REAL INTERPOLATION, LORENTZ SPACES AND THE NAVIER-STOKES EQUATION

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Abstract. In this note we give a brief sketch of real interpolations, the Lorentz spaces and their applications to the Navier-Stokes equations.

Introduction.

This note is a brief sketch of real interpolation and the Lorentz spaces, and its application to the Navier-Stokes equations.

There are two methods of interpolation: complex method and real method. Both of them survive because each has its advantages to the other. Namely, the advantage of complex interpolation is the following:

(i) Multilinear operators can be treated
(ii) The operator may depend holomorphically with respect to the interpolation parameter.

On the other hand, the advantage of real interpolation is the following:

(i) The function spaces need not be normed.
(ii) The operators need not be linear.
(iii) One can "improve" the function space in the course of interpolation.

Indeed, in the proof of the Mikhlin-Hörmander multiplier theorem we use properties (i) and (iii) of real interpolation, and in the proof of the boundedness of the Hardy-Littlewood maximal function, we use all of them.

Section 1 to Section 5 are devoted to the sketch of real interpolation and the Lorentz spaces. Generally I followed [1], but I avoided to describe the whole theory and limited ourselves to the necessary part of the theory, and some proofs are modified accordingly. Beside, all of the aforementioned properties of real interpolation are employed in the application to the Navier-Stokes equations, and hence I emphasized such properties.

Section 6 is devoted to the application of the theory above to the Navier-Stokes equations. There are two methods to the proof of the key inequality (6.7) in Lemma 6.2. One is the cut-off method developed by Prof. Shibata et al. This method can be applied to more general operators; namely, the operator need not be sectorial. This method is employed by Hishida and Shibata [2]. Here we employ a simpler method, which relies on the coercive estimate of the fractional power of the Stokes operator as in [3], which is based on complex interpolation. For related works, see the references therein.

1. Quasi-normed spaces.

Let X be an abelian group. A functional $\|\cdot\|$ on X is called a quasi-norm if there exists a $C \geq 1$ such that the following three conditions are satisfied:

$$\|x\| \geq 0, \tag{1.1}$$

$$\|x\| = 0 \text{ if and only if } x = 0, \tag{1.2}$$

$$\|-x\| = \|x\|, \tag{1.3}$$

$$\|x+y\| \leq C(\|x\| + \|y\|). \tag{1.4}$$

If X is a vector space on $\mathbb{R}$ or $\mathbb{C}$, we often assume the following condition, which is a generalization of (1.3):

$$\|ax\| = |a|\|x\| \text{ for } a \in \mathbb{R} \text{ or } \mathbb{C}. \tag{1.5}$$

Suppose that $F(u)$ is a quasi-norm which satisfies the triangle inequality $F(u+v) \leq F(u) + F(v)$ and the condition

$$F(ax) = |a|^p F(x) \text{ for } a \in \mathbb{R} \text{ or } \mathbb{C} \tag{1.6}$$

with some $p \in (0,1)$. (An example of such a quasi-norm is $\|u\|_p^p$ of the Lebesgue space L^p with $0 < p < 1$.) Then the function $\|a\| = F(x)^{1/p}$ satisfies (1.1), (1.2) and (1.5). Moreover, by Hölder's inequality we have

$$\begin{aligned} \|x+y\| &= F(a+b)^{1/p} \leq \big(F(a) + F(b)\big)^{1/p} \\ &\leq \big(F(a)^{1/p} + F(b)^{1/p}\big)(1+1)^{(1-p)/p} = (\|a\| + \|b\|)2^{1/p-1}. \end{aligned}$$

Namely, the functional $\|x\|$ enjoys (1.4) with $C = 2^{1/p-1}$. Conversely, we have the following theorem.

Theorem 1.1. *Choose $p \in (0,1)$ such that $C = 2^{1/p-1}$, and put*

$$F(x) = \inf\left\{ \sum_{j=1}^{N} \|x_j\|^p \;\middle|\; \sum_{j=1}^{N} x_j = x \right\}.$$

Then F is a quasi-norm on X, which enjoys the inequalities $F(x) \leq \|x\|^p \leq 2F(x)$ and the triangle inequality $F(x+y) \leq F(x) + F(y)$. If $\|\cdot\|$ enjoys (1.5), then we have (1.6).

In view of this theorem we see that the quasi-norm defines a metric on X, and the fundamental neighborhood system of 0 is given by the sets $\{x \in X \mid \|x\| < \varepsilon\}$ for $\varepsilon > 0$.

