned}$$

Since $\phi \cdot \nu = 0$ on $\partial\Omega$, we see that $\nabla(\phi \cdot \nu)$ is parallel to ν on $\partial\Omega$, which yields $u \cdot \nabla(\phi \cdot \nu) = 0$ on $\partial\Omega$. Hence, the third term of the right hand side of (2.11) vanishes, and the identity (2.10) follows. Now we have by (2.10) that

$$|(\nabla u, \nabla\phi)| \leqq (\|\text{rot } u\|_r + \|\text{div } u\|_r)\|\nabla\phi\|_{r'} + C\|u\|_{L^r(\partial\Omega)}\|\phi\|_{L^{r'}(\partial\Omega)}$$

for all $\phi \in \hat{\mathcal{X}}(\Omega)$. By the trace theorem, for every $\varepsilon > 0$ there is a constant $C_\varepsilon = C_\varepsilon(r)$ such that

$$\|u\|_{L^r(\partial\Omega)} \leqq \varepsilon\|\nabla u\|_r + C_\varepsilon\|u\|_r,$$

which yields

$$|(\nabla u, \nabla\phi)| \leqq C(\|\text{rot } u\|_r + \|\text{div } u\|_r + \varepsilon\|\nabla u\|_r + C_\varepsilon\|u\|_r)(\|\nabla\phi\|_{r'} + \|\phi\|_{r'}) \tag{2.12}$$

for all $\phi \in \hat{\mathcal{X}}(\Omega)$ with $C = C(r)$. Taking ε sufficiently small in (2.12), we obtain the desired estimate (2.8) from (2.6) in Lemma 2.1.

(2) The proof is quite similar to that of (1). Since $\hat{\mathcal{V}}(\Omega)$ is dense in $V^r(\Omega)$, it suffices to prove (2.9) for $u \in \hat{\mathcal{V}}(\Omega)$. Compared with (2.10), we make use of the following identity

$$(2.13) \qquad (\nabla u, \nabla \psi) = (\text{rot } u, \text{rot } \psi) + (\text{div } u, \text{div } \psi) + \int_{\partial\Omega} u \cdot (\psi \cdot \nabla \nu - \psi \text{div } \nu) dS$$

for all $u, \psi \in \hat{\mathcal{V}}^(\Omega)$. Indeed, since $u \times \nu = 0$ on $\partial\Omega$, in the same manner as in (2.11) we have

$$\begin{aligned} & (\nabla u, \nabla \psi) \\ (2.14) \qquad = \; & (\text{rot } u, \text{rot } \psi) + (\text{div } u, \text{div } \psi) + \int_{\partial\Omega} u \cdot (-\nu \text{div } \psi + \nu \cdot \nabla \psi) dS \\ = \; & (\text{rot } u, \text{rot } \psi) + (\text{div } u, \text{div } \psi) + \int_{\partial\Omega} u \cdot (\text{rot } (\psi \times \nu) + \psi \cdot \nabla \nu - \psi \text{div } \nu) dS. \end{aligned}$$

Since $u \times \nu = 0$ on $\partial\Omega$, we see that $u \cdot \text{rot}$ is a tangential derivation on $\partial\Omega$. Since $\psi \times \nu = 0$ on $\partial\Omega$, it holds $u \cdot \text{rot } (\psi \times \nu) = 0$ on $\partial\Omega$. Hence the third term of (2.14) vanishes, which yields (2.13). Now we have by (2.13) that

$$|(\nabla u, \nabla \psi)| \leqq (\|\text{rot } u\|_r + \|\text{div } u\|_r)\|\nabla \psi\|_{r'} + C\|u\|_{L^r(\partial\Omega)}\|\psi\|_{L^{r'}(\partial\Omega)}$$

for all $\psi \in \hat{\mathcal{V}}(\Omega)$. It is easy to see that the proof is quite parallel to that of the above (1) since we have (2.4) in Lemma 2.1. This proves Lemma 2.2.

By Lemma 2.2, we may identify $\hat{X}^r(\Omega)$ with $X^r(\Omega)$ and $\hat{V}^r(\Omega)$ with $V^r(\Omega)$ for all $1 < r < \infty$, respectively. Let us recall the spaces $X^r_{har}(\Omega)$ and $V^r_{har}(\Omega)$ defined by (1.8). Then by Lemma 2.2 we have

$$\begin{aligned} X^r_{har}(\Omega) &= \{u \in W^{1,r}(\Omega); \text{div } u = 0, \text{rot } u = 0 \quad \text{in } \Omega, \quad u \cdot \nu = 0 \quad \text{on } \partial\Omega\}, \\ V^r_{har}(\Omega) &= \{u \in W^{1,r}(\Omega); \text{div } u = 0, \text{rot } u = 0 \quad \text{in } \Omega, \quad u \times \nu = 0 \quad \text{on } \partial\Omega\}. \end{aligned}$$

By (2.8) and (2.9) we see that both $X^r_{har}(\Omega)$ and $V^r_{har}(\Omega)$ are of finite dimension. Indeed, if we regard $X^r_{har}(\Omega)$ and $V^r_{har}(\Omega)$ as subspaces in $L^r(\Omega)$, then by (2.8) and (2.9) their unit sphere with respect to the L^r-norm is a bounded set in $W^{1,r}(\Omega)$. By the Rellich theorem, it is a compact subset in $L^r(\Omega)$, which implies that the dimensions of $X^r_{har}(\Omega)$ and $V^r_{har}(\Omega)$ are finite. Let us define the space $X_{har}(\Omega)$ and $V_{har}(\Omega)$ by

$$(2.15) \qquad X_{har}(\Omega) \equiv \bigcap_{1<r<\infty} X^r_{har}(\Omega) \quad \text{and} \quad V_{har}(\Omega) \equiv \bigcap_{1<r<\infty} V^r_{har}(\Omega),$$

respectively. Then we have

Lemma 2.3 (1) *For every fixed q with $1 < q < \infty$, it holds*

$$X^q_{har}(\Omega) = X_{har}(\Omega), \quad V^q_{har}(\Omega) = V_{har}(\Omega).$$

(2) (i) *(in the case $\gamma_\nu u = 0$) Let $\dim. X_{har}(\Omega) = N$ and let $\{\varphi_1, \cdots, \varphi_N\}$ be a basis of $X_{har}(\Omega)$. For ever $1 < r < \infty$, there is a constant $C = C(r)$ such that*

$$(2.16) \qquad \|\nabla u\|_r + \|u\|_r \leqq C(\|\text{div } u\|_r + \|\text{rot } u\|_r + \sum_{j=1}^{N} |(u, \varphi_j)|)$$

holds for all $u \in X^r(\Omega)$.

(ii) *(in the case* $\tau_\nu u = 0$*) Let* $\dim.V_{har}(\Omega) = L$ *and let* $\{\psi_1, \cdots, \psi_L\}$ *be a basis of* $V_{har}(\Omega)$*. For ever* $1 < r < \infty$*, there is a constant* $C = C(r)$ *such that*

$$\|\nabla u\|_r + \|u\|_r \leqq C(\|\text{div } u\|_r + \|\text{rot } u\|_r + \sum_{j=1}^{L} |(u, \psi_j)|) \tag{2.17}$$

holds for all $u \in V^r(\Omega)$.

Proof. (1) Let us first prove that for every fixed $1 < q < \infty$ it holds $X^q_{har}(\Omega) = X_{har}(\Omega)$. Since Ω is bounded, it suffices to show that

$$X^q_{har}(\Omega) \subset X^r_{har}(\Omega) \quad \text{for all } r \text{ with } q \leqq r < \infty.$$

We may assume that $1 < q < 3$. Let us first show for $r = r_1$ with $1/r_1 = 1/q - 1/3$. We take p so that $1/p = 3/2q - 1/2$. Then it holds $1/p = 1/q - \frac{1}{2}(1 - 1/q)$ and $1/p' = 1/r_1' - \frac{1}{2}(1 - 1/r_1')$. Since $\partial\Omega$ is a two-dimensional surface, the Sobolev embedding and the trace theorem state that

$$\gamma_0\left(W^{1,q}(\Omega)\right) = W^{1-\frac{1}{q},q}(\partial\Omega) \subset L^p(\partial\Omega),\ \gamma_0\left(W^{1,r_1'}(\Omega)\right) = W^{1-\frac{1}{r_1'},r_1'}(\partial\Omega) \subset L^{p'}(\partial\Omega), \tag{2.18}$$

where γ_0 is the usual trace operator defined by (1.1) and (1.2). By density argument, the identity (2.10) holds for all $u \in X^q(\Omega)$ and all $\phi \in X^{q'}(\Omega)$. Hence, if $u \in X^q_{har}(\Omega)$, then we have by (2.18) that

$$\begin{aligned} |(\nabla u, \nabla\phi)| &\leqq C\|u\|_{L^p(\partial\Omega)}\|\phi\|_{L^{p'}(\partial\Omega)} \\ &\leqq C\|u\|_{W^{1-\frac{1}{q},q}(\partial\Omega)}\|\phi\|_{W^{1-\frac{1}{r_1'},r_1'}(\partial\Omega)} \\ &\leqq C\|u\|_{W^{1,q}(\Omega)}\|\phi\|_{W^{1,r_1'}(\Omega)} \end{aligned} \tag{2.19}$$

for all $\phi \in \hat{\mathcal{X}}(\Omega)$. Since $1/q' = 1/r_1' - 1/3$, we have $W^{1,r_1'}(\Omega) \subset L^{q'}(\Omega)$, which yields

$$|(u, \phi)| \leqq \|u\|_q\|\phi\|_{q'} \leqq C\|u\|_q\|\phi\|_{W^{1,r_1'}} \quad \text{for all } \phi \in \hat{\mathcal{X}}(\Omega). \tag{2.20}$$

We obtain from (2.19) and (2.20) that

$$\sup_{\phi\in\hat{\mathcal{X}}(\Omega)} \frac{|(\nabla u, \nabla\phi) + (u, \phi)|}{\|\nabla\phi\|_{r_1'} + \|\phi\|_{r_1'}} \leqq C\|u\|_{W^{1,q}(\Omega)}$$

Hence it follows from (2.7) in Lemma 2.1 that $u \in X^{r_1}_{har}(\Omega)$. Repeating this argument again with q replaced by r_1, we have

$$u \in X^{r_2}_{har}(\Omega) \quad \text{for } r_2 \text{ with } 1/r_2 = 1/r_1 - 1/3 = 1/q - 2/3.$$

Again by the same procedure, we conclude within finitely many steps that

$$u \in X^r_{har}(\Omega) \quad \text{for all } r \text{ with } q \leqq r < \infty.$$

We next show that for every fixed $1 < q < \infty$ it holds $V^q_{har}(\Omega) = V_{har}(\Omega)$. It should be noted that density argument yields the identity (2.13) for all $u \in V^q(\Omega)$ and all $\phi \in V^{q'}(\Omega)$. Hence it is easy to see that if $u \in V^q_{har}(\Omega)$, then similar estimates such as (2.19) and (2.20) hold for all $\phi \in \hat{\mathcal{V}}(\Omega)$. By (2.5) together with the bootstrap argument as above we obtain $u \in V^r_{har}(\Omega)$ for all r with $q \leqq r < \infty$.

(2) The proof of (i) and (ii) stems from (2.8) and (2.9), respectively. So, we may only prove (i). The proof of (ii) is quite the same. We make use of contradiction argument. Suppose that (2.16) dose not hold. Then there is a sequence $\{u_m\}_{m=1}^{\infty}$ such that

$$\|\nabla u_m\|_r + \|u_m\|_r \equiv 1, \quad \|\text{div } u_m\|_r + \|\text{rot } u_m\|_r + \sum_{j=1}^{N} |(u_m, \phi_j)| \leqq 1/m$$

for all $m = 1, 2, \cdots$. By the Rellich theorem, we may assume that there is $u \in X^r(\Omega)$ such that

$$u_m \to u \quad \text{strongly in } L^r(\Omega) \text{ as } m \to \infty.$$

By (2.8) we see that $\{u_m\}_{m=1}^{\infty}$ is a Cauchy sequence in $W^{1,r}(\Omega)$. Hence

$$\nabla u_m \to \nabla u \quad \text{strongly in } L^r(\Omega) \text{ as } m \to \infty.$$

Since div $u_m \to 0$ and rot $u_m \to 0$, we have $u \in X^r_{har}(\Omega)$. Moreover, since $(u_m, \phi_j) \to 0$ for all $j = 1, \cdots N$, we have

$$(u, \phi_j) = 0 \quad \text{for all } j = 1, \cdots, N,$$

which yields $u = 0$. This contradicts $\|\nabla u_m\|_r + \|u_m\|_r \equiv 1$ for all $m = 1, 2, \cdots$. ∎

Let us recall the spaces $X^r_\sigma(\Omega)$ and $V^r_\sigma(\Omega)$ defined by (1.7). The following variational inequalities on $X^r_\sigma(\Omega)$ and $V^r_\sigma(\Omega)$ play an essential role for the proof of Theorem 1.1.

Lemma 2.4 *Let Ω be as in the Assumption and let $1 < r < \infty$.*

(1)(in case $\gamma_\nu w = 0$) Let $\{\varphi_1, \cdots, \varphi_N\}$ be a basis of $X_{har}(\Omega)$. There is a constant $C = C(r)$ such that the estimate

$$\|\nabla w\|_r + \|w\|_r \leqq C \sup\left\{\frac{|(\text{rot } w, \text{rot } \varphi)|}{\|\nabla\varphi\|_{r'} + \|\varphi\|_{r'}}; \varphi \in X^{r'}_\sigma(\Omega), \varphi \neq 0\right\} + \sum_{j=1}^{N} |(w, \varphi_j)| \tag{2.21}$$

holds for all $w \in X^r_\sigma(\Omega)$.

(2)(in case $\tau_\nu w = 0$) Let $\{\psi_1, \cdots, \psi_L\}$ be a basis of $V_{har}(\Omega)$. There is a constant $C = C(r)$ such that the estimate

$$\|\nabla w\|_r + \|w\|_r \leqq C \sup\left\{\frac{|(\text{rot } w, \text{rot } \psi)|}{\|\nabla\psi\|_{r'} + \|\psi\|_{r'}}; \psi \in V^{r'}_\sigma(\Omega), \psi \neq 0\right\} + \sum_{j=1}^{L} |(w, \psi_j)| \tag{2.22}$$

holds for all $w \in V^r_\sigma(\Omega)$.

Let us first introduce the following uniqueness property. For that purpose, recall the spaces $\hat{\mathcal{X}}(\Omega)$ and $\hat{\mathcal{V}}(\Omega)$ defined by (2.1) and (2.2).

Proposition 2.1 *Let Ω be as in the Assumption.*

(1)(in case $\gamma_\nu u = 0$) Let $u \in X^r_\sigma(\Omega)$ for some $1 < r < \infty$. If u satisfies

$$(2.23) \qquad (\mathrm{rot}\ u, \mathrm{rot}\ \varphi) = 0 \quad \textit{for all } \varphi \in \hat{\mathcal{X}}(\Omega) \textit{ with } \mathrm{div}\ \varphi = 0,$$

then we have $u \in X_{har}(\Omega)$.

(2) (in case $\tau_\nu u = 0$) Let $u \in V^r_\sigma(\Omega)$ for some $1 < r < \infty$. If u satisfies

$$(2.24) \qquad (\mathrm{rot}\ u, \mathrm{rot}\ \psi) = 0 \quad \textit{for all } \psi \in \hat{\mathcal{V}}(\Omega) \textit{ with } \mathrm{div}\ \psi = 0,$$

then we have $u \in V_{har}(\Omega)$.

Proof. (1) By (2.23), we see that u fulfills

$$(2.25) \qquad (\mathrm{rot}\ u, \mathrm{rot}\ \Phi) = 0 \quad \text{for all } \Phi \in \hat{\mathcal{X}}(\Omega).$$

Indeed, for every $\Phi \in \hat{\mathcal{X}}(\Omega)$, we can choose a scalar function $p \in C^\infty(\bar{\Omega})$ in such a way that

$$(2.26) \qquad \Delta p = \mathrm{div}\ \Phi \quad \text{in } \Omega, \quad \frac{\partial p}{\partial \nu} = 0 \quad \text{on } \partial\Omega.$$

Taking $\varphi \equiv \Phi - \nabla p$, we have $\varphi \in \hat{\mathcal{X}}(\Omega)$ with $\mathrm{div}\ \varphi = 0$. Since $\mathrm{rot}\ (\nabla p) = 0$, we see that (2.23) implies (2.25).

For the proof, it suffices to show that $u \in X^2(\Omega)$. In such a case, since $\hat{\mathcal{X}}(\Omega)$ is dense in $X^2(\Omega)$, we may take $\Phi = u$ in (2.25), which yields $\mathrm{rot}\ u = 0$. Since Ω is bounded, we may assume that $1 < r < 2$. We make use of the identity (2.10) which holds also for $u \in X^r_\sigma(\Omega)$ and $\phi \in \hat{\mathcal{X}}(\Omega)$. Since $\mathrm{div}\ u = 0$, we have by (2.25) and (2.10) that

$$(\nabla u, \nabla \Phi) = -\int_{\partial\Omega} u \cdot (\Phi \cdot \nabla \nu - \Phi \times \mathrm{rot}\ \nu) dS \quad \text{for all } \Phi \in \hat{\mathcal{X}}(\Omega).$$

Then it is easy to see that the same argument as in (2.19) and (2.20) with $q = r$ in the proof of Lemma 2.3 yields the variational inequality

$$(2.27) \qquad \sup_{\Phi \in \hat{\mathcal{X}}(\Omega)} \frac{|(\nabla u, \nabla \Phi) + (u, \Phi)|}{\|\nabla \Phi\|_{r_1'} + \|\Phi\|_{r_1'}} \leqq C\|u\|_{W^{1,r}(\Omega)} \quad \text{for } r_1 \text{ with } 1/r_1 = 1/r - 1/3.$$

It follows from (2.7) that $u \in X^{r_1}(\Omega)$. If $5/6 \leqq r < 2$, then we have $r_1 \geqq 2$ and $u \in X^2(\Omega)$ follows. In case $1 < r < 5/6$, we repeat the above procedure again starting from $u \in X^{r_1}(\Omega)$ to see that $u \in X^{r_2}(\Omega)$ for r_2 with $1/r_2 = 1/r_1 - 1/3 = 1/r - 2/3$. Since $r_2 > 2$, we obtain $u \in X^2(\Omega)$, too.