Proof of Theorem 1.1. It is easy to see (1.1). Putting $N = 1$ and $x_1 = x$, we see $F(x) \le \|x\|^p$. Next, for x, $y \in X$, choose $\{x_j\}_{j=1}^M$ and $\{y_k\}_{k=1}^N$ such that $x = \sum_{j=1}^M x_j$ and $y = \sum_{k=1}^N y_k$. Then we have

$$x + y = \sum_{j=1}^M x_j + \sum_{k=1}^N y_j.$$

Hence, by the definition of $\|x + y\|$, we see that

$$F(x + y) \le \sum_{j=1}^M \|x_j\|^p + \sum_{k=1}^N \|y_j\|^p.$$

Taking the infimum for the sequences $\{x_j\}_{j=1}^M$ and $\{y_k\}_{k=1}^N$, we conclude that $F(x + y) \le F(x) + F(y)$. In the same way we can see (1.1) and (1.3). Furthermore, if $\|\cdot\|$ satisfies (1.5), we can see in the same way that F satisfies (1.6).

It remains to show that

$$\|x\|^p \le 2F(x), \tag{1.7}$$

which immediately yields (1.2),

Suppose that $x = \sum_{j=1}^N x_j$, and put $M = \sum_{j=1}^N \|x_j\|^p$. Then, for every $j = 1, \dots, N$, we can choose a positive integer n_j such that

$$2^{-n_j} \le \frac{\|x_j\|^p}{M} \le 2^{1-n_j}.$$

Admitting the following lemma, we conclude

$$\|x\|^p \le \max_{j=1,\dots,N} 2^{n_j} \|x_j\|^p \le 2M.$$

Taking the infimum for $x_1, \dots, x_N$ we obtain (1.7). □

Lemma 1.2. *Suppose that $x_1, \dots, x_N \in X$ and that $n_1, \dots, n_N$ be positive integers such that*

$$\sum_{j=1}^N 2^{-n_j} \le 1 \text{ and } x = \sum_{j=1}^N x_j. \tag{1.8}$$

Then we have

$$\|x\|^p \le \max_{j=1,\dots,N} 2^{n_j} \|x_j\|^p. \tag{1.9}$$

Proof. We proceed by induction on N. The statement for $N = 1$ is trivial. Suppose that the statement holds for $1, \dots, N - 1$. Then, for every $n_1, \dots, n_N$ satisfying (1.8), we can divide $\{1, \dots, n\}$ into two disjoint groups I_1 and I_2 such that $\sum_{j \in I_\ell} 2^{1-n_j} \le 1$.

Indeed, suppose that $n_1 \le n_2 \le \cdots n_N$, and suppose that $n_1, \ldots, n_{j-1}$ were divided. Then we put n_j into I_ℓ such that the sum $\sum_{k\in I_\ell, k\le j-1} 2^{-n_k}$ is smaller. Then we have $\sum_{k\in I_\ell, k\le j-1} 2^{1-n_k} + 2^{1-n_j} \le 1$. Proceeding in the same way we can obtain the required estimate.

Now put $x^{(\ell)} = \sum_{j\in I_\ell} x_j$ for $\ell = 1, 2$. Then the induction hypothesis imply

$$\|x^{(\ell)}\|^p \le \max_{j\in I_\ell} 2^{n_j-1}\|x_j\|^p. \tag{1.10}$$

On the other hand, we have

$$\|x\|^p \le C^p \left(\|x^{(1)}\| + \|y^{(2)}\|\right)^p \le C^p 2^p \max_{\ell=1,2} \|x^{(\ell)}\|^p = 2 \max_{\ell=1,2} \|x^{(\ell)}\|^p$$

Say that $\|x^{(1)}\| \ge \|x^{(2)}\|$. Then, substituting (1.10) into the right-hand side we obtain $\|x\|^p \le \max_{j\in I_1} 2^{n_j}\|x_j\|^p \le \max_{j=1,\ldots,N} 2^{n_j}\|x_j\|^p$. This completes the proof. □

As we have seen before, a quasi-normed abelian group X becomes a topological group with respect to the metric defined by $d(u, v) = F(u - v)$. If X is a vector space on $\mathbb{R}$ or $\mathbb{C}$ and $F(u)$ enjoys (1.6), then X becomes a topological vector space. A complete topological vector space topologized in this way is called a quasi-Banach spaces.

2. Real interpolation between quasi-normed spaces.

Suppose that X_0 and X_1 be quasi-normed abelian groups. Suppose moreover that X_0 and X_1 are contained in a common topological abelian group. (For topological abelian groups X and Y, the notation $X \subset Y$ means not only the inclusion relation but also the inclusion is a continuous mapping from X to Y in the sequel.) For $x \in X_0 + X_1$ and $t \in (0, \infty)$, we put

$$\begin{aligned} K(x,t) &= K(x,t,X_0,X_1) \\ &= \inf\left\{ \|y\|_{X_0} + \frac{1}{t}\|z\|_{X_1} \,\middle|\, y \in X_0, z \in X_1, x = y + z \right\}. \end{aligned}$$

Then it is easy to see that

$$K(x,t) \le K(x,s) \le \frac{t}{s} K(x,t) \text{ if } s \le t, \tag{2.1}$$

which implies that $K(x,t)$ is continuous with respect to $t \in (0,\infty)$. Next, for $\theta \in (0,1)$ and $q \in (0,\infty]$ we put

$$\|x\|_{\theta,q} = \begin{cases} \left(\int_0^\infty \left(t^\theta K(x,t)\right)^q \dfrac{dt}{t} \right)^{1/q} & \text{if } 0 < q < \infty, \\ \sup_{t>0} t^\theta K(x,t) & \text{if } q = \infty, \end{cases}$$

and put $(X_0, X_1)_{\theta,q} = \{x \in X_0 + X_1 \mid \|x\|_{\theta,q} < \infty\}$. In view of (2.1). the quasi-norm above is equivalent to $\left\| \left\{ 2^{\theta j} K(x, 2^j) \right\}_{j=-\infty}^{\infty} \right\|_{\ell^q}$. We also have the following proposition.

Proposition 2.1. *Suppose that X_0 and X_1 are complete. Then $(X_0, X_1)_{\theta,q}$ is also complete.*

Hence, if X_0 and X_1 are quasi-Banach spaces, then so is $(X_0, X_1)_{\theta,q}$. On the other hand, if X_0 and X_1 are normed spaces and $q \geq 1$, then $(X_0, X_1)_{\theta,q}$ is a normed space. From these facts we see that, if X_0 and X_1 are Banach spaces and $q \geq 1$, then $(X_0, X_1)_{\theta,q}$ is a Banach space.

Proof of Proposition 2.1. Suppose that $\{x_j\}_{j=1}^{\infty}$ is a Cauchy sequence in $X = (X_0, X_1)_{\theta,q}$. Then we can choose a subsequence $\{X_{j(n)}\}_{n=1}^{\infty}$ such that the inequality $\|x_k - x_{j(n)}\|_X < 1/2^n$ holds for every $k > j(n)$. Put

$$y_n = \begin{cases} x_{j(1)} & \text{for } n = 1, \\ x_{j(n)} - x_{j(n-1)} & \text{for } n \geq 2. \end{cases}$$

Then we have

$$\left\{ \sum_{k=-\infty}^{\infty} \left(2^{\theta k q} K(y_n, 2^k) \right)^q \right\}^{1/q} \leq C \|y_n\|_X \leq \begin{cases} C \|x_{j(1)}\|_X & \text{for } n = 1, \\ \dfrac{C}{2^n} & \text{for } n \geq 2. \end{cases}$$

Fix a positive number $\varepsilon > 0$. Then, for every $n \in \mathbb{N}$ and every $k \in \mathbb{Z}$, we can choose $w_{n,k} \in X_0$ and $z_{n,k} \in X_1$ such that $w_{n,k} + z_{n,k} = y_n$ and that

$$\|w_{n,k}\|_{X_0} + 2^{-k} \|z_{n,k}\|_{X_1} \leq 2K(y_n, 2^k).$$

Then, for every $k \in \mathbb{Z}$, the series $\displaystyle\sum_{n=1}^{\infty} w_{n,k}$ and $\displaystyle\sum_{n=1}^{\infty} z_{n,k}$ converge in X_0 and X_1 respectively. Indeed, let $F(w)$ be another quasi-norm on X_0 satisfying the triangle inequality such that $F(w) \leq \|w\|_{X_0}^p \leq 2F(w)$ with some $p \in (0, 1]$. Then, for every positive integers M, N such that $N < M$, we have

$$\begin{aligned} \left\| \sum_{n=1}^{M} w_{n,k} - \sum_{n=1}^{N} w_{n,k} \right\|_{X_0}^p &\leq 2F\left(\sum_{n=N+1}^{M} w_{n,k} \right) \\ &\leq 2 \sum_{n=N+1}^{M} F\big(w_{n,k}\big) \leq 2 \sum_{n=N+1}^{M} \big\|w_{n,k}\big\|_{X_0}^p \leq 2^{1+p} \sum_{n=N+1}^{M} K(y_n, 2^k)^p \\ &\leq \sum_{n=N+1}^{\infty} \frac{2^{1+p}}{2^{np}} = \frac{2^{1+p}}{(1 - 2^{-p}) 2^{(N+1)p}} . \end{aligned}$$

Hence $\sum_{n=1}^{\infty} w_{n,k}$ converges in X_0. Let u_k denote the limit. Then, for every N we have

$$\left\| u_k - \sum_{n=1}^{N} w_{n,k} \right\|_{X_0} \le \frac{2^{1+1/p}}{(1-2^{-p})^{1/p}} \sum_{n=N+1}^{\infty} K(y_n, 2^k) \le \frac{C}{2^{N+1}} . \tag{2.2}$$