(2) The proof is quite similar to that of the above (1). By (2.24), we see that u fulfills

$$(2.28) \qquad (\mathrm{rot}\ u, \mathrm{rot}\ \Psi) = 0 \quad \text{for all } \Psi \in \hat{\mathcal{V}}(\Omega).$$

Indeed, for every $\Psi \in \hat{\mathcal{V}}(\Omega)$, we choose a scalar function $p \in C^\infty(\bar{\Omega})$ in such a way that

$$(2.29) \qquad \Delta p = \mathrm{div}\ \Psi \quad \text{in } \Omega, \quad p = 0 \quad \text{on } \partial\Omega.$$

Taking $\psi \equiv \Phi - \nabla p$, we have $\psi \in \hat{\mathcal{V}}(\Omega)$ with div $\psi = 0$. Since rot $(\nabla p) = 0$, we see that (2.24) implies (2.28).

For the proof, in the same way as that of the above (1), it suffices to show that $u \in V^r(\Omega)$ for $1 < r < 2$ implies $u \in V^2(\Omega)$. We make use of the identity (2.13) which holds also for $u \in V^r_\sigma(\Omega)$ and $\phi \in \hat{\mathcal{V}}(\Omega)$. Since div $u = 0$, we have by (2.28) and (2.13) that

$$(\nabla u, \nabla \Psi) = \int_{\partial\Omega} u \cdot (\Psi \cdot \nabla \nu - \Psi \mathrm{div}\ \nu) dS \quad \text{for all } \Psi \in \hat{\mathcal{V}}(\Omega).$$

Then it is easy to see that the same argument as in (2.27) yields

$$\sup_{\Phi \in \hat{\mathcal{V}}(\Omega)} \frac{|(\nabla u, \nabla \Phi) + (u, \Phi)|}{\|\nabla \Phi\|_{r_1'} + \|\Phi\|_{r_1'}} \leqq C\|u\|_{W^{1,r}(\Omega)} \quad \text{for } r_1 \text{ with } 1/r_1 = 1/r - 1/3.$$

It follows from (2.5) that $u \in V^{r_1}(\Omega)$. If $5/6 \leqq r < 2$, then we have $r_1 \geqq 2$ and $u \in V^2(\Omega)$ follows. In case $1 < r < 5/6$, we repeat the above procedure again starting from $u \in V^{r_1}(\Omega)$ to see that $u \in V^{r_2}(\Omega)$ for r_2 with $1/r_2 = 1/r - 2/3$. Since $r_2 > 2$, we obtain $u \in V^2(\Omega)$, too. This proves Proposition 2.1.

Proof of Lemma 2.4.

(1) Let us first show that there is a constant $C = C(r)$ such that

$$\|\nabla u\|_r \leqq C \sup_{\varphi \in X^{r'}_\sigma(\Omega)} \frac{|(\text{rot } u, \text{rot } \varphi)|}{\|\nabla \varphi\|_{r'} + \|\varphi\|_{r'}} + C\|u\|_r \tag{2.30}$$

holds for all $u \in X^r_\sigma(\Omega)$. Since div $u = 0$ and since the identity (2.10) holds for $\phi \in X^{r'}(\Omega)$, we have similarly to (2.12) that for every $\varepsilon > 0$ there is a constant C_ε such that

$$|(\nabla u, \nabla \Phi)| \leqq |(\text{rot } u, \text{rot } \Phi)| + (\varepsilon\|\nabla u\|_r + C_\varepsilon\|u\|_r)\|\Phi\|_{W^{1,r'}(\Omega)} \quad \text{for all } \Phi \in X^{r'}(\Omega).$$

Hence taking ε sufficiently small, we obtain from (2.6) in Lemma 2.1 that

$$\begin{aligned}
\|\nabla u\|_r &\leqq C \sup_{\Phi \in X^{r'}(\Omega)} \frac{|(\text{rot } u, \text{rot } \Phi)|}{\|\Phi\|_{W^{1,r'}(\Omega)}} + C\|u\|_r \\
&\leqq C \sup_{\Phi \in X^{r'}(\Omega)} \frac{|(\text{rot } u, \text{rot } \Phi)|}{\|\text{rot } \Phi\|_{r'}} + C\|u\|_r \\
&= C \sup_{\varphi \in X^{r'}_\sigma(\Omega)} \frac{|(\text{rot } u, \text{rot } \varphi)|}{\|\text{rot } \varphi\|_{r'}} + C\|u\|_r \\
&= C \sup \left\{ \frac{|(\text{rot } u, \text{rot } \tilde{\varphi})|}{\|\text{rot } \tilde{\varphi}\|_{r'}}; \tilde{\varphi} \in X^{r'}_\sigma(\Omega) \text{ with } (\tilde{\varphi}, \varphi_j) = 0 \text{ for } j = 1, \cdots, N \right\} \\
&\quad + C\|u\|_r
\end{aligned} \tag{2.31}$$

Notice that by taking such p as in (2.26) for every $\Phi \in X^{r'}(\Omega)$ we may choose $\varphi \in X^{r'}_\sigma(\Omega)$ with rot $\varphi =$ rot Φ. Furthermore, defining $\tilde{\varphi} \equiv \varphi - \sum_{j=1}^N (\varphi, \varphi_j)\varphi_j$, we see that $\tilde{\varphi} \in X^{r'}_\sigma(\Omega)$ satisfies rot $\varphi =$ rot $\tilde{\varphi}$ with $(\tilde{\varphi}, \varphi_j) = 0$ for all $j = 1, \cdots, N$. By (2.16) with r replaced by r', it holds

$$\|\nabla \tilde{\varphi}\|_{r'} + \|\tilde{\varphi}\|_{r'} \leqq C\|\text{rot } \tilde{\varphi}\|_{r'}$$

Hence it follows from (2.31) that

$$
\begin{aligned}
&\|\nabla u\|_r \\
\leqq\ & C\sup\left\{\frac{|(\mathrm{rot}\ u, \mathrm{rot}\ \tilde{\varphi})|}{\|\nabla\tilde{\varphi}\|_{r'}+\|\tilde{\varphi}\|_{r'}}; \tilde{\varphi}\in X^{r'}_\sigma(\Omega) \text{ with } (\tilde{\varphi},\varphi_j)=0 \text{ for } j=1,\cdots,N\right\} \\
&+C\|u\|_r \\
\leqq\ & C\sup_{\varphi\in X^{r'}_\sigma(\Omega)}\frac{|(\mathrm{rot}\ u,\mathrm{rot}\ \varphi)|}{\|\nabla\varphi\|_{r'}+\|\varphi\|_{r'}}+C\|u\|_r,
\end{aligned}
$$

which implies (2.30).

We prove (2.21) by contradiction. Suppose that (2.21) does not hold. Then there is a sequence $\{w_m\}_{m=1}^\infty$ in $X^r_\sigma(\Omega)$ such that

(2.32) $$\|\nabla w_m\|_r + \|w_m\|_r \equiv 1 \quad \text{for all } m=1,2,\cdots,$$

(2.33) $$\varepsilon_m \equiv \sup_{\varphi\in X^{r'}_\sigma(\Omega)}\frac{|(\mathrm{rot}\ w_m,\mathrm{rot}\ \varphi)|}{\|\nabla\varphi\|_{r'}+\|\varphi\|_{r'}}\to 0 \quad \text{as } m\to\infty,$$

(2.34) $$(w_m,\varphi_j)\to 0 \quad \text{for } j=1,\cdots,N \text{ as } m\to\infty.$$

By (2.32) and the Rellich theorem, we may assume that there is $w\in X^r_\sigma(\Omega)$ such that

$$\nabla w_m \rightharpoonup \nabla w \text{ weakly in } L^r(\Omega), \quad w_m\to w \text{ strongly in } L^r(\Omega).$$

Hence by (2.33) w satisfies that

$$(\mathrm{rot}\ w, \mathrm{rot}\ \varphi)=0 \quad \text{for all } \varphi\in X^{r'}_\sigma.$$

It follows from Proposition 2.1 (1) that $w\in X_{har}(\Omega)$. By (2.34) we have

$$(w,\varphi_j)=0 \quad \text{for } j=1,\cdots N,$$

which yields $w=0$. On the other hand, we obtain from (2.30) that

$$\|\nabla w_m\|_r \leqq C\varepsilon_m + C\|w_m\|_r.$$

Since $w_m\to 0$ strongly in $L^r(\Omega)$, we have by (2.33) that

$$\nabla w_m \to 0 \quad \text{strongly in } L^r(\Omega),$$

which contradicts (2.32).

(2) The proof is quite similar to that of the above (1). Compared with (2.30), we shall show that there is a constant $C=C(r)$ such that

(2.35) $$\|\nabla u\|_r \leqq C\sup_{\psi\in V^{r'}_\sigma(\Omega)}\frac{|(\mathrm{rot}\ u,\mathrm{rot}\ \psi)|}{\|\nabla\psi\|_{r'}+\|\psi\|_{r'}}+C\|u\|_r$$

holds for all $u\in V^r_\sigma(\Omega)$. Since div $u=0$ and since the identity (2.13) holds for $\phi\in V^{r'}(\Omega)$, we have similarly to (2.12) that for every $\varepsilon>0$ there is a constant C_ε such that

$$|(\nabla u,\nabla\Psi)|\leqq|(\mathrm{rot}\ u,\mathrm{rot}\ \Psi)|+(\varepsilon\|\nabla u\|_r+C_\varepsilon\|u\|_r)\|\Psi\|_{W^{1,r'}(\Omega)} \quad \text{for all } \Psi\in V^{r'}(\Omega).$$

Hence taking ε sufficiently small, we obtain from (2.4) in Lemma 2.1 that

$$
\begin{aligned}
\|\nabla u\|_r &\leqq C \sup_{\Psi\in V^{r'}(\Omega)} \frac{|(\text{rot } u, \text{rot } \Psi)|}{\|\Psi\|_{W^{1,r'}(\Omega)}} + C\|u\|_r \\
&\leqq C \sup_{\Psi\in V^{r'}(\Omega)} \frac{|(\text{rot } u, \text{rot } \Psi)|}{\|\text{rot } \Psi\|_{r'}} + C\|u\|_r \\
&= C \sup_{\psi\in V_\sigma^{r'}(\Omega)} \frac{|(\text{rot } u, \text{rot } \psi)|}{\|\text{rot } \psi\|_{r'}} + C\|u\|_r \\
&= C \sup\left\{ \frac{|(\text{rot } u, \text{rot } \tilde{\psi})|}{\|\text{rot } \tilde{\psi}\|_{r'}}; \tilde{\psi} \in V_\sigma^{r'}(\Omega) \text{ with } (\tilde{\psi}, \psi_j) = 0 \text{ for } j = 1, \cdots, L \right\} \\
&\qquad + C\|u\|_r
\end{aligned}
\tag{2.36}
$$

Notice that by taking such p as in (2.29) for every $\Psi \in V^{r'}(\Omega)$ we may choose $\psi \in V_\sigma^{r'}(\Omega)$ with rot $\Psi =$ rot ψ. Furthermore, defining $\tilde{\psi} \equiv \psi - \sum_{j=1}^{L}(\psi, \psi_j)\psi_j$, we see that $\tilde{\psi} \in V_\sigma^{r'}(\Omega)$ satisfies rot $\psi =$ rot $\tilde{\psi}$ with $(\tilde{\psi}, \psi_j) = 0$ for all $j = 1, \cdots, L$. By (2.17) with r replaced by r', it holds

$$\|\nabla\tilde{\psi}\|_{r'} + \|\tilde{\psi}\|_{r'} \leqq C\|\text{rot } \tilde{\psi}\|_{r'}$$

Hence it follows from (2.36) that

$$
\begin{aligned}
&\|\nabla u\|_r \\
&\leqq C \sup\left\{ \frac{|(\text{rot } u, \text{rot } \tilde{\psi})|}{\|\nabla\tilde{\psi}\|_{r'} + \|\tilde{\psi}\|_{r'}}; \tilde{\psi} \in V_\sigma^{r'}(\Omega) \text{ with } (\tilde{\psi}, \psi_j) = 0 \text{ for } j = 1, \cdots, L \right\} \\
&\qquad + C\|u\|_r \\
&\leqq C \sup_{\psi\in V_\sigma^{r'}(\Omega)} \frac{|(\text{rot } u, \text{rot } \psi)|}{\|\nabla\psi\|_{r'} + \|\psi\|_{r'}} + C\|u\|_r,
\end{aligned}
$$

which implies (2.35).

Now, we prove (2.22) again by contradiction. Suppose that (2.22) does not hold. Then there is a sequence $\{w_m\}_{m=1}^{\infty}$ in $V_\sigma^r(\Omega)$ such that

$$\|\nabla w_m\|_r + \|w_m\|_r \equiv 1 \quad \text{for all } m = 1, 2, \cdots, \tag{2.37}$$

$$\varepsilon_m \equiv \sup_{\psi\in V_\sigma^{r'}(\Omega)} \frac{|(\text{rot } w_m, \text{rot } \psi)|}{\|\nabla\psi\|_{r'} + \|\psi\|_{r'}} \to 0 \quad \text{as } m \to \infty, \tag{2.38}$$

$$(w_m, \psi_j) \to 0 \quad \text{for } j = 1, \cdots, L \text{ as } m \to \infty. \tag{2.39}$$

By (2.37) and the Rellich theorem, we may assume that there is $w \in V_\sigma^r(\Omega)$ such that

$$\nabla w_m \rightharpoonup \nabla w \text{ weakly in } L^r(\Omega), \quad w_m \to w \text{ strongly in } L^r(\Omega).$$

Hence by (2.38) w satisfies that

$$(\text{rot } w, \text{rot } \psi) = 0 \quad \text{for all } \psi \in V_\sigma^{r'}.$$

It follows from Proposition 2.1 (2) that $w \in V_{har}(\Omega)$. By (2.39) we have

$$(w, \psi_j) = 0 \quad \text{for } j = 1, \cdots L,$$

which yields $w = 0$. On the other hand, we obtain from (2.35) that

$$\|\nabla w_m\|_r \leqq C\varepsilon_m + C\|w_m\|_r.$$

Since $w_m \to 0$ strongly in $L^r(\Omega)$, we have by (2.38) that

$$\nabla w_m \to 0 \quad \text{strongly in } L^r(\Omega),$$

which contradicts (2.37). This proves Lemma 2.4.

Based on Lemma 2.4, we have the following result on existence of weak solution to (0.3).

Lemma 2.5 *Let Ω be as in the Assumption and let $1 < r < \infty$.*
(1)(in case $\gamma_\nu w = 0$) For every $u \in L^r(\Omega)$ there is a function $w \in X^r_\sigma(\Omega)$ such that

$$(2.40) \qquad (\text{rot } w, \text{rot } \varphi) = (u, \text{rot } \varphi) \quad \textit{for all } \varphi \in X^{r'}_\sigma(\Omega)$$

with the estimate

$$(2.41) \qquad \|\nabla w\|_r + \|w\|_r \leqq C\|u\|_r,$$

where $C = C(r)$. If there is another $\tilde{w} \in X^r_\sigma(\Omega)$ satisfying (2.40), then we have rot $w =$ rot $\tilde{w}$.
(2)(in case $\tau_\nu w = 0$) For every $u \in L^r(\Omega)$ there is a function $w \in V^r_\sigma(\Omega)$ such that

$$(2.42) \qquad (\text{rot } w, \text{rot } \psi) = (u, \text{rot } \psi) \quad \textit{for all } \psi \in V^{r'}_\sigma(\Omega)$$

with the estimate

$$(2.43) \qquad \|\nabla w\|_r + \|w\|_r \leqq C\|u\|_r,$$

where $C = C(r)$. If there is another $\tilde{w} \in V^r_\sigma(\Omega)$ satisfying (2.42), then we have rot $w =$ rot $\tilde{w}$.

Proof. Since the proof of (1) and (2) is quite parallel, we may give it simultaneously. To this end, let us introduce the spaces $Y^r_\sigma(\Omega)$, $Y_{har}(\Omega)$ and $Z^r_\sigma(\Omega)$ by

$$Y^r_\sigma(\Omega) \equiv X^r_\sigma(\Omega), \text{and } V^r_\sigma(\Omega), \quad Y_{har}(\Omega) \equiv X_{har}(\Omega), \text{and } V_{har}(\Omega)$$
$$Z^r_\sigma(\Omega) \equiv \begin{cases} \{w \in X^r_\sigma(\Omega); (w, \varphi_j) = 0 \text{ for } j = 1, \cdots, N\}, \\ \{w \in V^r_\sigma(\Omega); (w, \psi_j) = 0 \text{ for } j = 1, \cdots, L\} \end{cases}$$

with the norm $\|w\|_{Z^r_\sigma} \equiv \|\text{rot } w\|_r$, where $\{\varphi_1, \cdots, \varphi_N\}$ and $\{\psi_1, \cdots, \psi_L\}$ are the bases of $X_{har}(\Omega)$ and $V_{har}(\Omega)$ in Lemma 2.4, respectively. By (2.21) and (2.22), we see that $Z^r_\sigma(\Omega)$ is a closed subspace in $W^{1,r}(\Omega)$, and hence it is a reflexive Banach space.

We consider the map $F : Z^r_\sigma(\Omega) \to Z^{r'}_\sigma(\Omega)^*$ defined by

$$\langle Fw, \phi\rangle \equiv (\text{rot } w, \text{rot } \phi) \quad \text{for } w \in Z^r_\sigma(\Omega) \text{ and } \phi \in Z^{r'}_\sigma(\Omega),$$

where $\langle\cdot,\cdot\rangle$ denotes the duality paring between $Z^{r'}_\sigma(\Omega)^*$ and $Z^{r'}_\sigma(\Omega)$. By the same argument as in (2.31) and (2.36), it holds that

$$\sup_{\phi\in Z^{r'}_\sigma(\Omega)} \frac{|(\mathrm{rot}\ w, \mathrm{rot}\ \phi)|}{\|\mathrm{rot}\ \phi\|_{r'}} = \sup_{\psi\in Y^{r'}_\sigma(\Omega)} \frac{|(\mathrm{rot}\ w, \mathrm{rot}\ \psi)|}{\|\mathrm{rot}\ \psi\|_{r'}}. \tag{2.44}$$

Hence Lemma 2.4 assures that the range $R(F)$ of F is closed in $Z^{r'}_\sigma(\Omega)^*$. In fact, we have

$$R(F) = Z^{r'}_\sigma(\Omega)^*. \tag{2.45}$$

Suppose that $R(F) \subset Z^{r'}_\sigma(\Omega)^*$. Then there exists a $g \in Z^{r'}_\sigma(\Omega)^{**}$ with $g \neq 0$ such that

$$g(Fw) = 0 \quad \text{for all } w \in Z^r_\sigma(\Omega). \tag{2.46}$$

Since $Z^{r'}_\sigma(\Omega)$ is reflexive, there is a unique $\phi \in Z^{r'}_\sigma(\Omega)$ with $\|g\|_{Z^{r'}_\sigma(\Omega)^{**}} = \|\mathrm{rot}\ \phi\|_r$ such that

$$g(f) = \langle f, \phi\rangle \quad \text{for all } f \in Z^{r'}_\sigma(\Omega)^*.$$

Hence, taking $f = Fw$ in the above identity, we have by (2.46) that

$$0 = g(Fw) = \langle Fw, \phi\rangle = (\mathrm{rot}\ w, \mathrm{rot}\ \phi) \quad \text{for all } w \in Z^r_\sigma(\Omega).$$

Replacing r by r' in (2.44) we obtain from Lemma 2.4 that $\phi = 0$, which yields $g = 0$. This causes contradiction.