In the same way we can see that

$$\left\| v_k - \sum_{n=1}^{N} z_{n,k} \right\|_{X_1} \le C2^k \sum_{n=N+1}^{\infty} K(y_n, 2^k) \le \frac{C2^k}{2^{N+1}} . \tag{2.3}$$

with some $v_k \in X_1$. Then we have

$$u_k + v_k - x_{j(N)} = u_k + v_k - \sum_{n=1}^{N} y_n = u_k - \sum_{n=1}^{N} w_{n,k} + v_k - \sum_{n=1}^{N} z_{n,k}$$

It follows from (2.2) and (2.3) that

$$\|u_k + v_k - x_{j(N)}\|_{X_0+X_1} \le \frac{C}{2^N} \to 0 \text{ as } N \to \infty.$$

This implies that the sequence $\{x_{j(n)}\}_{n=1}^{\infty}$ converges to $u_k + v_k$ in $X_0 + X_1$, which is topologized by the quasi-norm

$$\|x\|_{X_0+X_1} = \inf\big\{ \|y\|_{X_0} + \|z\|_{X_1} \mid y \in X_0,\ z \in X_1,\ x = y + z \big\}.$$

It follows that $u_k + v_k$ is independent of $k \in \mathbb{Z}$. Put $x = u_k + v_k$. Then it suffices to show that $x \in X$ and that $\{x_j\}_{j=1}^{\infty}$ converges in X. To this end we see that

$$\begin{aligned}
&\left\| x - x_{j(N)} \right\|_X \\
&\le C \left(\sum_{k=-\infty}^{\infty} 2^{\theta k q} K(x - x_{j(N)}, 2^k)^q \right)^{1/q} \\
&\le C \left(\sum_{k=-\infty}^{\infty} 2^{\theta k q} \left(\left\| u_k - \sum_{n=1}^{N} w_{n,k} \right\|_{X_0} + \frac{1}{2^k} \left\| v_k - \sum_{n=1}^{N} z_{n,k} \right\|_{X_1} \right)^q \right)^{1/q} \\
&\le C \left(\sum_{k=-\infty}^{\infty} 2^{\theta k q} \sum_{n=N+1}^{\infty} K(y_n, 2^k)^q \right)^{1/q} \\
&= C \left(\sum_{n=N+1}^{\infty} \sum_{k=-\infty}^{\infty} 2^{\theta k q} K(y_n, 2^k)^q \right)^{1/q} = C \left(\sum_{n=N+1}^{\infty} \|y_n\|_X{}^q \right)^{1/q} \\
&\le C \left(\sum_{n=N+1} \frac{1}{2^{nq}} \right)^{1/q} \to 0 \text{ as } N \to \infty.
\end{aligned}$$

This implies that $x \in X$ and that the subsequence $\{x_{j(n)}\}_{n=1}^{\infty}$ converges to x in X. It follows from this and the fact that the sequence $\{x_j\}_{j=1}^{\infty}$ is a Cauchy sequence in X that $\{x_j\}_{j=1}^{\infty}$ converges to x in X. □

3. Another interpolation and interpolation of operators.

In this section we introduce another method of real interpolation for complete quasi-normed abelian groups. For $x \in X_0 \cap X_1$ and $t > 0$ we put

$$J(x,t) = J(x,t,X_0,X_1) = \max\left\{\|x\|_{X_0}, \frac{1}{t}\|x\|_{X_1}\right\}$$

Then we have

$$J(x,t) \le J(x,s) \le \frac{t}{s} \text{ if } s \le t. \tag{3.1}$$

We introduce $(X_0, X_1)^{\theta,q}$ intuitively as the collection of

$$\int_0^\infty u(t)\,\frac{dt}{t}$$

for an $X_0 \cap X_1$-valued function $u(t)$ on $(0,\infty)$ such that

$$\left(\int_0^\infty \left\{t^\theta J\big(x(t),t\big)\right\}^q \frac{dt}{t}\right)^{1/q} < \infty.$$

In order to avoid the difficulty of the convergence of improper integrals, we use the infinite sum in view of (3.1). For a finite sequence $\{x_j\}_{j=-N}^{N}$ in $X_0 \cap X_1$, we put $\left\|\{x_j\}_{j=-N}^{N}\right\| = \left\|\left\{2^{\theta j} J(x_j, 2^j)\right\}_{j=-N}^{N}\right\|_{\ell^q}$. Then, for $x \in X_0 \cap X_1$, we introduce the quasi-norm by

$$\inf\left\{\left\|\{x_j\}_{j=-N}^{N}\right\| \;\middle|\; x = \sum_{j=-N}^{N} x_j\right\}.$$

This quasi-norm is weaker than the standard quasi-norm $\|x\|_{X_0} + \|x\|_{X_1}$ in $X_0 \cap X_1$. It is natural to consider the completion of $X_0 \cap X_1$ with respect to the norm corresponding to this quasi-norm. This coincides with the space $(X_0, X_1)_*^{\theta,q}$ defined as the infinite sum of the series $\sum_{j=-\infty}^{\infty} x_j$ such that

$$\left\|\{x_j\}_{j=-\infty}^{\infty}\right\| = \left\|\left\{2^{\theta j} J(x_j, 2^j)\right\}_{j=-\infty}^{\infty}\right\|_{\ell^q} < \infty, \tag{3.2}$$

where the convergence is considered with respect to the quasi-norm

$$\|x\|_{(X_0,X_1)^{\theta,q}} = \inf\left\{\left\|\{x_j\}_{j=-\infty}^{\infty}\right\| \;\middle|\; x = \sum_{j=-\infty}^{\infty} x_j\right\}. \tag{3.3}$$

There is another method. Let $(X_0, X_1)^{\theta,q}$ denote the infinite sum of the series $\sum_{j=-\infty}^{\infty} x_j$ satisfying (3.2), where the infinite sum $\sum_{j=0}^{\infty} x_j$ converges in X_0 and the sum $\sum_{j=-\infty}^{-1} x_j$ converges in X_1. We equip $(X_0, X_1)^{\theta,q}$ with the quasi-norm defined by (3.3). Then $(X_0, X_1)^{\theta,q}$ is identical with $(X_0, X_1)^{\theta,q}_*$ for $q \in (0, \infty)$. However, the space $X_0 \cap X_1$ is not dense in $(X_0, X_1)^{\theta,\infty}$, and $(X_0, X_1)^{\theta,\infty}_*$ coincides with the closure of $X_0 \cap X_1$ in $(X_0, X_1)^{\theta,\infty}$.

Then we have the following theorem.

Theorem 3.1. *Let X_0 and X_1 be complete quasi-normed abelian groups. Then, for every $\theta \in (0,1)$ and every $q \in (0, \infty]$, the space $(X_0, X_1)^{\theta,q}$ coincides with the space $(X_0, X_1)_{\theta,q}$, and their quasi-norms are equivalent.*

Proof. We first show the inclusion relation $(X_0, X_1)^{\theta,q} \subset (X_0, X_1)_{\theta,q}$. Suppose that $x \in (X_0, X_1)^{\theta,q}$. Choose a sequence $\{x_j\}_{j=-\infty}^{\infty}$ such that $x = \sum_{j=0}^{\infty} x_j$ in X_0, $x = \sum_{j=-\infty}^{-1} x_j$ in X_1 and

$$\left\| \left\{ 2^{\theta j} J(x_j, 2^j) \right\}_{j=-\infty}^{\infty} \right\| < 2 \|x\|_{(X_0,X_1)^{\theta,q}} .$$

For every h, $k \in \mathbb{Z}$, we put $w_{h,k} = \sum_{j=h}^{k} x_j$. Recall that there exists a quasi-norm F on X_0 satisfying the triangle inequality such that $F(w) \leq \|w\|^p \leq 2F(w)$ holds for some $p \in (0, 1]$. We hence have

$$\|w_{h,k}\|_{X_0}^p \leq \sum_{j=h}^{k} 2^{-jp\theta} \|x\|_{(X_0,X_1)^{\theta,\infty}}{}^p \leq C 2^{-kp\theta} \|x\|_{(X_0,X_1)^{\theta,q}}{}^p .$$

This implies $\|w_{h,k}\|_{X_0} \leq C 2^{-k\theta} \|x\|_{(X_0,X_1)^{\theta,q}}$. In the same way we have $\|w_{h,k}\|_{X_1} \leq C 2^{(1-\theta)h} \|x\|_{(X_0,X_1)^{\theta,q}}$.

Hence the completeness of X_0 and X_1 implies the existence of

$$y_k = \sum_{j=k+1}^{\infty} x_j \in X_0, \quad z_k = \sum_{j=-\infty}^{k} x_j \in X_1,$$

and they satisfy the equality $x = y_k + z_k$. It follows that

$$K(x, 2^k) \leq \|y_k\|_{X_0} + \frac{1}{2^k} \|z_k\|_{X_1} ,$$

and hence

$$\|x\|_{\theta,q} \leq C \left(\sum_{k=-\infty}^{\infty} \left\{ 2^{k\theta} \left(\|y_k\|_{X_0} + \frac{1}{2^k} \|z_k\|_{X_1} \right) \right\}^q \right)^{1/q} \leq I_1 + I_2, \tag{3.4}$$

where

$$I_1 = C\left(\sum_{k=-\infty}^{\infty} 2^{kq\theta}\|y_k\|_{X_0}^q\right)^{1/q} \qquad I_2 = \left(\sum_{k=-\infty}^{\infty} 2^{kq(\theta-1)}\|z_k\|_{X_1}^q\right)^{1/q}.$$