For every $u \in L^r(\Omega)$, we define $f_u \in Z^{r'}_\sigma(\Omega)^*$ by

$$\langle f_u, \phi\rangle = (u, \mathrm{rot}\ \phi) \quad \text{for } \phi \in Z^{r'}_\sigma(\Omega).$$

Obviously, we have $\|f_u\|_{Z^{r'}_\sigma(\Omega)^*} \leqq \|u\|_r$. Now it follows from (2.45) that there is $w \in Z^r_\sigma(\Omega)$ such that $Fw = f_u$, which implies that

$$(\mathrm{rot}\ w, \mathrm{rot}\ \phi) = (u, \mathrm{rot}\ \phi) \quad \text{for all } \phi \in Z^{r'}_\sigma(\Omega).$$

Again by the similar argument to (2.31) and (2.36), this identity yields

$$(\mathrm{rot}\ w, \mathrm{rot}\ \psi) = (u, \mathrm{rot}\ \psi) \quad \text{for all } \psi \in Y^{r'}_\sigma(\Omega).$$

Since $(w, \varphi_j) = 0$ for $j = 1, \cdots, N$ and $(w, \psi_j) = 0$ for $j = 1, \cdots, L$, from (2.21) and (2.22) we obtain (2.41) and (2.43), respectively.

Finally, suppose that $\tilde{w} \in Y^r_\sigma(\Omega)$ satisfies (2.40) and (2.42). Then we have

$$(\mathrm{rot}\ (w - \tilde{w}), \mathrm{rot}\ \psi) = 0 \quad \text{for all } \psi \in Y^{r'}_\sigma(\Omega).$$

From Proposition 2.1, we conclude that $w - \tilde{w} \in Y_{har}(\Omega)$. This proves Lemma 2.5.

To show that $X^r_{har}(\Omega) \subset C^\infty(\bar{\Omega})$ and $V^r_{har}(\Omega) \subset C^\infty(\bar{\Omega})$ for all $1 < r < \infty$, we need to consider the following two elliptic systems of generalized boundary value problem in the sense of Agmon-Douglis-Nirenberg [1].

Lemma 2.6 (1) *Let $s \geq 2$ be an integer and let $1 < r < \infty$. Suppose that $f = (f^1, f^2, f^3) \in W^{s-2,r}(\Omega)$, $\phi \in W^{s-1-1/r,r}(\partial\Omega)$, $\Phi = (\Phi^1, \Phi^2, \Phi^3) \in W^{s-1/r,r}(\partial\Omega)$. Then the boundary value problem*

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ \text{div } u = \phi & \text{on } \partial\Omega, \\ u \times \nu = \Phi & \text{on } \partial\Omega \end{cases} \tag{2.47}$$

takes the form of the uniformly elliptic operator with the complementing boundary conditions in the sense of Agmon-Douglis-Nirenberg [1, Theorem 10.5], and hence it holds

$$\|u\|_{W^{s,r}(\Omega)} \leqq C(\|f\|_{W^{s-2,r}(\Omega)} + \|\phi\|_{W^{s-1-1/r,r}(\partial\Omega)} + \|\Phi\|_{W^{s-1/r,r}(\partial\Omega)} + \|u\|_r), \tag{2.48}$$

where $C = C(\Omega, s, r)$.

(2) *Let $s \geq 2$ be an integer and let $1 < r < \infty$. Suppose that $f = (f^1, f^2, f^3) \in W^{s-2,r}(\Omega)$, $\psi \in W^{s-1/r,r}(\partial\Omega)$, $\Psi = (\Psi^1, \Psi^2, \Psi^3) \in W^{s-1-1/r,r}(\partial\Omega)$. Then the boundary value problem*

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u \cdot \nu = \psi & \text{on } \partial\Omega, \\ \text{rot } u \times \nu = \Psi & \text{on } \partial\Omega. \end{cases} \tag{2.49}$$

takes the form of the uniformly elliptic operator with the complementing boundary conditions in the sense of Agmon-Douglis-Nirenberg [1, Theorem 10.5], and hence it holds

$$\|u\|_{W^{s,r}(\Omega)} \leqq C(\|f\|_{W^{s-2,r}(\Omega)} + \|\Psi\|_{W^{s-1-1/r,r}(\partial\Omega)} + \|\psi\|_{W^{s-1/r,r}(\partial\Omega)} + \|u\|_r), \tag{2.50}$$

where $C = C(\Omega, s, r)$.

An immediate consequence of Lemma 2.6 is the following $W^{s,r}$-bounds via operators rot and div .

Lemma 2.7 *Let $s \geq 2$ and let $1 < r < \infty$.*

(1) *(in case $\gamma_\nu u$). Suppose that $u \in L^r(\Omega)$ with* div $u \in W^{s-1,r}(\Omega)$, rot $u \in W^{s-1,r}(\Omega)$ *and $\gamma_\nu u \in W^{s-1/r,r}(\partial\Omega)$. Then we have $u \in W^{s,r}(\Omega)$ with the estimate*

$$\|u\|_{W^{s,r}(\Omega)} \leqq C(\|\text{div } u\|_{W^{s-1,r}(\Omega)} + \|\text{rot } u\|_{W^{s-1,r}(\Omega)} + \|\gamma_\nu u\|_{W^{s-1/r,r}(\partial\Omega)} + \|u\|_r), \tag{2.51}$$

where $C = C(\Omega, s, r)$.

(2) *(in case $\tau_\nu u$) Suppose that $u \in L^r(\Omega)$ with* div $u \in W^{s-1,r}(\Omega)$, rot $u \in W^{s-1,r}(\Omega)$ *and $\tau_\nu u \in W^{s-1/r,r}(\partial\Omega)$. Then we have $u \in W^{s,r}(\Omega)$ with the estimate*

$$\|u\|_{W^{s,r}(\Omega)} \leqq C(\|\text{div } u\|_{W^{s-1,r}(\Omega)} + \|\text{rot } u\|_{W^{s-1,r}(\Omega)} + \|\tau_\nu u\|_{W^{s-1/r,r}(\partial\Omega)} + \|u\|_r), \tag{2.52}$$

where $C = C(\Omega, s, r)$.

Proof. (1) In (2.49) we may take $f = -\Delta u = \text{rot rot } u - \nabla(\text{div } u)$, $\Psi = \text{rot } u \times \nu$ and $\psi = \gamma_\nu u$. Indeed, we have

$$\|f\|_{W^{s-2,r}(\Omega)} \leqq C(\|\text{rot } u\|_{W^{s-1,r}(\Omega)} + \|\text{div } u\|_{W^{s-1,r}(\Omega)}),$$

and the trace theorem yields

$$\|\Psi\|_{W^{s-1-1/r,r}(\partial\Omega)} \leqq C\|\text{rot } u\|_{W^{s-1-1/r,r}(\partial\Omega)} \leqq C\|\text{rot } u\|_{W^{s-1,r}(\Omega)},$$

from which and (2.50) we obtain (2.51).

(2) In (2.47) we may take $f = -\Delta u = \text{rot rot } u - \nabla(\text{div } u)$, $\phi = \text{div } u$ and $\Phi = \tau_\nu u$. Since

$$\|\phi\|_{W^{s-1-1/r,r}(\partial\Omega)} = \|\text{div } u\|_{W^{s-1-1/r,r}(\partial\Omega)} \leqq C\|\text{div } u\|_{W^{s-1,r}(\Omega)},$$

we see that (2.48) implies (2.52). This proves Lemma 2.7.

2.2 L^r-Helmholtz-Weyl decomposition; Proof of Theorems 1.1 and 1.2

Proof of Theorem 1.1: (1) Let $u \in X^r_{har}(\Omega)$ for some $1 < r < \infty$. By Lemma 2.2 (1), we have $u \in W^{1,r}(\Omega)$, which yields $u \in W^{1-1/r,r}(\partial\Omega)$ with $\gamma_\nu u = u \cdot \nu|_{\partial\Omega} = 0$. Then it follows from (2.51) that $u \in W^{s,r}(\Omega)$ for all $s \geq 2$, which implies that $u \in C^\infty(\bar{\Omega})$. Similarly, the fact that $V^r_{har}(\Omega) \subset C^\infty(\bar{\Omega})$ follows from Lemma 2.2 (2) and (2.52).

(2) Let $u \in L^r(\Omega)$. The scalar potential $p \in W^{1,r}(\Omega)$ is taken is such a way that

$$(\nabla p, \nabla\eta) = (u, \nabla\eta) \quad \text{for all } \eta \in W^{1,r'}(\Omega) \tag{2.53}$$

with the estimate

$$\|\nabla p\|_r \leqq C\|u\|_r, \tag{2.54}$$

where $C = C(r)$. Such a scalar function p as (2.53) is unique up to an additive constant. This was proved by Simader-Sohr [41], [42].

The vector potential $w \in V^r_\sigma(\Omega)$ in (1.9) can be derived from Lemma 2.5 (2). For $u \in L^r(\Omega)$ we take w such that (2.42) and (2.43) are fulfilled. Note that rot $w \in L^r_\sigma(\Omega)$. To see this, we may verify that $\gamma_\nu(\text{rot } w) = 0$. Since the usual trace operator $\gamma_0 : W^{1,r'}(\Omega) \to W^{1-1/r',r'}(\partial\Omega)$ is surjective, by the generalized Stokes formula (1.1), it suffices to show that

$$(\text{rot } w, \nabla q) = 0 \quad \text{for all } q \in W^{1,r'}\Omega).$$

Since rot $(\nabla q) = 0$ and since $\tau_\nu w = 0$, by (1.2) we have this identity.

Let us define $h \equiv u - \nabla p - \text{rot } w$. Then we see

$$(h, \nabla\eta) = 0 \quad \text{for all } \eta \in C_0^\infty(\Omega), \tag{2.55}$$

$$(h, \text{rot } \phi) = 0 \quad \text{for all } \phi \in C_0^\infty(\Omega). \tag{2.56}$$

We take $\tilde{\phi} \equiv \phi - \nabla q$ with $q \in C^\infty(\bar{\Omega})$ satisfying $\Delta q = \text{div } \phi$ in Ω, $q = 0$ on $\partial\Omega$. Since $\tilde{\phi} \in V^{r'}_\sigma(\Omega)$ with rot $\phi = \text{rot } \tilde{\phi}$, we have by (1.1), (1.2), (2.42) and (2.53) that

$$(h, \nabla\eta) = (u - \nabla p, \nabla\eta) - (\text{rot } w, \nabla\eta) = 0,$$
$$(h, \text{rot } \phi) = (u - \text{rot } w, \text{rot } \tilde{\phi}) - (\nabla p, \text{rot } \phi) = (p, \text{div } (\text{rot } \phi)) - \langle\gamma_0 p, \gamma_\nu(\text{rot } \phi)\rangle_{\partial\Omega} = 0,$$

which yields (2.55) and (2.56). This implies that div $h = 0$ and rot $h = 0$ in the sense of distributions in Ω. Since $\gamma_\nu h = \gamma_\nu(u - \nabla p) - \gamma_\nu(\text{rot } w) = 0$, we obtain $h \in X_{har}(\Omega)$. Let $\{\varphi_1, \cdots, \varphi_N\}$ be an orthogonal basis in $X_{har}(\Omega)$. Then similarly to (2.55) and (2.56) we see that $(h, \varphi_j) = (u, \varphi_j)$ for all $j = 1, \cdots, N$, from which it follows that

$$h = \sum_{j=1}^{N} (u, \varphi_j)\varphi_j. \tag{2.57}$$

Finally, we obtain the following representation of u:

$$u = \sum_{j=1}^{N} (u, \varphi_j)\varphi_j + \text{rot } w + \nabla p,$$

which yields (1.9). The estimate (1.10) is a consequence of (2.54), (2.43) and (2.57).

We next show the uniqueness such as (1.11). Suppose that u has another expression

$$u = \tilde{h} + \text{rot } \tilde{w} + \nabla \tilde{p},$$

with $\tilde{h} \in X_{har}(\Omega)$, $\tilde{w} \in V^r_\sigma(\Omega)$ and $\tilde{p} \in W^{1,r}(\Omega)$. Similarly to (2.55) and (2.56), we see easily that

$$(h - \tilde{h}, \varphi_j) = 0, \quad j = 1, \cdots, N,$$

which implies $h = \tilde{h}$. Hence we have

$$\text{rot } (w - \tilde{w}) = -\nabla(p - \tilde{p}).$$

Since $\gamma_\nu(\text{rot } \psi) = 0$ for all $\psi \in \hat{\mathcal{V}}(\Omega)$, we obtain from (1.1) that

$$(\text{rot } (w - \tilde{w}), \text{rot } \psi) = (-\nabla(p - \tilde{p}), \text{rot } \psi) = (p - \tilde{p}, \text{div } (\text{rot } \psi)) - \langle \gamma_0(p - \tilde{p}), \gamma_\nu(\text{rot } \psi) \rangle_{\partial\Omega} = 0$$

for all $\psi \in \hat{\mathcal{V}}(\Omega)$. Then it follows from Proposition 2.1 (2) that $w - \tilde{w} \in V_{har}(\Omega)$, which means rot $w =$ rot $\tilde{w}$. As a result, we get $\nabla p = \nabla \tilde{p}$.

(3) Let $u \in L^r(\Omega)$. Compared with (2.53), the scalar potential $p \in W^{1,r}_0(\Omega)$ is taken is as the weak solution of the Dirichlet problem for Δ in Ω, i.e.,

$$(\nabla p, \nabla \eta) = (u, \nabla \eta) \quad \text{for all } \eta \in W^{1,r'}_0(\Omega) \tag{2.58}$$

with the estimate

$$\|\nabla p\|_r \leqq C\|u\|_r, \tag{2.59}$$

where $C = C(r)$. Such a scalar function p as (2.58) is unique and moreover, p is subject to the estimate (2.59). Since Ω is a bounded domain, this was proved for all $1 < r < \infty$ by Simader-Sohr [41], [42]. It should be noted that

$$\tau_\nu(\nabla p) = 0 \tag{2.60}$$

Indeed, since $\gamma_0 p = 0$, we see by (1.1) and (1.2) that

$$\langle \tau_\nu(\nabla p), \gamma_0 \phi \rangle_{\partial\Omega} = (\nabla p, \text{rot } \phi) = \langle \gamma_0 p, \tau_\nu(\text{rot } \phi) \rangle_{\partial\Omega} = 0$$

for all $\phi \in W^{1,r'}(\Omega)$. Since the trace operator $\gamma_0 : W^{1,r'}(\Omega) \mapsto W^{1-1/r',r'}(\partial\Omega)$ is surjective, the above identity implies (2.60).

The vector potential $w \in X^r_\sigma(\Omega)$ in (1.12) can be derived from Lemma 2.5 (1). For $u \in L^r(\Omega)$ we take $w \in X^r_\sigma(\Omega)$ such that (2.40) and (2.41) are fulfilled. Let us define $h \equiv u - \nabla p - \text{rot}\ w$. Then, similarly to the proof of the above (1) we see that h satisfies (2.55) and (2.56). Indeed, the identity (2.55) is a consequence of (2.58). As for the proof of (2.56), for every $\phi \in C^\infty_0(\Omega)$, we take $\tilde{\phi} = \phi - \nabla q$, where $q \in C^\infty(\bar{\Omega})$ is a solution of $\Delta q = \text{div}\ \phi$ in Ω, $\partial q/\partial\nu = 0$ on $\partial\Omega$. Since $\tilde{\phi} \in X^{r'}_\sigma(\Omega)$ with rot $\phi =$ rot $\tilde{\phi}$, from (2.40) we obtain (2.56). To see $h \in V_{har}(\Omega)$, by (2.60) we may show that

$$\tau_\nu(u - \text{rot}\ w) = 0. \tag{2.61}$$

Since rot $(u - \text{rot}\ w) = \text{rot}\ (h + \nabla p) = 0$ in the sense of distributions in Ω, we have by (2.40) and (1.2) that

$$\langle \tau_\nu(u - \text{rot}\ w), \gamma_0\phi\rangle_{\partial\Omega} = (u - \text{rot}\ w, \text{rot}\ \phi) = (u - \text{rot}\ w, \text{rot}\ \tilde{\phi}) = 0$$

for all $\phi \in W^{1,r'}(\Omega)$, where $\tilde{\phi} \in X^{r'}_\sigma(\Omega)$ is defined as $\tilde{\phi} = \phi - \nabla q$ with $q \in W^{2,r'}(\Omega)$ such that $\Delta q = 0$ in Ω, $\partial q/\partial\nu = \phi \cdot \nu$ on $\partial\Omega$. This implies $\tau_\nu(u - \text{rot}\ w) = 0$.

Let $\{\psi_1, \cdots, \psi_L\}$ be an orthogonal basis in $V_{har}(\Omega)$. By (1.1) and (1.2) we can easily verify that $(u, \psi_j) = (h, \psi_j)$ for $j = 1, \cdots, L$, which yields

$$u = \sum_{j=1}^{L}(u, \psi_j)\psi_j + \text{rot}\ w + \nabla p.$$

This implies (1.12). The estimate (1.13) is a consequence of (2.59), (2.41) and the representation of $h = \sum_{j=1}^{L}(u, \psi_j)\psi_j$.

Finally we show the uniqueness such as (1.14). Suppose that u has another expression

$$u = \tilde{h} + \text{rot}\ \tilde{w} + \nabla\tilde{p},$$

with $\tilde{h} \in V_{har}(\Omega)$, $\tilde{w} \in X^r_\sigma(\Omega)$ and $\tilde{p} \in W^{1,r}_0(\Omega)$. Similarly to (2.55) and (2.56), we see easily that

$$(h - \tilde{h}, \psi_j) = 0, \quad j = 1, \cdots, L,$$

which implies $h = \tilde{h}$. Hence we have

$$\text{rot}\ (w - \tilde{w}) = -\nabla(p - \tilde{p}).$$

Since $\gamma_0 p = \gamma_0\tilde{p} = 0$, we obtain from (1.1) that

$$(\text{rot}\ (w - \tilde{w}), \text{rot}\ \varphi) = (-\nabla(p - \tilde{p}), \text{rot}\ \varphi) = -\langle \gamma_0(p - \tilde{p}), \gamma_\nu(\text{rot}\ \varphi)\rangle_{\partial\Omega} = 0$$

for all $\varphi \in \hat{\mathcal{X}}(\Omega)$. Then it follows from Proposition 2.1 (1) that $w - \tilde{w} \in X_{har}(\Omega)$, which means rot $w =$ rot $\tilde{w}$. Consequently, we get $\nabla p = \nabla\tilde{p}$. This proves Theorem 1.1.