Here we have $\|y_k\|_{X_0}^p \le C \sum_{j=k+1}^{\infty} \|x_j\|_{X_0}^p \le C \sum_{j=k+1}^{\infty} J(x_j, 2^j)^p$. It follows that

$$I_1 \le C\left(\sum_{k=-\infty}^{\infty} 2^{kq\theta}\left\{\sum_{j=k+1}^{\infty} J(x_j, 2^j)^p\right\}^{q/p}\right)^{1/q}.$$

If $p \ge q$, we have

$$\begin{aligned}
I_1 &\le C\left(\sum_{k=-\infty}^{\infty} 2^{kq\theta} \sum_{j=k+1}^{\infty} J(x_j, 2^j)^q\right)^{1/q}. \\
&= C\left(\sum_{j=-\infty}^{\infty} J(x_j, 2^j)^q \sum_{k=-\infty}^{k-1} 2^{kq\theta}\right)^{1/q} \\
&\le C\left(\sum_{j=-\infty}^{\infty} 2^{jq\theta} J(x_j, 2^j)^q\right)^{1/q} \le C\|x\|_{(X_0,X_1)^{\theta,q}}.
\end{aligned}$$

On the other hand, if $p < q$, we choose s so that $1 = 1/s + p/q$. Then Hölder's inequality yields

$$\begin{aligned}
I_1 &\le C\left(\sum_{k=-\infty}^{\infty}\left\{\sum_{j=k+1}^{\infty} 2^{(k-j)sp\theta/2}\right\}^{q/sp} \sum_{j=-\infty}^{k} \left(2^{(k+j)\theta/2} J(x_j, 2^j)\right)^q\right)^{1/q} \\
&\le C\left(\sum_{j=-\infty}^{\infty}\left\{\sum_{k=-\infty}^{k-1} 2^{(j-k)q\theta/2}\right\}\left(2^{j\theta} J(x_j, 2^j)\right)^q\right)^{1/q} \\
&\le C\left(\sum_{j=-\infty}^{\infty} 2^{jq\theta} J(x_j, 2^j)^q\right)^{1/q} \le C\|x\|_{(X_0,X_1)^{\theta,q}}.
\end{aligned}$$

Hence in both cases we have $I_1 \le C\|x\|_{(X_0,X_1)^{\theta,q}}$. In the same way we have $I_2 \le C\|x\|_{(X_0,X_1)^{\theta,q}}$. Substituting these inequalities into (3.4) we conclude that $\|x\|_{\theta,q} \le C\|x\|_{(X_0,X_1)^{\theta,q}}$. This implies the inclusion relation $(X_0, X_1)^{\theta,q} \subset (X_0, X_1)_{\theta,q}$.

We turn to the proof of the converse inclusion relation $(X_0, X_1)_{\theta,q} \subset (X_0, X_1)^{\theta,q}$. Suppose that $x \in (X_0, X_1)_{\theta,q}$. Then, for every $j \in \mathbb{Z}$, there exists $y_j \in X_0$ satisfying $z_j = x - y_j \in X_1$ such that

$$\|y_j\|_{X_0} + \frac{1}{2^j}\|z_j\|_{X_1} < 2K(x, 2^j).$$

Now put $w_j = y_{j+1} - y_j$. Then we have $w_j = z_j - z_{j+1}$, and hence we have

$$\|w_j\|_{X_0} \le C(\|y_{j+1}\|_{X_0} + \|y_j\|_{X_0}) \le C\big(K(x,2^j) + K(x,2^{j+1})\big) \le CK(x,2^j).$$

In the same way we have $\|w_j\|_{X_1} \le C2^jK(x,2^j)$. It follows that $J(w_j,2^j) \le CK(x,2^j)$ with an absolute constant C, and hence

$$\|\{w_j\}_{j=-\infty}^{\infty}\| = \left(\sum_{j=-\infty}^{\infty} \left\{2^{\theta j}J(w_j,2^j)\right\}^q\right)^{1/q} \le C\left(\sum_{j=-\infty}^{\infty} \left\{2^{\theta j}K(x,2^j)\right\}^q\right)^{1/q} = C\|x\|_{\theta,q}. \tag{3.5}$$

Next, in view of the equality

$$\sum_{j=-M}^{N} w_j = y_{N+1} - y_{-M} = z_{N+1} - z_{-M}$$

and the inequality

$$2^{\theta j}\|y_j\|_{X_0} \le 2^{\theta j+1}K(x,2^j) \le C\|x\|_{\theta,\infty} \le C\|x\|_{\theta,q}\,,$$

we obtain

$$\|y_j\|_{X_0} \le C\|x\|_{\theta,q}2^{-\theta j} \to 0 \text{ as } j \to \infty,$$

which implies that

$$\lim_{N\to\infty} y_{N+1} = 0, \lim_{N\to\infty} z_{N+1} = x \text{ in } X_0.$$

In the same way we can show

$$\lim_{M\to\infty} y_{-M} = x, \lim_{N\to-\infty} z_{-M} = 0 \text{ in } X_1.$$

From these facts and the estimate (3.5) we have $\|x\|_{(X_0,X_1)^{\theta,q}} \le C\|x\|_{\theta,q}$, and hence the inclusion relation $(X_0,X_1)_{\theta,q} \subset (X_0,X_1)^{\theta,q}$ holds. □

This theorem yields the following corollary.

Corollary 3.2. *Suppose that X_0 and X_1 are complete, and let E_j denote the closure of $X_0 \cap X_1$ in X_j for $j = 0,1$. If F_j is a closed subgroup of X_j containing E_j for $j = 0,1$, then $(F_0,F_1)_{\theta,q}$ coincides with $(X_0,X_1)_{\theta,q}$. In particular, we have*

$$(E_0,E_1)_{\theta,q} = (E_0,X_1)_{\theta,q} = (X_0,E_1)_{\theta,q} = (X_0,X_1)_{\theta,q}.$$

Proof. The inclusion relation $(E_0,E_1)_{\theta,q} \subset (F_0,F_1)_{\theta,q} \subset (X_0,X_1)_{\theta,q}$ and the equality of the quasi-norms are clear. On the other hand, it follows from the definition of $(X_0,X_1)^{\theta,q}$ the inclusion relation $(X_0,X_1)_{\theta,q} = (X_0,X_1)^{\theta,q} \subset E_0 + E_1$. Hence we have the equality $(E_0,E_1)_{\theta,q} = (X_0,X_1)_{\theta,q}$, which implies the conclusion $(F_0,F_1)_{\theta,q} = (X_0,X_1)_{\theta,q}$. □

At the end of this section we consider the situation that an operator is defined from a pair of quasi-normed abelian groups to another pair, and show that the operator maps an interpolation space of the first pair to the corresponding interpolation space of the second pair.

Let X_0, X_1, Y_0 and Y_1 be quasi-normed abelian groups, and let T be a mapping from $X_0 + X_1$ to $Y_0 + Y_1$ satisfying the following:

(i) For $j = 0, 1$, the mapping T maps X_j to Y_j, and there exist nonnegative constants A_0 and A_1 such that the estimates $\|Tx\|_{Y_j} \le A_j \|x\|_{X_j}$ hold for $x \in X_j$.

(ii) For every $x = x_0 + x_1$ such that $x_j \in X_j$, there exists $y_j \in Y_j$ such that $Tx = y_0 + y_1$ and that the estimates $\|y_j\|_{Y_j} \le A_j \|Tx_j\|_{X_j}$ hold.

A typical example is the case that Y_j are function spaces such that $|f(x)| \le |g(x)|$ for every x implies $\|f\|_{Y_j} \le \|g\|_{Y_j}$, and that the operator T satisfies the inequality

$$\left|\big(T\{f_0 + f_1\}\big)(x) - (Tf_0)(x)\right| \le \left|(Tf_1)(x)\right|$$

for every x. A typical example is the maximal function. Another example, which will be employed for the Navier-Stokes equation, is the norm $\|f(t)\|_X$ of an X-valued function of t.

Then we have the following theorem.

Theorem 3.3. *Suppose that $0 < \theta < 1$ and that $0 < q \le \infty$. Then, under these assumptions, the operator T is bounded from $X = (X_0, X_1)_{\theta,q}$ to $Y = (Y_0, Y_1)_{\theta,q}$ with*

$$\|Tx\|_Y \le A_0^{1-\theta} A_1^{\theta} \|x\|_X.$$

Proof. We first put $S = A_1/A_0$, and let ε be an arbitrary positive number.

Suppose that $x \in X$. Then, for every $t > 0$, we can choose $y(t) \in X_0$ such that $z(t) = x - y(t) \in X_1$ and that

$$\|y(t)\|_{X_0} + \frac{1}{t}\|z(t)\|_{X_1} \le (1+\varepsilon)K(x,t). \tag{3.6}$$

Then, from the assumption we can take $u(t) \in Y_0$ and $v(t) \in Y_1$ such that $Tx = u(t) + v(t)$,

$$\begin{aligned} \|u(t)\|_{Y_0} &\le \|Ty(t)\|_{Y_0} \le A_0\|y(t)\|_{X_0}, \\ \|v(t)\|_{Y_1} &\le \|Tz(t)\|_{Y_1} \le A_1\|z(t)\|_{X_1}. \end{aligned}$$

Then (3.6) yields

$$\begin{aligned} K(Tx, St) &\le \|u(t)\|_{Y_0} + \frac{1}{St}\|v(t)\|_{Y_1} \le A_0\|y(t)\|_{X_0} + \frac{A_1}{St}\|z(t)\|_{X_1} \\ &\le A_0\left(\|y(t)\|_{X_0} + \frac{1}{t}\|z(t)\|_{X_1}\right) \le (1+\varepsilon)A_0K(x,t). \end{aligned}$$