Proof of Corollary 1.1: The direct sums (1.15) and (1.16) are consequences of the representation formulas (1.9) and (1.12) with uniqueness properties as (1.11) and (1.14), respectively. Hence the operators S_r, R_r and Q_r are well-defined by (1.17). Their continuity in $L^r(\Omega)$ stems from the estimates (1.10) and (1.13). The properties $S_r^2 = S_r$, $R_r^2 = R_r$ and $Q_r^2 = Q_r$ are guaranteed by the uniqueness (1.11) and (1.14).

Suppose that $u \in L^r(\Omega)$ and $\tilde{u} \in L^{r'}(\Omega)$ are decomposed as

$$
\begin{aligned}
&u = h + \text{rot } w + \nabla p, \\
&\ \text{where } h \in X_{har}(\Omega) \text{ (or } V_{har}(\Omega)),\ w \in V_\sigma^r(\Omega)(\text{or } X_\sigma^r(\Omega)),\ p \in W^{1,r}(\Omega) \text{ or } (W_0^{1,r}(\Omega)), \\
&\tilde{u} = \tilde{h} + \text{rot } \tilde{w} + \nabla \tilde{p}, \\
&\ \text{where } \tilde{h} \in X_{har}(\Omega) \text{ (or } V_{har}(\Omega)),\ \tilde{w} \in V_\sigma^{r'}(\Omega)(\text{or } X_\sigma^{r'}(\Omega)),\ \tilde{p} \in W^{1,r'}(\Omega) \text{ or } (W_0^{1,r'}(\Omega)).
\end{aligned}
$$

By (1.17) we have

$$
\begin{aligned}
&S_r u = h, \quad R_r u = \text{rot } w, \quad Q_r u = \nabla p, \\
&S_{r'} \tilde{u} = \tilde{h}, \quad R_{r'} \tilde{u} = \text{rot } \tilde{w}, \quad Q_{r'} \tilde{u} = \nabla \tilde{p}
\end{aligned}
$$

In the same way as in (2.55) and (2.56) it is easy to show that

$$
\begin{aligned}
&(S_r u, \tilde{u}) = (u, S_{r'} \tilde{u}) = (h, \tilde{h}), \\
&(R_r u, \tilde{u}) = (u, R_{r'} \tilde{u}) = (\text{rot } w, \text{rot } \tilde{w}), \\
&(Q_r u, \tilde{u}) = (u, Q_{r'} \tilde{u}) = (\nabla p, \nabla \tilde{p}),
\end{aligned}
$$

from which we conclude that

$$
S_r^* = S_{r'}, \quad R_r^* = R_{r'}, \quad Q_r^* = Q_{r'}.
$$

This completes the proof of Corollary 1.1.

Proof of Theorem 1.2; $W^{s,p}$-bounds via rot *and* div . Let us first show that both (2.51) and (2.52) in Lemma 2.7 hold also for $s = 1$.

Lemma 2.8 *Let $1 < r < \infty$.*

(1) *(in case $\gamma_\nu u$). Suppose that $u \in L^r(\Omega)$ with* div $u \in L^r(\Omega)$, rot $u \in L^r(\Omega)$ *and $\gamma_\nu u \in W^{1-1/r,r}(\partial\Omega)$. Then we have $u \in W^{1,r}(\Omega)$ with the estimate*

$$
\|\nabla u\|_r + \|u\|_r \leqq C(\|\text{div } u\|_r + \|\text{rot } u\|_r + \|\gamma_\nu u\|_{W^{1-1/r,r}(\partial\Omega)} + \|u\|_r), \tag{2.62}
$$

where $C = C(\Omega, r)$.

(2) *(in case $\tau_\nu u$) Suppose that $u \in L^r(\Omega)$ with* div $u \in L^r(\Omega)$, rot $u \in L^r(\Omega)$ *and $\tau_\nu u \in W^{1-1/r,r}(\partial\Omega)$. Then we have $u \in W^{1,r}(\Omega)$ with the estimate*

$$
\|\nabla u\|_r + \|u\|_r \leqq C(\|\text{div } u\|_r + \|\text{rot } u\|_r + \|\tau_\nu u\|_{W^{1-1/r,r}(\partial\Omega)} + \|u\|_r), \tag{2.63}
$$

where $C = C(\Omega, r)$.

Proof. We shall reduce (2.62) and (2.63) to the homogeneous condition on $\partial\Omega$ such as (2.8) and (2.9), respectively.

(1) To show (2.62), we consider the Neumann problem

$$\text{(2.64)} \qquad \begin{cases} \Delta p = \text{div } u \quad \text{in } \Omega, \\ \dfrac{\partial p}{\partial \nu} = u \cdot \nu \quad \text{on } \partial\Omega. \end{cases}$$

Since div $u \in L^r(\Omega)$ with $u \cdot \nu = \gamma_\nu u \in W^{1-1/r,r}(\partial\Omega)$, there exist a uniquely solution $p \in W^{2,r}(\Omega)$, up to an additive constant, of (2.64) with

$$\text{(2.65)} \qquad \|\nabla^2 p\|_r + \|\nabla p\|_r \leqq C(\|\text{div } u\|_r + \|\gamma_\nu u\|_{W^{1-1/r,r}(\partial\Omega)}),$$

where $C = C(\Omega, r)$. Taking $v \equiv u - \nabla p$, we have by (2.64) and (2.65) that $v \in L^r(\Omega)$ with div $v = 0$, rot $v =$ rot $u \in L^r(\Omega)$ and $\gamma_\nu v = 0$. Hence, applying (2.8) to v, we see from (2.65) that

$$\text{(2.66)} \qquad \begin{aligned} \|\nabla v\|_r + \|v\|_r &\leqq C(\|\text{rot } u\|_r + \|u\|_r + \|\nabla p\|_r) \\ &\leqq C(\|\text{div } u\|_r + \|\text{rot } u\|_r + \|\gamma_\nu u\|_{W^{1-1/r,r}(\partial\Omega)} + \|u\|_r), \end{aligned}$$

where $C = C(\Omega, r)$. Since $u = v + \nabla p$, from (2.65) and (2.66) we obtain (2.62).

(2) We shall next show (2.63). Let us consider the boundary value problem

$$\text{(2.67)} \qquad \begin{cases} -\Delta w = \text{rot } u \quad \text{in } \Omega, \\ w \cdot \nu = 0 \quad \text{on } \partial\Omega, \\ \text{rot } w \times \nu = \tau_\nu u \quad \text{on } \partial\Omega. \end{cases}$$

Under our assumption, there exists a unique solution $w \in W^{2,r}(\Omega)$ of (2.67), up to modulo the space $X_{har}(\Omega)$, such that

$$\text{(2.68)} \qquad \|\nabla(\text{rot } w)\|_r + \|\text{rot } w\|_r \leqq C(\|\text{rot } u\|_r + \|\tau_\nu u\|_{W^{1-1/r,r}(\partial\Omega)}),$$

where $C = C(\Omega, r)$.

For a moment, let us assume the existence of $w \in W^{2,r}(\Omega)$ satisfying (2.67) and (2.68). Then taking $v = u -$ rot w, we have by (2.67) and (2.68) that div $v =$ div $u \in L^r(\Omega)$, rot $v =$ rot $u -$ rot rot $w \in L^r(\Omega)$ and $\tau_\nu v = \tau_\nu u - \tau_\tau(\text{rot } w) = u \times \nu - \text{rot } w \times \nu = 0$. Applying (2.9) to v, we see from (2.68) that

$$\text{(2.69)} \qquad \begin{aligned} \|\nabla v\|_r + \|v\|_r &\leqq C(\|\text{div } v\|_r + \|\text{rot } v\|_r + \|v\|_r) \\ &\leqq C(\|\text{div } u\|_r + \|\text{rot } u\|_r + \|\tau_\nu u\|_{W^{1-1/r,r}(\partial\Omega)} + \|u\|_r), \end{aligned}$$

where $C = C(\Omega, r)$. Since $u = v + \text{rot } w$, from (2.68) and (2.69) we obtain (2.63).

Let us now construct the solution $w \in W^{2,r}(\Omega)$ of (2.67) and (2.68). We first reduce (2.67) to the problem with the homogeneous boundary condition on $\partial\Omega$. Since $\tau_\nu u = u \times \nu \in W^{1-1/r,r}(\partial\Omega)$, there extension $\tilde{w} \in W^{2,r}(\Omega)$ such that

$$\text{(2.70)} \qquad \tilde{w} = 0, \quad \text{rot } \tilde{w} \times \nu = u \times \nu \quad \text{on } \partial\Omega.$$

Indeed, it follows from Triebel [49] that there exists $\tilde{w} \in W^{2,r}(\Omega)$ such that

$$\text{(2.71)} \qquad \tilde{w} = 0, \quad \frac{\partial \tilde{w}}{\partial \nu} = u \times \nu \quad \text{on } \partial\Omega$$

with

$$\|\tilde{w}\|_{W^{2,r}(\Omega)} \leqq C\|\tau_\nu u\|_{W^{1-1/r,r}(\partial\Omega)}, \tag{2.72}$$

where $C = C(\Omega, r)$. Since $\tilde{w} = 0$ on $\partial\Omega$, it holds

$$\text{rot } \tilde{w} \times \nu = P_{\tan}\left(\frac{\partial \tilde{w}}{\partial \nu}\right) \quad \text{on } \partial\Omega, \tag{2.73}$$

where $P_{\tan}$ is the projection onto the direction of the tangent space of $\partial\Omega$, i.e., $P_{\tan}f = f - (f \cdot \nu)f$ for $f = (f^1, f^2, f^3) \in W^{1-1/r,r}(\partial\Omega)$. Hence we have by (2.71) and (2.73) that

$$\begin{aligned} \text{rot } \tilde{w} \times \nu = P_{\tan}\left(\frac{\partial \tilde{w}}{\partial \nu}\right) &= P_{\tan}(u \times \nu) \\ &= u \times \nu - (u \times \nu \cdot \nu)\nu \\ &= u \times \nu \quad \text{on } \partial\Omega, \end{aligned}$$

which implies (2.70). Defining $w' \equiv w - \tilde{w}$, we see from (2.70) that the equation (2.67) can be reduced to the following problem for w'.

$$\begin{cases} -\Delta w' = g \equiv \text{rot } u + \Delta \tilde{w} \quad \text{in } \Omega, \\ w' \cdot \nu = 0 \quad \text{on } \partial\Omega, \\ \text{rot } w' \times \nu = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{2.74}$$

Since rot $u \in L^r(\Omega)$, it follows from (2.72) that $g \in L^r(\Omega)$. On the other hand, we have

$$\begin{aligned} \tilde{X}^r_{har}(\Omega) &\equiv \{h \in W^{2,r}(\Omega); -\Delta h = 0 \quad \text{in } \Omega,\ h \cdot \nu = 0, \text{ rot } h \times \nu = 0 \text{ on } \partial\Omega\} \\ &= X_{har}(\Omega) \end{aligned} \tag{2.75}$$

for all $1 < r < \infty$. Let us assume for a moment (2.75). By (2.70) and the Stokes formula such as (1.1) and (1.2), we have

$$\begin{aligned} (g, h) &= (\text{rot } u, h) + (-\text{rot rot } \tilde{w} + \nabla \text{div } \tilde{w}, h) \\ &= \langle \nu \times u, h\rangle_{\partial\Omega} - \langle \nu \times \text{rot } \tilde{w}, h\rangle_{\partial\Omega} + \langle \text{div } \tilde{w}, h \cdot \nu\rangle_{\partial\Omega} \\ &= 0 \end{aligned}$$

for all $h \in X_{har}(\Omega)$. Hence by the Fredholm alternative, there exists a solution $w' \in W^{2,r}(\Omega)$ of (2.74) which is unique up to modulo $X_{har}(\Omega)$. From (2.72) and Lemma 2.6 (2) with $s = 2$, we see that such w' is subject to the estimate

$$\begin{aligned} \|w'\|_{W^{2,r}(\Omega)} &\leqq C\|g\|_r \\ &\leqq C(\|\text{rot } u\|_r + \|\tilde{w}\|_{W^{2,r}(\Omega)}) \\ &\leqq C(\|\text{rot } u\|_r + \|\tau_\nu u\|_{W^{1-1/r,r}(\partial\Omega)}). \end{aligned} \tag{2.76}$$

Since $w = w' + \tilde{w}$, the desired estimate (2.68) is a consequence of (2.72) and (2.76).

It remains to show (2.75). The inclusion $X_{har}(\Omega) \subset \tilde{X}^r_{har}(\Omega)$ is obvious. Let $h \in \tilde{X}^r_{har}(\Omega)$ for $1 < r < \infty$. Then Lemma 2.6 (2) yields $h \in C^\infty(\bar{\Omega})$. Hence by the Stokes formula as in (1.1) and (1.2), we have

$$
\begin{aligned}
0 = (-\Delta h, h) &= (\text{rot rot } h - \nabla \text{div } h, h) \\
&= \|\text{rot } h\|_2^2 + \|\text{div } h\|_2^2 + \int_{\partial\Omega} (\nu \times \text{rot } h \cdot h - (\text{div } h) h \cdot \nu) dS \qquad (2.77)\\
&= \|\text{rot } h\|_2^2 + \|\text{div } h\|_2^2,
\end{aligned}
$$

which yields that rot $h = 0$ and div $h = 0$ in Ω. Hence we have $h \in X_{har}(\Omega)$. This proves Lemma 2.8.

Completion of the proof of Theorem 1.2. The estimates (1.27) and (1.29) are consequences of (2.16) and (2.17) in Lemma 2.3, respectively. Hence it suffices to show (1.28) and (1.30). The proof of (1.28) follows from (2.51) and (2.62), while the proof (1.30) follows from (2.52) and (2.63). Let us first prove (1.28). We make use of a contradiction argument. Suppose that (1.28) does not hold. Then there is a sequence $\{u_m\}_{m=1}^\infty$ in $W^{s,r}(\Omega)$ such that

$$\|u_m\|_{W^{s,r}(\Omega)} \equiv 1, \qquad (2.78)$$

$$\|\text{div } u_m\|_{W^{s-1,r}(\Omega)} + \|\text{rot } u_m\|_{W^{s-1,r}(\Omega)} + \|\gamma_\nu u_m\|_{W^{s-1/r,r}(\partial\Omega)} + \sum_{j=1}^N |(u_m, \varphi_j)| \leqq \frac{1}{m}, \qquad (2.79)$$

holds for all $m = 1, 2, \cdots$. Since $W^{s,r}(\Omega)$ is compactly embedded into $L^r(\Omega)$, we may assume that $\{u_m\}_{m=1}^\infty$ is a strong convergence sequence in $L^r(\Omega)$. Hence from (2.51) and (2.62), we see that there is $u \in W^{s,r}(\Omega)$ such that

$$u_m \to u \quad \text{in } W^{s,r}(\Omega) \text{ as } m \to \infty. \qquad (2.80)$$

On the other hand, from (2.79) it follows that

$$u \in X_{har}(\Omega) \quad \text{with} \quad (u, \varphi_j) = 0 \quad \text{for } j = 1, \cdots, N,$$

which implies that $u \equiv 0$ in Ω. This contradicts (2.78) and (2.80), and we obtain (1.28).

On account of (2.52) and (2.63), it is easy to see that the same argument as above holds also for the proof of (1.30). So we may omit it.

2.3 Characterization of the harmonic vector fields $X_{har}(\Omega)$ and $V_{har}(\Omega)$

In this subsection, we construct the bases $\{\varphi_1, \cdots, \varphi_N\}$ and $\{\psi_1, \cdots, \psi_L\}$ of the harmonic vector spaces $X_{har}(\Omega)$ and $V_{har}(\Omega)$, respectively, provided the domain Ω satisfies the conditions (1.23) and (1.22) in Definition 1.1. We also characterize the range of the operator rot such as (1.26). The characterization of $X_{har}(\Omega)$ is due to Foias-Temam [15] and Temam [48, Appendix I](see also Martensen [36]). On the other hand, our characterization of $V_{har}(\Omega)$ seems to be new.

Theorem 2.1 *Let Ω be a bounded domain in $\mathbb{R}^3$ with the first and the second Betti numbers N and L as in Definition 1.1, respectively. Let $X_{har}(\Omega)$ and $V_{har}(\Omega)$ be as in Theorem 1.1(1).*

(1)*(in case $\gamma_\nu h = 0$) There exist N functions $p_1, \cdots, p_N$ in $C^\infty(\dot{\Omega})$ such that*

$$\begin{cases} \Delta p_i = 0 \quad \text{in } \dot{\Omega}, \\ \dfrac{\partial p_i}{\partial \nu} = 0 \quad \text{on } \partial\Omega, \\ \left[\dfrac{\partial p_i}{\partial \nu_j}\right]_j = 0, \quad [p_i]_j = \delta_{ij} \qquad \text{for } i, j = 1, \cdots, N, \end{cases} \tag{2.81}$$

where $[f]_j$ denotes the jump of the value f on Σ_j defined by

$$[f]_j \equiv f|_{\Sigma_j^+} - f|_{\Sigma_j^-}$$

with Σ_j^+ and Σ_j^- denoting two sides of Σ_j, and where ν_j is the unit outward normal to Σ_j with its direction from Σ_j^- to Σ_j^+. Moreover, for such $p_1, \cdots, p_N$, the set $\{\varphi_1, \cdots, \varphi_N\}$ of vector fields given by

$$\varphi_i \equiv \nabla p_i, \qquad i = 1, \cdots, N$$

forms a basis of $X_{har}(\Omega)$.

(2) *(in case $\tau_\nu h = 0$) There exist L functions $q_1, \cdots, q_L$ in $C^\infty(\Omega)$ such that*

$$\begin{cases} \Delta q_i = 0 \quad \text{in } \Omega, \\ q_i|_{\Gamma_0} = 0, \quad q_i|_{\Gamma_j} = \delta_{ij} \qquad \text{for } i, j = 1, \cdots, L. \end{cases} \tag{2.82}$$

Moreover, for such $q_1, \cdots, q_L$, the set $\{\psi_1, \cdots, \psi_L\}$ of vector fields given by

$$\psi_i \equiv \nabla q_i, \qquad i = 1, \cdots, L$$

forms a basis of $V_{har}(\Omega)$.

Proof. (1) The proof of (1) is essentially due to Temam [48, Appendix I, Lemmata 1.1, 1.2]. Indeed, the solutions $p_1, \cdots, p_N$ of (2.81) can be found in the space

$$X_i \equiv \{h \in W^{1,2}(\dot{\Omega}); [h]_i = \text{const.}, \quad [h]_j = 0 \quad \text{for } i \neq j\}, \quad i = 1, \cdots, N.$$

Furthermore, it is shown in [48] that $\{\nabla p_1, \cdots, \nabla p_N\}$ forms a basis in $X^2_{har}(\Omega)$. Since $X^2_{har}(\Omega) = X_{har}(\Omega)$, implied by Lemma 2.3, we obtain the desired result.

(2) For the proof of (2), we need the following Proposition.

Proposition 2.2 *Let Ω be a bounded domain in $\mathbb{R}^3$ with the first and the second Betti numbers N and L as in Definition 1.1, respectively. For any $\psi \in C^\infty(\bar{\Omega})$ with* rot $\psi = 0$ *in Ω and $\psi \times \nu = 0$ on $\partial\Omega$ there is a single valued function $q \in C^\infty(\bar{\Omega})$ such that*

$$\psi = \nabla q \quad \text{in } \bar{\Omega}.$$

For a moment, let us assume Proposition 2.2. It is well-known that there are solutions $q_1, \cdots, q_N$ of (2.82) in $C^\infty(\bar{\Omega})$. Obviously, it holds

$$\text{div } \psi_i = \Delta q_i = 0, \quad \text{rot } \psi_i = \text{rot } (\nabla q_i) = 0, \quad i = 1, \cdots, L, \quad \text{in } \Omega.$$

Since $q_i = \text{const.}(i = 1, \cdots, L)$ on each $\Gamma_0, \Gamma_1, \cdots, \Gamma_L$, we see that $\psi_i = \nabla q_i$ is parallel to ν on $\partial\Omega$, which yields $\psi_i \times \nu = 0$ on $\partial\Omega$. This shows that $\psi_i \in V_{har}(\Omega)$ for all $i = 1, \cdots, L$. Furthermore, $\{\psi_1, \cdots, \psi_L\}$ is linearly independent. In fact, if $\sum_{i=1}^L \lambda_i \psi_j(x) = 0$ for all $x \in \bar{\Omega}$ for constants $\lambda_1, \cdots, \lambda_L$ in $\mathbb{R}$, then we see that $\sum_{i=1}^L \lambda_i q_i(x) = \text{const.}$ for all $x \in \bar{\Omega}$. Letting x run over $\Gamma_0, \Gamma_1, \cdots$ and Γ_L, we see from (2.82) that $\lambda_1 = \lambda_2 = \cdots = \lambda_L = 0$.

We next show that $V_{har}(\Omega)$ is spanned by $\{\psi_1, \cdots, \psi_L\}$. Let $h \in V_{har}(\Omega)$. Then by Proposition 2.2 we see that $h = \nabla q$ with some $q \in C^\infty(\bar{\Omega})$. Since $\nabla q \times \nu = h \times \nu = 0$ on $\partial\Omega$, ∇q is parallel to ν on $\partial\Omega$, which means that

$$q = c_i \quad \text{on } \Gamma_i, \quad i = 0, 1, \cdots, L$$

with some constants $c_0, c_1, \cdots, c_L$. Without loss of generality, we may assume that $c_0 = 0$. Defining $\tilde{q} \equiv q - \sum_{i=1}^L c_i q_i$, we have

$$\begin{cases} \Delta\tilde{q} = \text{div } h - \Sigma_{i=1}^L c_i \Delta q_i = 0 \quad \text{in } \Omega, \\ \tilde{q} = 0 \quad \text{on } \partial\Omega, \end{cases}$$

which yields $\tilde{q} = 0$, and hence $h = \nabla q = \sum_{i=1}^L c_i \psi_i$. Since $h \in V_{har}(\Omega)$ is arbitrary, we see that $\{\psi_1, \cdots, \psi_L\}$ forms a basis of $V_{har}(\Omega)$.

Now, it remains to prove Proposition 2.2.

Proof of Proposition 2.2. Let us fix some point $x_0 \in \dot{\Omega}$. For every point $x \in \dot{\Omega}$ we denote by $l_{x_0 \to x}$ the piecewise smooth curve connected from x_0 to x. For $\psi \in C^\infty(\bar{\Omega})$ with rot $\psi = 0$ in Ω and $\psi \times \nu = 0$ on $\partial\Omega$, the scalar potential $q(x)$ can be defined as the line integral of ψ along the curve $l_{x_0 \to x}$;

$$q(x) = \int_{l_{x_0 \to x}} \psi \cdot ds.$$

Since $\dot{\Omega}$ is simply connected and since rot $\psi = 0$, we see that the line integral on the above right hand side is determined independently of choice of the curve $l_{x_0 \to x}$. So, $q(x)$ is well defined on $\dot{\Omega}$. To see that $q(x)$ is a smooth single-valued function defined on $\bar{\Omega}$, we may show that $q(x)$ does not have any jump at each point $x \in \Sigma_i$ across from Σ_i^- to Σ_i^+. Since $\nu \times \nabla q = \nu \times \psi = 0$ on $\partial\Omega$ and since $\nu \times \nabla$ gives a tangential derivation on $\partial\Omega$, we have that

$$q = \text{ const. } \quad \text{on each } \Gamma_i, \quad i = 0, 1, \cdots, L. \tag{2.83}$$

For every $x \in \Sigma_i$, we denote by x^+ and x^- the points of $x \in \dot{\Omega}$ on the sides of Σ_i^+ and Σ_i^-, respectively. Taking $x_* \in \bar{\Sigma}_i \cap \partial\Omega$, we have

$$\int_{l_{x_*^+ \to x^+}} \psi \cdot ds = \int_{l_{x_*^- \to x^-}} \psi \cdot ds \tag{2.84}$$

because ψ is a single valued smooth vector function on $\bar{\Omega}$. Since $\nabla q = \psi$ on each side of Σ_i^+ and Σ_i^-, we have by (2.84) that

$$
\begin{aligned}
& q(x^+) - q(x_*^+) \\
(2.85) \qquad = \;& \int_{l_{x_*^+ \to x^+}} \nabla q \cdot ds = \int_{l_{x_*^+ \to x^+}} \psi \cdot ds = \int_{l_{x_*^- \to x^-}} \psi \cdot ds = \int_{l_{x_*^- \to x^-}} \nabla q \cdot ds \\
= \;& q(x^-) - q(x_*^-).
\end{aligned}
$$

Since $x_* \in \partial\Omega$, we see from (2.83) that $q(x_*^+) = q(x_*^-)$, from which it follows that

$$q(x^+) = q(x^-).$$

Since $x \in \Sigma_i$ is arbitrary, this implies that q does not have any jump on each Σ_i for $i = 1, \cdots, N$. This proves Proposition 2.2. Hence the proof of Theorem 2.1 is now complete.

Finally, we characterize the range of the operator rot with the domain $W^{1,r}(\Omega)$ and $W^{s,r}(\Omega)$ for $s \geqq 2$.

Proposition 2.3 *Let Ω be a bounded domain in $\mathbb{R}^3$ with the first and the second Betti numbers N and L as in Definition 1.1, respectively. Let $1 < r < \infty$ and $s \geqq 2$. Then it holds*

$$
\begin{aligned}
& \{\mathrm{rot}\ w; w \in X_\sigma^r(\Omega)\} \\
(2.86) \qquad = \;& \{v \in L^r(\Omega); \mathrm{div}\ v = 0 \quad \textit{in}\ \Omega\ , \quad \langle \gamma_\nu v, 1\rangle_{\Gamma_j} = 0 \quad \textit{for all}\ j = 0, 1, \cdots, L\}, \\
& \{\mathrm{rot}\ w; w \in W^{s,r}(\Omega) \cap X_\sigma^r(\Omega)\} \\
(2.87) \qquad = \;& \{v \in W^{s-1,r}(\Omega); \mathrm{div}\ v = 0 \quad \textit{in}\ \Omega\ , \quad \int_{\Gamma_j} v \cdot \nu dS = 0 \quad \textit{for all}\ j = 0, 1, \cdots, L\}.
\end{aligned}
$$

Proof. By the generalized Stokes formula (1.1) and (1.2), it is easy to see that the sets of the right hand sides of (2.86) and (2.87) are included into those of the right hand sides. So, it suffices to show the converse inclusion. Let $v \in L^r(\Omega)$ with div $v = 0$ in Ω and $\langle \gamma_\nu v, 1\rangle_{\Gamma_j} = 0$ for all $j = 0, 1, \cdots, L$. Then it follows from Theorem 1.1 (2) that

$$v = h + \mathrm{rot}\ w \quad \text{for some } h \in V_{har}(\Omega) \text{ and } w \in X_\sigma^r(\Omega).$$

Taking a basis $\{\psi_1, \cdots, \psi_L\}$ given by Theorem 2.1 (2), we have by (1.1), (2.82) and the assumption on v that

$$(h, \psi_j) = (v, \psi_j) = (v, \nabla q_j) = \sum_{i=1}^{L} \langle \gamma_\nu v, q_j\rangle_{\Gamma_i} = \langle \gamma_\nu v, 1\rangle_{\Gamma_j} = 0 \quad \text{for all } j = 1, \cdots, L,$$

which yields $h = 0$. Hence it holds $v = \mathrm{rot}\ w$. This implies (2.86).

If in addition, $v \in W^{s-1,r}(\Omega)$, then w can be taken as the solution

$$
\begin{cases}
-\Delta w = \mathrm{rot}\ v & \text{in } \Omega, \\
w \times \nu = 0, \quad \mathrm{div}\ w = 0 & \text{on } \partial\Omega.
\end{cases}
$$

Hence by Lemma 2.6(1), we see that $w \in W^{s,r}(\Omega) \cap X_\sigma^r(\Omega)$, which yields (2.87). This proves Proposition 2.3.

3 Stationary Navier-Stokes equations under the general flux condition

3.1 Leray's problem

We first explain Leray's problem on the stationary Navier-Stokes equations in multi-connected domains Ω with the first and the second Betti numbers N and L as is Definition 1.1. In Ω we consider the boundary value problem for the stationary Navier-Stokes equations:

$$\text{(N-S)} \qquad \begin{cases} -\mu\Delta v + v\cdot\nabla v + \nabla p = 0 \quad \text{in } \Omega, \\ \text{div } v = 0 \quad \text{in } \Omega, \\ v = \beta \quad \text{on } \partial\Omega, \end{cases}$$

where $v = v(x) = (v_1(x), v_2(x), v_3(x))$ and $p = p(x)$ denote the unknown velocity vector and the unknown pressure at the point $x = (x_1, x_2, x_3) \in \Omega$, while $\mu > 0$ is the given viscosity constant, and $\beta = (\beta_1, \beta_2, \beta_3)$ is the given boundary data on $\partial\Omega$. We use the standard notation as $\Delta v = \sum_{j=1}^{3} \frac{\partial^2 v}{\partial x_j^2}$, $\nabla p = \left(\frac{\partial p}{\partial x_1}, \frac{\partial p}{\partial x_2}, \frac{\partial p}{\partial x_3}\right)$, div $v = \sum_{j=1}^{3} \frac{\partial v_j}{\partial x_j}$, and $v\cdot\nabla v = \sum_{j=1}^{3} v_j \frac{\partial v}{\partial x_j}$. Since the solution v satisfies div $v = 0$ in Ω, the given boundary data β on $\partial\Omega$ is required to fulfill the following compatibility condition which we call the general flux condition:

$$\text{(G.F.)} \qquad \sum_{j=0}^{L} \int_{\Gamma_j} \beta\cdot\nu dS = 0,$$

where ν denotes the unit outer normal to $\partial\Omega$. Leray [33] proposed to solve the following problem.

Leray's problem. Suppose that $\beta \in H^{1/2}(\partial\Omega)$ satisfies the general flux condition (G.F.). Does there exist at least one weak solution $v \in H^1(\Omega)$ of (N-S) ?

Up to now, we are not yet successful to give a complete answer to this question. However, some partial answer has been proved by Leray [33], Fujita[16] and Ladyzehenskaya [32] under *the restricted flux condition* (R.F.) on β:

$$\text{(R.F.)} \qquad \int_{\Gamma_j} \beta\cdot\nu dS = 0 \quad \text{for all } j = 0, 1, \cdots, L.$$

Indeed, under the restricted flux condition (R.F.) on β, they showed that there exists at least one weak solution v of (N-S).

If the given boundary data β satisfies the general flux condition (G.F.), then there exists an extension b into Ω with $b|_{\partial\Omega} = \beta$ such that div $b = 0$. We call such b a solenoidal extension into Ω of β. Introducing a new unknown variable $u \equiv v - b$, we can reduce the original equations (N-S) to the following ones with the *homogeneous* boundary condition:

$$\text{(N-S')} \qquad \begin{cases} -\mu\Delta u + b\cdot\nabla u + u\cdot\nabla b + u\cdot\nabla u + \nabla p = \mu\Delta b - b\cdot\nabla b \quad \text{in } \Omega, \\ \text{div } u = 0 \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega. \end{cases}$$

To solve (N-S') we need to handle the linear convection term $b\cdot\nabla u+u\cdot\nabla b$. Let us take L harmonic functions $q_1(x),\cdots,q_L(x)$ in Ω so that

$$\Delta q_j=0 \quad \text{in } \Omega, \quad q_j|_{\Gamma_0}=0, \quad q_j|_{\Gamma_i}=\delta_{ji}, \quad i,j=1,\cdots,L, \tag{3.1}$$

and set $\psi_j(x)=\nabla q_j(x)$, $j=1,\cdots,L$. Then it holds that

$$V_{har}(\Omega)=\mathrm{Span}\{\psi_1,\cdots,\psi_L\}.$$

We choose $\{\varphi_1,\cdots,\varphi_L\}$ as the orthogonal basis of $V_{har}(\Omega)$ in L^2-sense;

$$V_{har}(\Omega)=\mathrm{Span.}\{\varphi_1,\cdots,\varphi_L\} \quad \text{with } (\varphi_i,\varphi_j)=\delta_{ij},\ i,j=1,\cdots,L.$$

Then there exists a regular $L\times L$ matrix $(\alpha_{jk})_{1\leqq j,k\leqq L}$ depending only on Ω such that

$$\varphi_j(x)=\sum_{k=1}^{L}\alpha_{jk}\psi_k(x), \quad j=1,\cdots,L \tag{3.2}$$

holds for all $x\in\Omega$.

Our main theorem on existence of weak solutions to (N-S) now reads:

Theorem 3.1 *Let Ω be a bounded domain in $\mathbb{R}^3$ with the first and second Betti numbers N and L as in Definition 1.1. Suppose that $\beta\in H^{1/2}(\partial\Omega)$ satisfies the general flux condition (G.F.). If*

$$\left\|\sum_{j,k=1}^{L}\alpha_{jk}\left(\int_{\Gamma_k}\beta\cdot\nu dS\right)\varphi_j\right\|_3<\mu C_s^{-1}, \tag{3.3}$$

then there exist $b\in H^1(\Omega)$ with $\mathrm{div}\, b=0$ in Ω, $b=\beta$ on $\partial\Omega$, and $u\in H^1_{0,\sigma}(\Omega)$ such that (N-S') is satisfied in the weak sense that

$$\mu(\nabla u,\nabla\varphi)+(b\cdot\nabla u+u\cdot\nabla b,\varphi)+(u\cdot\nabla u,\varphi)=\mu(\nabla b,\nabla\varphi)-(b\cdot\nabla\varphi,b) \tag{3.4}$$

holds for all $\varphi\in C^\infty_{0,\sigma}(\Omega)$. Here $C_s=3^{-\frac{1}{2}}2^{\frac{2}{3}}\pi^{-\frac{2}{3}}$ is the best constant of the Sobolev embedding $H^1_0(\Omega)\subset L^6(\Omega)$.

Remark 3.1 (1) The regular matrix $(\alpha_{jk})_{1\leqq j,k\leqq L}$ in (3.2) and (3.3) can be expressed by means of the harmonic functions $\{q_1,\cdots,q_L\}$ in (3.1). Indeed, we have

$$\alpha_{jk}=\begin{cases}\frac{1}{\sqrt{\Delta_{j-1}\Delta_j}}E_{jk}, & 1\leqq k\leqq j,\\ 0 & j+1\leqq k\leqq L,\end{cases} \tag{3.5}$$

where E_{jk} denotes the (j,k)-cofactor of

$$E_j=\begin{pmatrix}c_{11} & \cdots & c_{1j}\\ \vdots & \cdots & \vdots\\ c_{j1} & \cdots & c_{jj}\end{pmatrix}, \quad 1\leqq k\leqq j\leqq L,$$

$$\Delta_j = \det E_j, \quad 1 \leqq j \leqq L,$$

with

$$c_{jk} = \int_{\Gamma_j} \frac{\partial q_k}{\partial \nu} dS, \quad j,k = 1,2,\cdots L.$$

Then the hypothesis (3.3) of the Theorem can be rewritten as

$$\left\| \sum_{k=1}^{L} \sum_{i=k}^{L} \alpha_{ik} \left(\sum_{j=1}^{i} \alpha_{ij} \int_{\Gamma_j} \beta \cdot \nu dS \right) \nabla q_k \right\|_3 < \mu C_s^{-1}.$$

(2) Galdi [21, VIII, Theorem 4.1] proved the existence of solutions to (N-S) under the stronger condition than (3.3) such as

$$\sum_{i=1}^{L} c_i \left| \int_{\Gamma_i} \beta \cdot \nu dS \right| < \mu,$$

where $c_1, \cdots, c_L$ are computable constants depending on Ω.

Corollary 3.1 *Let* $\Omega = A_{R_0,R_1} \equiv \{x \in \mathbb{R}^3; R_1 < |x| < R_0\}$ *with* $0 < R_1 < R_0$. *Let* $\Gamma_0 = \{x \in \mathbb{R}^3; |x| = R_0\}$ *and* $\Gamma_1 = \{x \in \mathbb{R}^3; |x| = R_1\}$. *Suppose that* $\beta \in H^{1/2}(\partial\Omega)$ *satisfies the general flux condition (G.F.) as*

$$\int_{\Gamma_0} \beta \cdot \nu dS + \int_{\Gamma_1} \beta \cdot \nu dS = 0.$$

If

$$\left| \int_{\Gamma_1} \beta \cdot \nu dS \right| < \mu 3^{\frac{5}{6}} 2^{\frac{2}{3}} \pi^{\frac{4}{3}} \frac{R_0 R_1}{(R_0^3 - R_1^3)^{\frac{1}{3}}}, \tag{3.6}$$

then there exists a weak solution $v \in H^1(A_{R_0,R_1})$ *of (N-S).*

Remark 3.2 Borchers-Pilekas [7] obtained a similar result to the above Corollary. Since the methods are different, it seems to be impossible to see the relation of inclusion to each other.

3.2 Solenoidal extension of the boundary data β

To solve (N-S) we first take a solenoidal extension b into Ω of β.

Lemma 3.1 *Let* Ω *be a bounded domain in* $\mathbb{R}^3$ *with the first and second Betti numbers* N *and* L *as in Definition 1.1. Let* β *satisfy the general flux condition (G.F.). Then there exists* $b \in H^1(\Omega)$ *with* $\operatorname{div} b = 0$ *in* Ω *and* $b = \beta$ *on* $\partial\Omega$ *such that* b *is decomposed as*

$$b = h + \operatorname{rot} w \tag{3.7}$$

with $h \in V_{har}(\Omega)$ *and* $w \in X_\sigma^2(\Omega) \cap H^2(\Omega)$. *Moreover,* h *is expressed as*

$$h(x) = \sum_{j,k=1}^{L} \alpha_{jk} \left(\int_{\Gamma_k} \beta \cdot \nu dS \right) \varphi_j(x) = \sum_{k=1}^{L} \sum_{i=k}^{L} \alpha_{ik} \left(\sum_{j=1}^{i} \alpha_{ij} \int_{\Gamma_j} \beta \cdot \nu dS \right) \nabla q_k(x) \tag{3.8}$$

for all $x \in \Omega$, where $\{\alpha_{jk}\}_{1\leqq j,k\leqq L}$, $\{\varphi_j\}_{1\leqq j\leqq L}$ and $\{q_j\}_{1\leqq j\leqq L}$ are the $L\times L$ matrix, the orthogonal basis of $V_{har}(\Omega)$ in the L^2-sense, and L harmonic functions in Ω appearing in (3.5), (3.2) and (3.1), respectively.

Remark 3.3 Although there are infinitely many solenoidal extensions b into Ω of β, it follows from (3.8) that the harmonic part h of b is uniquely determined only in terms of the flux $\int_{\Gamma_j} \beta\cdot\nu dS$ through Γ_j for $j = 1,\cdots,L$.

Proof of Lemma 3.1. By the trace theorem, there exists $f \in H^1(\Omega)$ such that $f = \beta$ on $\partial\Omega$. Let us consider the equation

$$\text{div } g = \text{div } f \quad \text{in } \Omega, \quad g = 0 \quad \text{on } \partial\Omega. \tag{3.9}$$

Since β satisfies the general flux condition (G.F.), we have

$$\int_\Omega \text{div } f dx = \int_{\partial\Omega} f\cdot\nu dS = \sum_{j=0}^{L}\int_{\Gamma_j}\beta\cdot\nu dS = 0.$$

Hence it follows from Bogovskii [6] and Borchers-Sohr [8, Theorem 4.1] that there exists $g \in H^1_0(\Omega)$ satisfying (3.9). Defining $b \equiv g - f$, we see by (3.9) that $b \in H^1(\Omega)$ with

$$\text{div } b = 0 \quad \text{in } \Omega, \quad b = \beta \quad \text{on } \partial\Omega.$$

By Theorem 1.1 (3) with the aid of the a priori estimate as in (2.67) and (2.68) with u replaced by b, there exist $h \in V_{har}(\Omega)$, $w \in X^2_\sigma \cap H^2(\Omega)$ and $p \in H^1_0(\Omega)$ such that

$$b = h + \text{rot } w + \nabla p.$$

Since the above scalar potential p is determined by (2.58) with u replaced by b, and since div $b = 0$ in Ω, we have $p \equiv 0$, which yields (3.7).

Since $\{\varphi_1,\cdots,\varphi_L\}$ is an orthogonal basis of $V_{har}(\Omega)$ in the L^2-sense, it follows from (3.1) and (3.2) that the above harmonic part h of b can be expressed as

$$\begin{aligned} h &= \sum_{j=1}^{L}(b,\varphi_j)\varphi_j = \sum_{j=1}^{L}(b,\sum_{k=1}^{L}\alpha_{jk}\psi_k)\varphi_j = \sum_{j,k=1}^{L}\alpha_{jk}(b,\nabla q_k)\varphi_j \\ &= -\sum_{j,k=1}^{L}\alpha_{jk}(\text{div } b, q_k)\varphi_j + \sum_{j,k=1}^{L}\alpha_{jk}(\int_{\partial\Omega}\beta\cdot\nu q_k dS)\varphi_j \\ &= \sum_{j,k=1}^{L}\alpha_{jk}\left(\sum_{l=0}^{L}\int_{\Gamma_l}\beta\cdot\nu q_k dS\right)\varphi_j \\ &= \sum_{j,k=1}^{L}\alpha_{jk}\left(\sum_{l=0}^{L}\int_{\Gamma_l}\beta\cdot\nu\delta_{kl} dS\right)\varphi_j \\ &= \sum_{j,k=1}^{L}\alpha_{jk}\left(\int_{\Gamma_k}\beta\cdot\nu dS\right)\varphi_j, \end{aligned}$$

which implies (3.8). This proves Lemma 3.1.

To investigate (N-S'), for every solenoidal extension $b \in H^1(\Omega)$ into Ω of β, let us introduce the perturbed Stokes operator $\mathcal{L}_b : H^1_{0,\sigma}(\Omega) \to H^1_{0,\sigma}(\Omega)^*$ defined by

$$\langle \mathcal{L}_b u, \varphi\rangle = \mu(\nabla u, \nabla\varphi) + (b\cdot\nabla u + u\cdot\nabla b, \varphi), \quad u, \varphi \in H^1_{0,\sigma}(\Omega), \tag{3.10}$$

where $\langle\cdot,\cdot\rangle$ denotes the duality paring between $H^1_{0,\sigma}(\Omega)$ and $H^1_{0,\sigma}(\Omega)^*$. We regard $H^1_{0,\sigma}(\Omega)$ as the Hilbert space with the Dirichlet norm $\|\nabla u\|_2$. Since $b \in H^1(\Omega)$ with div $b = 0$, we see that $\mathcal{L}_b$ is a bounded linear operator from $H^1_{0,\sigma}(\Omega)$ to $H^1_{0,\sigma}(\Omega)^*$. More precisely, we have

$$\|\mathcal{L}_b u\|_{H^1_\sigma(\Omega)^*} \leqq (\mu + 2C_s\|b\|_3)\|\nabla u\|_2 \tag{3.11}$$

for all $u \in H^1_{0,\sigma}(\Omega)$, where $C_s = 3^{-\frac{1}{2}}2^{\frac{2}{3}}\pi^{-\frac{2}{3}}$ is the best constant of the Sobolev inequality $\|u\|_6 \leqq C_s\|\nabla u\|_2$. Indeed, by the Hölder and the Sobolev inequalities it holds

$$\begin{aligned} |(b\cdot\nabla u, \varphi)| &\leqq \|b\|_3\|\nabla u\|_2\|\varphi\|_6 \leqq C_s\|b\|_3\|\nabla u\|_2\|\nabla\varphi\|_2 \\ |(u\cdot\nabla b, \varphi)| &= |(u\cdot\nabla\varphi, b)| \leqq \|u\|_6\|\nabla\varphi\|_2\|b\|_3 \leqq C_s\|b\|_3\|\nabla u\|_2\|\nabla\varphi\|_2, \end{aligned}$$

from which it follows that

$$|\langle \mathcal{L}_b u, \varphi\rangle| \leqq (\mu + 2C_s\|b\|_3)\|\nabla u\|_2\|\nabla\varphi\|_2$$

for all $u, \varphi \in H^1_{0,\sigma}(\Omega)$. This implies (3.11).

The following invertibility of $\mathcal{L}_b$ with some solenoidal extension b into Ω of β plays an essential role for solvability of (3.4). Indeed, we have

Lemma 3.2 *Let Ω be a bounded domain in $\mathbb{R}^3$ with the first and second Betti numbers N and L as in Definition 1.1. Suppose that $\beta \in H^{\frac{1}{2}}(\Omega)$ satisfies the general flux condition (G.F.). If β fulfills (3.3), then there exists $b_* \in H^1(\Omega)$ with* div $b_* = 0$ *in Ω, $b_* = \beta$ on $\partial\Omega$, and a positive constant δ such that*

$$\langle \mathcal{L}_{b_*} u, u\rangle \geqq \delta\|\nabla u\|_2^2$$

holds for all $u \in H^1_{0,\sigma}(\Omega)$.

Proof. Since β satisfies the general flux condition (G.F.), it follows from Lemma 3.1 that there are $h \in V_{har}(\Omega)$ with its expression as in (3.8) and $w \in X^2_\sigma(\Omega) \cap H^2(\Omega)$ such that $b \equiv h + \text{rot } w$ fulfills

$$\text{div } b = 0 \quad \text{in } \Omega, \quad b = \beta \quad \text{on } \partial\Omega.$$

Let us take a family $\{\theta_\varepsilon\}_{\varepsilon>0}$ of C^∞-cut-off functions in Ω so that

$$\theta_\varepsilon(x) = \begin{cases} 1 & \text{for } d(x) \equiv \text{dist.}(x, \partial\Omega) \leqq e^{-\frac{2}{\varepsilon}}, \\ 0 & \text{for } d(x) > 2e^{-\frac{1}{\varepsilon}}, \end{cases} \tag{3.12}$$

with

$$|\nabla\theta_\varepsilon(x)| \leqq \frac{\varepsilon}{d(x)} \quad \text{for } d(x) < 2e^{-\frac{1}{\varepsilon}}. \tag{3.13}$$

For every $\varepsilon > 0$ we define b_ε by

$$b_\varepsilon \equiv h + \text{rot }(\theta_\varepsilon w).$$

Since $\theta_\varepsilon \equiv 1$ in a neighbourhood of $\partial\Omega$, it holds that

$$\text{div } b_\varepsilon = 0 \quad \text{in } \Omega, \quad b_\varepsilon = \beta \quad \text{on } \partial\Omega \text{ for all } \varepsilon > 0, \tag{3.14}$$

and that

$$|(u \cdot \nabla b_\varepsilon, u)| = |(u \cdot \nabla u, b_\varepsilon)| \leqq |(u \cdot \nabla u, h)| + |(u \cdot \nabla u, \text{rot }(\theta_\varepsilon w))| \tag{3.15}$$

for all $u \in H^1_{0,\sigma}(\Omega)$. By the Hölder and the Sobolev inequalities we have

$$|(u \cdot \nabla u, h)| \leqq \|u\|_6 \|\nabla u\|_2 \|h\|_3 \leqq C_s \|\nabla u\|_2^2 \|h\|_3 \quad \text{for all } u \in H^1_{0,\sigma}(\Omega). \tag{3.16}$$

Similarly to Temam [48, Chapter II, Lemma 1.8], for every $\gamma > 0$ we can choose $\varepsilon = \varepsilon(\gamma) > 0$ in such a way that

$$|(u \cdot \nabla u, \text{rot }(\theta_\varepsilon w))| \leqq \gamma \|\nabla u\|_2^2 \quad \text{for all } u \in H^1_{0,\sigma}(\Omega). \tag{3.17}$$

For reader's convenience, we show (3.17). Let $a_\varepsilon(x) = \text{rot }(\theta_\varepsilon(x) w(x))$. By (3.12) and (3.13) it holds that

$$\text{supp } a_\varepsilon = \{x \in \Omega; d(x) \equiv \text{dist.}(x, \partial\Omega) \leqq 2e^{-\frac{1}{\varepsilon}}\},$$

and that

$$|a_\varepsilon(x)| \leqq \frac{\varepsilon}{d(x)} |w(x)| + |\nabla w(x)| \quad \text{for } x \in \Omega \text{ with } d(x) < 2e^{-\frac{1}{\varepsilon}},$$

which yields

$$\begin{aligned} |(u \cdot \nabla u, a_\varepsilon)| &\leqq \|\nabla u\|_2 \|a_\varepsilon u\|_2 \\ &\leqq \|\nabla u\|_2 \left(\varepsilon \|w\|_\infty \left\| \frac{u}{d} \right\|_2 + \|u \cdot \nabla w\|_{L^2(\{x \in \Omega; d(x) < 2e^{-\frac{1}{\varepsilon}}\})} \right). \end{aligned} \tag{3.18}$$

Since $u = 0$ on $\partial\Omega$, we have by the Hardy inequality that

$$\left\| \frac{u}{d} \right\|_2 \leqq C \|\nabla u\|_2. \tag{3.19}$$

Since $w \in H^2(\Omega) \subset W^{1,6}(\Omega) \subset L^\infty(\Omega)$, we have by the Hölder and the Sobolev inequalities that

$$\begin{aligned} \|u \cdot \nabla w\|_{L^2(\{x \in \Omega; d(x) < 2e^{-\frac{1}{\varepsilon}}\})} &\leqq \|u\|_6 \|\nabla w\|_{L^3(\{x \in \Omega; d(x) < 2e^{-\frac{1}{\varepsilon}}\})} \\ &\leqq C_s \|\nabla u\|_2 \|\nabla w\|_{L^3(\{x \in \Omega; d(x) < 2e^{-\frac{1}{\varepsilon}}\})} \end{aligned} \tag{3.20}$$

Hence it follows from (3.18), (3.19) and (3.20) that

$$\begin{aligned} |(u \cdot \nabla u, a_\varepsilon)| &\leqq C \left(\varepsilon \|w\|_\infty + \|\nabla w\|_{L^3(\{x \in \Omega; d(x) < 2e^{-\frac{1}{\varepsilon}}\})} \right) \|\nabla u\|_2^2 \\ &\quad \text{(by } \varepsilon \to 0) \\ &\leqq \gamma \|\nabla u\|_2^2 \end{aligned}$$

for all $u \in H^1_{0,\sigma}(\Omega)$, which implies (3.17).

Hence it follows from (3.15), (3.16) and (3.17) that

$$|(u \cdot \nabla b_\varepsilon, u)| \leqq (C_s \|h\|_3 + \gamma)\|\nabla u\|_2^2 \quad \text{for all } u \in H^1_{0,\sigma}(\Omega). \tag{3.21}$$

Since h is expressed as in (3.8), it follows from the assumption (3.3) that

$$\|h\|_3 < C_s^{-1}\mu, \tag{3.22}$$

Now we choose $\gamma = \frac{1}{2}(\mu - C_s\|h\|_3)$, and fix $\varepsilon = \varepsilon(\gamma)$ so that (3.17) is satisfied. Then taking $b_* \equiv b_{\varepsilon(\gamma)}$, we obtain from (3.21) that

$$\begin{aligned} \langle \mathcal{L}_{b_*} u, u\rangle &= \mu(\nabla u, \nabla u) + (b_* \cdot \nabla u + u \cdot \nabla b_*, u) \\ &= \mu\|\nabla u\|_2^2 + (u \cdot \nabla b_*, u) \\ &\geqq (\mu - C_s\|h\|_3 - \gamma)\|\nabla u\|_2^2 \\ &= \delta\|\nabla u\|_2^2, \qquad \text{for all } u \in H^1_{0,\sigma}(\Omega) \end{aligned}$$

with $\delta \equiv \frac{1}{2}(\mu - C_s\|h\|_3)$, which yields the desired estimate from below. This proves Lemma 3.2.

3.3 Existence of solutions; Proof of Theorem 3.1

We are now in a position to show the existence of the solution $u \in H^1_{0,\sigma}(\Omega)$ of (3.4) with b_* as in Lemma 3.2. Let us take the solenoidal extension b_* into Ω of β as in Lemma 3.2. For such b_*, we define the operator $\mathcal{L}_{b_*} : H^1_{0,\sigma}(\Omega) \to H^1_{0,\sigma}(\Omega)^*$ by (3.10), and introduce $f_{b_*} \in H^1_{0,\sigma}(\Omega)^*$ by

$$\langle f_{b_*}, \varphi\rangle \equiv \mu(\nabla b_*, \nabla\varphi) - (b_* \cdot \nabla\varphi, b_*), \quad \varphi \in H^1_{0,\sigma}(\Omega).$$

Moreover, we consider the nonlinear mapping $B : H^1_{0,\sigma}(\Omega) \to H^1_{0,\sigma}(\Omega)^*$ defined by

$$\langle Bu, \varphi\rangle \equiv (u \cdot \nabla u, \varphi), \quad u, \varphi \in H^1_{0,\sigma}(\Omega).$$

Then it is easy to see that (3.4) can be reformulated as

$$\mathcal{L}_{b_*} u + Bu = f_{b_*}.$$

By Lemma 3.2, $\mathcal{L}_{b_*}$ has a bounded inverse $\mathcal{L}_{b_*}^{-1} : H^1_{0,\sigma}(\Omega)^* \to H^1_{0,\sigma}(\Omega)$, and hence the above equation for u is equivalent to

$$u + \mathcal{L}_{b_*}^{-1} Bu = \mathcal{L}_{b_*}^{-1} f_{b_*}. \tag{3.23}$$

We show that $\mathcal{L}_{b_*}^{-1} B$ is a compact mapping from $H^1_{0,\sigma}(\Omega)$ into itself. Indeed, suppose that $\{u_m\}_{m=1}^\infty$ is a bounded sequence in $H^1_{0,\sigma}(\Omega)$. We take $M > 0$ so that $\sup_{m=1,\cdots} \|\nabla u_m\|_2 \leqq M$. By the Rellich theorem, there is a subsequence of $\{u_m\}_{m=1}^\infty$, which we denote by $\{u_m\}_{m=1}^\infty$ itself for notational simplicity, such that $\{u_m\}_{m=1}^\infty$ converges strongly in $L^3(\Omega)$. Let $v_m \equiv \mathcal{L}_{b_*}^{-1} Bu_m$. Then it holds that

$$\mathcal{L}_{b_*}(v_m - v_k) = Bu_m - Bu_k, \quad m, k = 1, \cdots. \tag{3.24}$$

By Lemma 3.2, we have

$$\langle \mathcal{L}_{b_*}(v_m - v_k), v_m - v_k\rangle \geqq \delta\|\nabla v_m - \nabla v_k\|_2^2. \tag{3.25}$$

Similarly to (3.16), we have by integration by parts that

$$
\begin{aligned}
& |\langle Bu_m - Bu_k, v_m - v_k\rangle| \\
= \;& |(u_m \cdot \nabla u_m - u_k \cdot \nabla u_k, v_m - v_k)| \\
= \;& |((u_m - u_k)\cdot \nabla u_m + u_k \cdot \nabla(u_m - u_k), v_m - v_k)| \\
= \;& |((u_m - u_k)\cdot \nabla u_m, v_m - v_k) - (u_k \cdot \nabla(v_m - v_k), u_m - u_k)| \\
\leqq \;& \|u_m - u_k\|_3 \|\nabla u_m\|_2 \|v_m - v_k\|_6 + \|u_k\|_6 \|\nabla v_m - \nabla v_k\|_2 \|u_m - u_k\|_3 \\
\leqq \;& C_s(\|\nabla u_m\|_2 + \|\nabla u_k\|_2)\|\nabla v_m - \nabla v_k\|_2 \|u_m - u_k\|_3 \\
\leqq \;& 2C_s M \|\nabla v_m - \nabla v_k\|_2 \|u_m - u_k\|_3.
\end{aligned} \tag{3.26}
$$

Hence it follows from (3.24), (3.25) and (3.26) that

$$\|\nabla v_m - \nabla v_k\|_2 \leqq 2\delta^{-1} C_s M \|u_m - u_k\|_3.$$

Since $\{u_m\}_{m=1}^{\infty}$ is a strong convergence sequence in $L^3(\Omega)$, the above estimate implies that $\{v_m\}_{m=1}^{\infty}$ converges strongly in $H^1_{0,\sigma}(\Omega)$. Hence $\mathcal{L}_{b_*}^{-1}B$ is a compact mapping from $H^1_{0,\sigma}(\Omega)$ into itself.

Now, we apply the Leray-Schauder fixed point theorem to show the existence of solutions to (3.23). To this end, we may prove that there is a constant $N > 0$ such that

$$\|\nabla u_\lambda\|_2 \leqq N \quad \text{for all } \lambda \in [0,1], \tag{3.27}$$

where u_λ is any solution of the equation

$$u_\lambda + \lambda \mathcal{L}_{b_*}^{-1} B u_\lambda = \lambda \mathcal{L}_{b_*}^{-1} f_{b_*}, \quad 0 \leqq \lambda \leqq 1.$$

Since u_λ satisfies $\mathcal{L}_{b_*} u_\lambda + \lambda B u = \lambda f_{b_*}$ and since $\langle B u_\lambda, u_\lambda\rangle = 0$, we have by Lemma 3.2 that

$$
\begin{aligned}
\delta \|\nabla u_\lambda\|_2^2 \leqq \langle \mathcal{L}_{b_*} u_\lambda, u_\lambda\rangle &= \lambda \langle f_{b_*}, u_\lambda\rangle \\
&= \lambda\mu(\nabla b_*, \nabla u_\lambda) - \lambda(b_* \cdot \nabla u_\lambda, b_*) \\
&\leqq \lambda(\mu\|\nabla b_*\|_2 + \|b_*\|_4^2)\|\nabla u_\lambda\|_2 \\
&\leqq (\mu\|\nabla b_*\|_2 + \|b_*\|_4^2)\|\nabla u_\lambda\|_2
\end{aligned}
$$

for all $\lambda \in [0,1]$. Hence we can choose N in (3.27) so that $N \equiv \delta^{-1}(\mu\|\nabla b_*\|_2 + \|b_*\|_4^2)$. This proves Theorem 3.1.

Proof of Corollary 3.1. Since $\dim. V_{har}(A_{R_0,R_1}) = 1$ with the base

$$
\begin{aligned}
\psi(x) &= \nabla q(x) = C\nabla\left(\frac{1}{|x|}\right) \quad \text{with} \quad C \equiv \frac{R_0 R_1}{R_0 - R_1}, \\
\varphi(x) &= \alpha\psi(x) \quad \text{with} \quad \alpha \equiv \frac{1}{\sqrt{4\pi C}}, \quad \|\varphi\|_2 = 1,
\end{aligned}
$$

by the direct calculation, we see that the hypothesis (3.6) is equivalent to that of (3.3). This proves the Corollary 3.1.

3.4 Leray's inequality

In this subsection, we consider the relation between Leray's inequality (L.I.) in Introduction and the restricted flux condition (R.F.). Let us first define that the boundary data $\beta \in H^{\frac{1}{2}}(\partial\Omega)$ satisfies Leray's inequality in Ω.

Definition 3.1 Let Ω be a bounded domain in $\mathbb{R}^3$ with the first and second Betti numbers N and L as in Definition 1.1. Suppose that $\beta \in H^{1/2}(\partial\Omega)$ fulfills (G.F.). We say that β satisfies Leray's inequality in Ω if for every $\varepsilon > 0$ there exists $b_\varepsilon \in H^1(\Omega)$ with div $b_\varepsilon = 0$ in Ω and $b_\varepsilon = \beta$ on $\partial\Omega$ such that

$$(\text{L.I.}) \qquad |(u \cdot \nabla b_\varepsilon, u)| \leqq \varepsilon \|\nabla u\|_2^2 \quad \text{for all } u \in H^1_{0,\sigma}(\Omega).$$

In what follows, we shall generalize Takeshita's result with a simple proof. Although our result is not altogether new, we do not need to impose any *topological restriction* on the boundary, while Takeshita [47] requires that each Γ_i, $i = 0, 1, \cdots, L$, is diffeomorphic to the sphere.

Our result now reads:

Theorem 3.2 *Let Ω be a bounded domain in $\mathbb{R}^3$ with the first and second Betti numbers N and L as in Definition 1.1. Suppose that $\beta \in H^{1/2}(\partial\Omega)$ and satisfies (G.F.). Assume that there is a sphere S in Ω such that $\Gamma_1, \cdots, \Gamma_k$ lie inside of S and such that the others $\Gamma_{k+1}, \cdots, \Gamma_L$ and Γ_0 lie outside of S. If β satisfies Leray's inequality in Ω as in Definition 3.1, then we have*

$$(3.28) \qquad \gamma_1 + \cdots + \gamma_k = 0, \quad \gamma_{k+1} + \cdots + \gamma_L + \gamma_0 = 0.$$

As an immediate consequence of this theorem, we obtain the following necessary and sufficient condition on Leray's inequality.

Corollary 3.2 *Let Ω be a bounded domain in $\mathbb{R}^3$ with the first and second Betti numbers N and L as in Definition 1.1. Suppose that $\beta \in H^{1/2}(\partial\Omega)$ satisfies (G.F.). Assume that there exist L spheres $S_1, \cdots, S_L$ in Ω such that S_i contains only Γ_i in its inside and the rests $\partial\Omega \setminus \Gamma_i$ lie in the outside of S_i for all $i = 1, \cdots, L$. Then β satiesfies Leray's inequality in Ω as in Definition 3.1 if and only if (R.F.) holds.*

Remark 3.4 (1) Corollary 3.2 may be regarded as a generalization of Takeshita [47, Theorem 2] since it is only assumed that each component Γ_i, $i = 1, \cdots, L$ is a smooth two-dimensional closed surface in $\mathbb{R}^3$.

(2) The assumption on regularity of the boundary $\partial\Omega$ can be relaxed so that the Stokes integral formula holds for vector fields on $\bar{\Omega}$. For instance, Theorem 1.1 holds for bounded locally Lipschitz domains Ω. More generally, we may treat the case when Ω is a bounded domain in $\mathbb{R}^3$ with locally finite perimeter as in Ziemer [54, Theorem 5.8.2].

(3) A similar argument to make use of the sphere covering each component of the boundary was established by Kobayashi [24] in the two-dimensional multi-connected domains. Indeed, he proved the corresponding result to Corollary 3.2 in the plane. However, it seems difficult to apply his method directly to our three-dimensional case.

(4) Under some hypothesis on symmetry of the multi-connected domain Ω in $\mathbb{R}^2$, Amick [2] showed an existence theorem for the solution $v \in H^1(\Omega)$ of (N-S). His method is based on a contradiction argument. Later on, Fujita [17] proved (L.I.) for all solenoidal vector fields u with symmetry, which yields necessarily the existence of solutions. See also Morimoto [37].

Proof of Theorem 3.2. Suppose that the boundary data $\beta \in H^{1/2}(\partial\Omega)$ satisfies Leray's inequality in Ω in the sense of Definition 3.1. Then, for every $\varepsilon > 0$ there exists $b_\varepsilon \in H^1(\Omega)$ with div $b_\varepsilon = 0$ in Ω and $b_\varepsilon = \beta$ such that (L.I.) holds. By the hypothesis on $\partial\Omega$, without loss of generality, we may take $0 < R < R'$ such that both spheres $S_R \equiv \{x \in \mathbb{R}^3; |x| = R\}$ and $S_{R'} \equiv \{x \in \mathbb{R}^3; |x| = R'\}$ are contained in Ω, $\Gamma_1, \cdots, \Gamma_k$ lie inside of S_R and such that $\Gamma_{k+1}, \cdots, \Gamma_L$ and Γ_0 lie outside of $S_{R'}$. Since $\sum_{i=0}^{L} \gamma_i = 0$, implied by (G.F.), and since div $b_\varepsilon = 0$ in Ω with $b_\varepsilon = \beta$ on $\partial\Omega$, it holds

$$\int_{S_R} b_\varepsilon \cdot \nu \, dS = \gamma \equiv \sum_{i=1}^{k} \gamma_i, \quad \int_{S_{R'}} b_\varepsilon \cdot \nu \, dS = -\gamma. \tag{3.29}$$

Now we reduce our problem to that in the concentric spherical domain $D \equiv \{x \in \mathbb{R}^3; R < |x| < R'\}$ and follow the argument given by Takeshita [47].

Let us take the mean $M(b_\varepsilon)$ of b_ε with respect to the normalized Haar measure dg on $SO(3)$-actions. That is,

$$\begin{aligned} M(b_\varepsilon) &= \int_{SO(3)} T_g b_\varepsilon \, dg, \\ T_g b_\varepsilon(x) &= g b_\varepsilon(g^{-1}x), \quad x \in D, g \in SO(3). \end{aligned}$$

By (3.29) it holds

$$\begin{cases} \text{div } M(b_\varepsilon) = 0 \quad \text{in } D, \\ \displaystyle\int_{S_R} M(b_\varepsilon) \cdot \nu \, dS = \gamma, \quad \int_{S_{R'}} M(b_\varepsilon) \cdot \nu \, dS = -\gamma. \end{cases} \tag{3.30}$$

Furthermore, by (L.I.) we have

$$\left| \int_D v \cdot \nabla M(b_\varepsilon) \cdot v \, dx \right| \leqq \varepsilon \int_D |\nabla v|^2 \, dx \quad \text{for all } v \in C_{0,\sigma}^\infty(D), \tag{3.31}$$

where $C_{0,\sigma}^\infty(D)$ is the set of all solenoidal vector fields with compact support in D. Indeed, since det $g = 1$, by changing the variable $x \in D \mapsto y = g^{-1}x \in D$, we have

$$\int_D v \cdot \nabla(T_g b_\varepsilon) \cdot v \, dx = \int_D T_g^{-1} v \cdot \nabla b_\varepsilon \cdot T_g^{-1} v \, dy$$

for all $g \in SO(3)$, which implies with the aid of the Fubini theorem that

$$\begin{aligned} \left| \int_D v \cdot \nabla(M b_\varepsilon) \cdot v \, dx \right| &= \left| \int_{SO(3)} \left(\int_D T_g^{-1} v \cdot \nabla b_\varepsilon \cdot T_g^{-1} v \, dy \right) dg \right| \\ &\leqq \int_{SO(3)} \left| \int_D T_g^{-1} v \cdot \nabla b_\varepsilon \cdot T_g^{-1} v \, dy \right| dg. \end{aligned} \tag{3.32}$$

Since $T_g^{-1}v \in C_{0,\sigma}^\infty(D)$ and since $|\nabla T_g^{-1}v(y)|^2 = |\nabla v(gy)|^2$ for all $y \in D$, we have by (L.I.) and again by changing variable $y \in D \mapsto x = gy \in D$ with $\det g^{-1} = 1$ that

$$
\begin{aligned}
\left|\int_D T_g^{-1}v \cdot \nabla b_\varepsilon \cdot T_g^{-1}v \, dy\right| &= \left|\int_\Omega T_g^{-1}v \cdot \nabla b_\varepsilon \cdot T_g^{-1}v \, dy\right| \\
&\leqq \varepsilon \int_\Omega |\nabla T_g^{-1}v|^2 dy \\
&= \varepsilon \int_D |\nabla T_g^{-1}v|^2 dy \\
&= \varepsilon \int_D |\nabla v|^2 dx
\end{aligned} \tag{3.33}
$$

for all $g \in SO(n)$. It follows from (3.32) and (3.33) that

$$
\left|\int_D v \cdot \nabla(Mb_\varepsilon) \cdot v \, dx\right| \leqq \varepsilon \int_{SO(3)} \left(\int_D |\nabla v|^2 dx\right) dg = \varepsilon \int_D |\nabla v|^2 dx,
$$

which implies (3.31).

In the next step, we test (3.31) by an appropriate $v \in C_{0,\sigma}^\infty(D)$. First, it follows from (3.30) that $M(b_\varepsilon)$ has the representation as

$$
M(b_\varepsilon) = \frac{\gamma}{4\pi r^3} x, \quad x \in D, \tag{3.34}
$$

where $r = |x|$. Now, we choose a test vector function v of (3.31) as

$$
v(x) = (-\rho(r)x_2, \rho(r)x_1, 0), \quad x = (x_1, x_2, x_3) \in D
$$

with $\rho \in C_0^\infty((R, R'))$. It is easy to see that $v \in C_{0,\sigma}^\infty(D)$ with the property that $v(x) \cdot x = 0$ for all $x \in D$. Since

$$
\frac{\partial}{\partial x_j} M(b_\varepsilon)_k = \frac{\gamma}{4\pi r^3} \left(\delta_{jk} - 3\frac{x_j}{r}\frac{x_k}{r}\right), \quad j, k = 1, 2, 3
$$

and since $v(x) \cdot x = 0$ for all $x \in D$, it holds that

$$
\begin{aligned}
v \cdot \nabla M(b_\varepsilon) \cdot v = \sum_{j,k=1}^3 v_j \frac{\partial}{\partial x_j} M(b_\varepsilon)_k v_k &= \frac{\gamma}{4\pi r^3}\left(|v|^2 - 3\left(\frac{v \cdot x}{r}\right)^2\right) \\
&= \frac{\gamma}{4\pi r^3}|v|^2
\end{aligned} \tag{3.35}
$$

in D. Hence it follows from (3.31) and (3.35) that

$$
\frac{|\gamma|}{4\pi} \int_D \frac{|v|^2}{r^3} dx \leqq \varepsilon \int_D |\nabla v|^2 dx \tag{3.36}
$$

for all $\varepsilon > 0$. Since v and the left hand side of (3.36) are independent of ε and since $\nabla v \neq 0$, by letting $\varepsilon \to 0$ we conclude from (3.36) that

$$
\gamma = 0.
$$

This proves Theorem 3.2.

4 Global Div-Curl lemma

4.1 Global convergence

Let Ω be an open set in $\mathbb{R}^3$. It is well-known that if $u_j \rightharpoonup u$, $v_j \rightharpoonup v$ weakly in $L^2(\Omega)$ and if $\{\text{div } u_j\}_{j=1}^{\infty}$ and $\{\text{rot } v_j\}_{j=1}^{\infty}$ are bounded in $L^2(\Omega)$, then it holds that $u_j \cdot v_j \rightharpoonup u \cdot v$ in the sense of distributions in Ω. This is the original Div-Curl lemma. For instance, we refer to Tartar [47]. The purpose of this section is to deal with a similar lemma to bounded domains where the convergence $u_j \cdot v_j \to u \cdot v$ holds in the sense that

$$\int_\Omega u_j \cdot v_j dx \to \int_\Omega u \cdot v dx \quad \text{as } j \to \infty. \tag{4.37}$$

Our result may be regarded as a *global* version of the Div-Curl lemma, which includes the previous one. To obtain such a global version, we need to pay an attention to the behaviour of $\{u_j\}_{j=1}^{\infty}$ and $\{v_j\}_{j=1}^{\infty}$ on the boundary $\partial\Omega$ of Ω. Indeed, an additional bound of $\{u_j \cdot \nu|_{\partial\Omega}\}_{j=1}^{\infty}$, or that of $\{v_j \times \nu|_{\partial\Omega}\}_{j=1}^{\infty}$ in $H^{\frac{1}{2}}(\partial\Omega)$ on the boundary $\partial\Omega$ plays an essential role for our convergence, where ν denotes the unit outward normal to $\partial\Omega$. We shall establish a global convergence in the whole domain Ω in $L^r(\Omega)$ and $L^{r'}(\Omega)$.

Our result now reads:

Theorem 4.1 *Let Ω be as in the Assumption. Let $1 < r < \infty$. Suppose that $\{u_j\}_{j=1}^{\infty} \subset L^r(\Omega)$ and $\{v_j\}_{j=1}^{\infty} \subset L^{r'}(\Omega)$ satisfy*

$$u_j \rightharpoonup u \quad \text{weakly in } L^r(\Omega), \quad v_j \rightharpoonup v \quad \text{weakly in } L^{r'}(\Omega) \tag{4.38}$$

for some $u \in L^r(\Omega)$ and $v \in L^{r'}(\Omega)$, respectively. Assume also that

$$\{\text{div } u_j\}_{j=1}^{\infty} \text{ is bounded in } L^q(\Omega) \text{ for some } q > \max\{1, 3r/(3+r)\} \tag{4.39}$$

and that

$$\{\text{rot } v_j\}_{j=1}^{\infty} \text{ is bounded in } L^s(\Omega) \text{ for some } s > \max\{1, 3r'/(3+r')\}, \tag{4.40}$$

respectively. If either

(i) $\{\gamma_\nu u_j\}_{j=1}^{\infty}$ *is bounded in* $W^{1-1/q,q}(\partial\Omega)$,

or

(ii) $\{\tau_\nu v_j\}_{j=1}^{\infty}$ *is bounded in* $W^{1-1/s,s}(\partial\Omega)$,

then it holds that

$$\int_\Omega u_j \cdot v_j dx \to \int_\Omega u \cdot v dx \quad \text{as } j \to \infty. \tag{4.41}$$

In particular, if either $\gamma_\nu u_j = 0$, or $\tau_\nu v_j = 0$ for all $j = 1, 2, \cdots$ is satisfied, then we have also (4.41).

As an immediate consequence of our theorem, we have the following Div-Curl lemma in an arbitrary open set in $\mathbb{R}^3$.

Corollary 4.1 (Tartar [47]) *Let D be an arbitrary open set in $\mathbb{R}^3$. Let $1 < r < \infty$. Suppose that $\{u_j\}_{j=1}^\infty \subset L^r(D)$ and $\{v_j\}_{j=1}^\infty \subset L^{r'}(D)$ satisfy*

$$u_j \rightharpoonup u \quad \text{weakly in } L^r(D), \quad v_j \rightharpoonup v \quad \text{weakly in } L^{r'}(D) \tag{4.42}$$

for some $u \in L^r(D)$ and $v \in L^{r'}(D)$, respectively. Assume also that

$$\{\text{div } u_j\}_{j=1}^\infty \text{ and } \{\text{rot } v_j\}_{j=1}^\infty \text{ are bounded in } L^r(D) \text{ and } L^{r'}(D), \tag{4.43}$$

respectively. Then it holds that

$$u_j \cdot v_j \rightharpoonup u \cdot v \quad \text{in the sense of distributions in } D. \tag{4.44}$$

Remark 4.1 (1) Since Ω is a bounded domain, we may assume that $3r/(3+r) < q \leqq r$ and $3r'/(3+r') < s \leqq r'$, and hence it holds that $\{u_j\}_{j=1}^\infty \subset E^q_{div}(\Omega)$ and that $\{v_j\}_{j=1}^\infty \subset E^s_{rot}(\Omega)$. Then we have that $\{\gamma_\nu u_j\}_{j=1}^\infty \subset W^{1-1/q',q'}(\partial\Omega)^*$ and $\{\tau_\nu v_j\}_{j=1}^\infty \subset W^{1-1/s',s'}(\partial\Omega)^*$.

(2) In Theorem 4.1, it is *un*necessary to assume both bounds of $\{\gamma_\nu u_j\}_{j=1}^\infty$ in $W^{1-1/r,r}(\partial\Omega)$ and $\{\tau_\nu v_j\}_{j=1}^\infty$ in $W^{1-1/r',r'}(\partial\Omega)$. Indeed, what we need is only one of these bounds.

4.2 More regularity of vector and scalar potentials

If u has an additional regularity such as div $u \in L^q(\Omega)$ and rot $u \in L^q(\Omega)$ for some $1 < q \leqq r$, then we may choose the scalar and the vector potentials p and w in (1.9) and (1.12) in the class $W^{2,q}(\Omega)$. More precisely, we have

Proposition 4.1 *Let Ω be as in the Assumption and let $1 < r < \infty$. Suppose that $u \in L^r(\Omega)$.*

(1) *Let us consider the decomposition (1.9).*

(i) *If, in addition,* rot $u \in L^q(\Omega)$ *for some $1 < q \leqq r$, then the vector potential w of u in (1.9) can be chosen as $w \in W^{2,q}(\Omega) \cap V^r_\sigma(\Omega)$ with the estimate*

$$\|w\|_{W^{2,q}} \leqq C(\|\text{rot } u\|_q + \|u\|_r). \tag{4.45}$$

(ii) *If, in addition,* div $u \in L^q(\Omega)$ *with $\gamma_\nu u \in W^{1-1/q,q}(\partial\Omega)$ for some $1 < q \leqq r$, then the scalar potential p of u in (1.9) can be chosen as $p \in W^{2,q}(\Omega) \cap W^{1,r}(\Omega)$ with the estimate*

$$\|p\|_{W^{2,q}} \leqq C(\|\text{div } u\|_q + \|u\|_r + \|\gamma_\nu u\|_{W^{1-1/q,q}(\partial\Omega)}). \tag{4.46}$$

(2) *Let us consider the decomposition (1.12).*

(i) *If, in addition,* div $u \in L^q(\Omega)$ *for some $1 < q \leqq r$, then the scalar potential p of u in (1.12) can be chosen as $p \in W^{2,q}(\Omega) \cap W^{1,r}_0(\Omega)$ with the estimate*

$$\|p\|_{W^{2,q}} \leqq C\|\text{div } u\|_q. \tag{4.47}$$

(ii) *If, in addition,* rot $u \in L^q(\Omega)$ *with $\tau_\nu u \in W^{1-1/q,q}(\partial\Omega)$ for some $1 < q \leqq r$, then the vector potential w of u in (1.12) can be chosen as $w \in W^{2,q}(\Omega) \cap X^r_\sigma(\Omega)$ with the estimate.*

$$\|w\|_{W^{2,q}} \leqq C(\|\text{rot } u\|_q + \|u\|_r + \|\tau_\nu u\|_{W^{1-1/q,q}(\partial\Omega)}). \tag{4.48}$$

Here $C = C(r, q)$ is the constant depending only on r and q.

Proof. (1) (i) In the decomposition (1.9), the vector potential $w \in V^r_\sigma(\Omega)$ is taken in such a way that

$$(\text{rot } w, \text{rot } \Psi) = (u, \text{rot } \Psi) \quad \text{for all } \Psi \in V^{r'}_\sigma(\Omega) \tag{4.49}$$

with the estimate

$$\|w\|_{W^{1,r}} \leqq C\|u\|_r, \tag{4.50}$$

where $C = C(r)$ is a constant depending only on r. See Lemma 2.5. Since div $w = 0$ in Ω and since rot $u \in L^q(\Omega)$, it follows from (4.49) that $-\Delta w = \text{rot } u$ in the sense of distributions in Ω, and we may regard w as a weak solution of the boundary value problem

$$\begin{cases} -\Delta w = \text{rot } u & \text{in } \Omega, \\ \text{div } w = 0 & \text{on } \partial\Omega, \\ w \times \nu = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.51}$$

Hence it follows from Lemma 2.6 (1) and the classical theory of Agmon-Douglis-Nirenberg [1] that the solution w of the *homogeneous* boundary value problem (4.51) belongs to $W^{2,q}(\Omega)$ and that the estimate

$$\|w\|_{W^{2,q}} \leqq C(\|\text{rot } u\|_q + \|w\|_q) \tag{4.52}$$

holds with a constant C depending only on q. Since $\|w\|_q \leqq |\Omega|^{\frac{1}{q}-\frac{1}{r}}\|w\|_r$, the desired estimate (4.45) follows from (4.50) and (4.52).

(ii) The scalar potential $p \in W^{1,r}(\Omega)$ in (1.9) is chosen in such a way that

$$(\nabla p, \nabla \psi) = (u, \nabla \eta) \quad \text{for all } \eta \in W^{1,r'}(\Omega) \tag{4.53}$$

with the estimate

$$\|p\|_{W^{1,r}} \leqq C\|u\|_r. \tag{4.54}$$

See (2.53) and (2.54). See also Simader-Sohr [41, Theorems 1.3, 1.4]. Since div $u \in L^q(\Omega)$ and since $\gamma_\nu(\nabla p - u) = 0$, we may regard p as a weak solution of

$$\Delta p = \text{div } u \quad \text{in } \Omega, \quad \frac{\partial p}{\partial \nu} = u \cdot \nu \quad \text{on } \partial\Omega.$$

Since $\gamma_\nu u \in W^{1-1/q,q}(\partial\Omega)$, the well-known a priori estimate for the inhomogeneous Neumann problem of the Poisson equation states that $p \in W^{2,q}(\Omega)$ with the estimate

$$\|p\|_{W^{2,q}} \leqq C(\|\text{div } u\|_q + \|p\|_q + \|\gamma_\nu u\|_{W^{1-1/q,q}(\partial\Omega)}).$$

Since $\|p\|_q \leqq |\Omega|^{\frac{1}{q}-\frac{1}{r}}\|p\|_r$, from (4.54) and the above estimate we obtain (4.46).

(2) (i) In the decomposition of (1.12), the scalar potential p is the solution of the Dirichlet problem of the Poisson equation

$$\Delta p = \text{div } u \quad \text{in } \Omega, \quad p = 0 \quad \text{on } \partial\Omega. \tag{4.55}$$

More precisely, we may choose $p \in W^{1,r}_0(\Omega)$ as in (2.58) and (2.59). Since div $u \in L^q(\Omega)$, we may take p in such a way that $p \in W^{2,q}(\Omega) \cap W^{1,r}_0(\Omega)$ with the estimate

$$\|p\|_{W^{2,q}} \leqq C\|\text{div } u\|_q,$$

which yields (4.47).

(ii) The vector potential $w \in X^r_\sigma(\Omega)$ in (1.12) is chosen in such a way that

$$(\text{rot } w, \text{rot } \Phi) = (u, \text{rot } \Phi) \quad \text{for all } \Phi \in X^{r'}_\sigma(\Omega) \tag{4.56}$$

with the estimate

$$\|w\|_{W^{1,r}} \leqq C\|u\|_r. \tag{4.57}$$

See 2.40 and (2.40). Since div $w = 0$ in Ω and since rot $u \in L^q(\Omega)$, it follows from (4.56) that $-\Delta w = \text{rot } u$ in the sense of distributions in Ω, and we may regard w as a weak solution of the boundary value problem

$$\begin{cases} -\Delta w = \text{rot } u \quad \text{in } \Omega, \\ \text{rot } w \times \nu = u \times \nu \quad \text{on } \partial\Omega, \\ w \cdot \nu = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{4.58}$$

Since $\tau_\nu u \in W^{1-1/q,q}(\partial\Omega)$, it follows from Lemma 2.6 (1) that the solution w of the *inhomogeneous* boundary value problem (4.58) belongs to $W^{2,q}(\Omega)$ and that the estimate

$$\|w\|_{W^{2,q}} \leqq C(\|\text{rot } u\|_q + \|w\|_q + \|\tau_\nu u\|_{W^{1-1/q,q}(\partial\Omega)})$$

holds with a constant $C = C(q)$ depending only on q. Since $\|w\|_q \leqq |\Omega|^{\frac{1}{q}-\frac{1}{r}}\|w\|_r$, from (4.57) and the above estimates we obtain the desired estimate (4.48). This completes the proof of Proposition 4.1.

4.3 L^r-global Div-Curl lemma; Proof of Theorem 4.1

(i) Let us first consider the case when $\{\gamma_\nu u_j\}_{j=1}^\infty$ is bounded in $W^{1-1/q,q}(\partial\Omega)$. In such a case, we make use of the decomposition (1.9). Let S_r, R_r and Q_r be the projection operators from $L^r(\Omega)$ onto $X_{har}(\Omega)$, rot $V^r_\sigma(\Omega)$ and $\nabla W^{1,r}(\Omega)$ defined by (1.17), respectively. Notice that the identity

$$(u, v) = (S_r u, S_{r'} v) + (R_r u, R_{r'} v) + (Q_r u, Q_{r'} v) \tag{4.59}$$

holds for all $u \in L^r(\Omega)$ and all $v \in L^{r'}(\Omega)$. Indeed, by the generalized Stokes formula (1.1) and (1.2), we have

$$(\nabla p, h) = -(p, \text{div } h) + \langle \gamma_\nu h, \gamma_0 p \rangle_{\partial\Omega} = 0,$$
$$(\text{rot } w, h) = (w, \text{rot } h) + \langle \tau_\nu w, \gamma_0 h \rangle_{\partial\Omega} = 0$$

for all $p \in W^{1,r}(\Omega)$, $w \in V^r_\sigma(\Omega)$ and $h \in X_{har}(\Omega)$, Similarly, we have

$$(\text{rot } w, \nabla p) = \langle \gamma_\nu(\text{rot } w), \gamma_0 p \rangle_{\partial\Omega} = 0 \quad \text{for all } w \in V^r_\sigma(\Omega),\ p \in W^{1,r'}(\Omega).$$

Thus we obtain (4.59).

Now, by (4.59), we see that the convergence (4.41) can be reduced to

$$(S_r u_j, S_{r'} v_j) \ \rightarrow \ (S_r u, S_{r'} v), \tag{4.60}$$
$$(R_r u_j, R_{r'} v_j) \ \rightarrow \ (R_r u, R_{r'} v), \tag{4.61}$$
$$(Q_r u_j, Q_{r'} v_j) \ \rightarrow \ (Q_r u, Q_{r'} v). \tag{4.62}$$

By Theorem 1.1 (1), the ranges of S_r and $S_{r'}$ are of finite dimension, which means that both S_r and $S_{r'}$ are finite rank operators. Hence, we have by (4.38) that

$$S_r u_j \to S_r u \quad \text{strongly in } L^r(\Omega), \quad S_{r'} v_j \to S_{r'} v \quad \text{strongly in } L^{r'}(\Omega),$$

from which it follows (4.60).

Next, we apply Proposition 4.1 (1) to (4.61) and (4.62). Since Ω is bounded, we may assume that

$$\max\left\{1, \frac{3r}{3+r}\right\} < q \leqq r, \quad \max\left\{1, \frac{3r'}{3+r'}\right\} < s \leqq r'.$$

By (4.39) and (4.45) with q and r replaced by s and r', respectively, we see that $R_{r'}v_j \equiv \text{rot } \tilde{w}_j$ with $\tilde{w}_j \in V_\sigma^{r'}(\Omega)$ satisfies $\tilde{w}_j \in W^{2,s}(\Omega) \cap V_\sigma^{r'}(\Omega)$ with the estimate

$$\|\tilde{w}_j\|_{W^{2,s}} \leqq C(\|\text{rot } v_j\|_s + \|v_j\|_{r'}) \leqq M, \quad \text{for all } j = 1, 2, \cdots$$

with a constant M independent of j. Since $1/r' > 1/s - 1/3$, the embedding $W^{2,s}(\Omega) \subset W^{1,r'}(\Omega)$ is compact, and hence we see that $\{\tilde{w}_j\}_{j=1}^\infty$ has a strongly convergent subsequence in $W^{1,r'}(\Omega)$. Since (4.38) yields rot $\tilde{w}_j = R_{r'}v_j \rightharpoonup R_{r'}v$ weakly in $L^{r'}(\Omega)$, it holds, in fact, that

$$R_{r'}v_j \to R_{r'}v \quad \text{strongly in } L^{r'}(\Omega). \tag{4.63}$$

Obviously by (4.38), $R_r u_j \rightharpoonup R_r u$ weakly in $L^r(\Omega)$, and hence it follows (4.61).

Since $\{\gamma_\nu u_j\}_{j=1}^\infty$ is bounded in $W^{1-1/q,q}(\partial\Omega)$, we see from (4.39) and (4.46) that $Q_r u_j = \nabla p_j$ satisfies that $p_j \in W^{2,q}(\Omega)$ with the estimate

$$\|p_j\|_{W^{2,a}} \leqq C(\|\text{div } u_j\|_q + \|u_j\|_r + \|\gamma_\nu u_j\|_{W^{1-1/q,q}(\partial\Omega)}) \leqq M \quad \text{for all } j = 1, 2, \cdots$$

with a constant M independent of j. Since $1/r > 1/q - 1/3$, again by the compact embedding $W^{2,q}(\Omega) \subset W^{1,r}(\Omega)$ and by the weak convergence $\nabla p_j = Q_r u_j \rightharpoonup Q_r u$ in $L^r(\Omega)$, implied by (4.38), it holds that

$$Q_r u_j \to Q_r u \quad \text{strongly in } L^r(\Omega). \tag{4.64}$$

Since (4.38) yields $Q_{r'}v_j \rightharpoonup Q_{r'}v$ weakly in $L^{r'}(\Omega)$, it follows (4.62).

(ii) We next consider the case when $\{\tau_\nu v_j\}_{j=1}^\infty$ is bounded in $W^{1-1/s,s}(\partial\Omega)$. In this case, we make use of the decomposition (1.12). Then the argument is quite similar to the former case (i) above. By the same notations S_r, R_r and Q_r, we denote the projection operators from $L^r(\Omega)$ onto $V_{har}(\Omega)$, rot $X_\sigma^r(\Omega)$ and $\nabla W_0^{1,r}(\Omega)$ defined by (1.17), respectively. By the generalized Stokes formula, it is easy to see that the identity (4.59) holds, and hence we may prove (4.60), (4.61) and (4.62). Since the range of S_r is $V_{har}(\Omega)$, it follows from Theorem 1.1 (1) that the convergence (4.60) holds.

Since $\{\tau_\nu v_j\}_{j=1}^\infty$ is bounded in $W^{1-1/s,s}(\partial\Omega)$, by (4.40) and (4.48) with q and r replaced by s and r', we find that $R_{r'}v_j = \text{rot } \tilde{w}_j$ with $\tilde{w}_j \in X_\sigma^{r'}(\Omega)$ satisfies, in fact, that $\tilde{w}_j \in W^{2,s}(\Omega) \cap X_\sigma^{r'}(\Omega)$ with the estimate

$$\|\tilde{w}_j\|_{W^{2,s}} \leqq C(\|\text{rot } v_j\|_s + \|v_j\|_{r'} + \|\tau_\nu v_j\|_{W^{1-1/s,s}(\partial\Omega)}) \leqq M, \quad \text{for all } j = 1, 2, \cdots$$

with a constant M independent of j. By the compact embedding $W^{2,s}(\Omega) \subset W^{1,r'}(\Omega)$ and by the weak convergence rot $\tilde{w}_j = R_{r'}v_j \rightharpoonup R_{r'}v$ in $L^{r'}(\Omega)$, implied by (4.38), it holds that

$$R_{r'}v_j \to R_{r'}v \quad \text{strongly in } L^{r'}(\Omega). \tag{4.65}$$

Since (4.38) yields $R_r u_j \rightharpoonup R_r u$ weakly in $L^r(\Omega)$, it follows (4.61).

From (4.39) and (4.47) we see that $Q_r u_j = \nabla p_j$ with $p_j \in W_0^{1,r}(\Omega)$ satisfies, in fact, that $p_j \in W^{2,q}(\Omega) \cap W_0^{1,r}(\Omega)$ with the estimate

$$\|p_j\|_{W^{2,q}} \leqq C\|\text{div } u_j\|_q \leqq M \quad \text{for all } j = 1, 2, \cdots$$

with a constant M independent of j. Hence again by the compact embedding $W^{2,q}(\Omega) \subset W^{1,r}(\Omega)$ and by the weak convergence $\nabla p_j = Q_r u_j \rightharpoonup Q_r u$ in $L^r(\Omega)$, implied by (4.38), it holds that

$$Q_r u_j \to Q_r u \quad \text{strongly in } L^r(\Omega). \tag{4.66}$$

Since (4.38) yields $Q_{r'}v_j \rightharpoonup Q_{r'}v$ weakly in $L^{r'}(\Omega)$, it follows (4.62). This proves Theorem 4.1.

Proof of Corollary 4.1. We may prove that for every $\varphi \in C_0^\infty(D)$

$$\int_D \varphi u_j \cdot v_j dx \to \int_D \varphi u \cdot v dx.$$

Let us take a bounded domain $\Omega \subset \mathbb{R}^3$ with the smooth boundary $\partial\Omega$ so that supp $\varphi \subset \Omega \subset D$. Then it suffices to prove that

$$\int_\Omega \varphi u_j \cdot v_j dx \to \int_\Omega \varphi u \cdot v dx. \tag{4.67}$$

Obviously by (4.42), it holds that

$$\varphi u_j \rightharpoonup \varphi u \quad \text{weakly } L^r(\Omega), \quad v_j \rightharpoonup v \quad \text{weakly } L^{r'}(\Omega). \tag{4.68}$$

Since div $(\varphi u_j) = \varphi$div $u_j + u_j \cdot \nabla\varphi$, we see by (4.42) and (4.43) that $\{\text{div } (\varphi u_j)\}_{j=1}^\infty$ is bounded in $L^r(\Omega)$ with

$$\gamma_\nu(\varphi u_j) = 0, \quad j = 1, 2, \cdots. \tag{4.69}$$

Since (4.43) states that $\{\text{rot } v_j\}_{j=1}^\infty$ is also bounded in $L^{r'}(\Omega)$, by taking $q = r$ and $s = r'$ in (4.39) and (4.40), respectively, we see that the convergence (4.67) follows from (4.68), (4.69) and Theorem 4.1 (i). This proves Corollary 4.1.

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