It follows that

$$
\begin{aligned}
&\|Tx\|_Y \\
&= \left(\int_0^\infty \left(s^\theta K(Tx,s)\right)^q \frac{ds}{s}\right)^{1/q} = \left(\int_0^\infty \left(S^\theta t^\theta K(Tx,St)\right)^q \frac{Sdt}{St}\right)^{1/q} \\
&\le (1+\varepsilon)A_0 S^\theta \left(\int_0^\infty \left(t^\theta K(x,t)\right)^q \frac{dt}{t}\right)^{1/q} = (1+\varepsilon)A_0^{1-\theta}A_1^\theta \|x\|_X.
\end{aligned}
$$

Since $\varepsilon > 0$ is arbitrary, we obtain the conclusion. □

4. Duality.

In this section we assume that X_0 and X_1 are Banach spaces, and we characterize the dual spaces of the real interpolation spaces. Let X_j' denote the dual space of X_j for $j = 0, 1$. We start with a preliminary lemma.

Lemma 4.1. *The mapping $X_0' \to (X_0 \cap X_1)'$ is injective if and only if $X_0 \cap X_1$ is dense in X_0.*

Proof. If the closure of $X_0 \cap X_1$ is $E \subset\neq X_0$, then there exists $\xi \in X_0'$ such that $\xi \neq 0$ and that $\xi_E = 0$. Hence the mapping $X_0' \to (X_0 \cap X_1)'$ is not injective.

On the other hand, if $X_0 \cap X_1$ is dense in X_0, then every $\xi \in (X_0 \cap X_1)'$ is bounded on X_0, and is uniquely extended to a bounded linear functional $\eta \in X_0'$. Then η is the only element in X_0' mapped to ξ. □

Lemma 4.1 implies that, if $X_0 \cap X_1$ is dense both in X_0 and in X_1, then we have $X_0' + X_1' \subset (X_0 \cap X_1)'$. The main result of this section is the following theorem.

Theorem 4.2. *Suppose that $X_0 \cap X_1$ is dense both in X_0 and in X_1. Then we have $\left((X_0, X_1)_{\theta,q}\right)' = (X_0', X_1')_{\theta,q/(q-1)}$ for $q \in [1, \infty)$, and $\left((X_0, X_1)_{\theta,\infty-}\right)' = (X_0', X_1')_{\theta,1}$.*

As we see in Section 5, we have $\left(L^{2/3}, L^2\right)_{1/2,1} = L^1$, which implies $\left(\left(L^{2/3}, L^2\right)_{1/2,1}\right)' = L^\infty$. On the other hand, since $L^{1,\infty}([0,1]) \subset L^{2/3}([0,1])$ and $\left(L^{1,\infty}([0,1])\right)' = \{0\}$, it follows that $\left(L^{2/3}([0,1])\right)' = \{0\}$. This implies $\left(\left(L^{2/3}([0,1])\right)', \left(L^2([0,1])\right)'\right)_{1/2,\infty} = \{0\}$. Hence we see that the assumption that X_0 and X_1 are Banach spaces is essential. In fact, we essentially use the Hahn-Banach theorem in the proof.

Proof of Theorem 4.2. We first show

$$
(X_0', X_1')_{\theta,q/(q-1)} \subset \left((X_0, X_1)_{\theta,q}\right)' \tag{4.1}
$$

for $q \in [1, \infty)$. To this end we first show that, for every $\xi \in X_0' + X_1'$ and every $t > 0$, we have the inequality

$$\sup_{x \in X_0 \cap X_1} \frac{\big|\langle x, \xi \rangle\big|}{J(x, 1/t, X_0, X_1)} \le K(\xi, t, X_0', X_1'). \tag{4.2}$$

Suppose that $\xi = \eta + \zeta$ such that $\eta \in X_0'$ and that $\zeta \in X_1'$. Then, for every $x \in X_0 \cap X_1$ we have

$$\begin{aligned}\big|\langle x, \xi \rangle\big| \le \big|\langle x, \eta \rangle\big| + \big|\langle x, \zeta \rangle\big| &\le \|x\|_{X_0}\|\eta\|_{X_0'} + \|x\|_{X_1}\|\zeta\|_{X_1'} \\ &\le J(x, t, X_0, X_1)\|\eta\|_{X_0'} + J(x, t, X_0, X_1)t\|\zeta\|_{X_1'}.\end{aligned}$$

Taking the infimum of the right-hand side, we obtain

$$\big|\langle x, \xi \rangle\big| \le J(x, t, X_0, X_1)K(\xi, 1/t, X_0', X_1'),$$

which implies (4.2).

Now we prove (4.1) for $q < \infty$. To this end we assume that $\xi \in (X_0', X_1')_{\theta, q/(q-1)} \subset X_0' + X_1'$ and that $\{x_j\}_{j=-M}^N$ is a finite sequence in $X_0 \cap X_1$. Put $x = \sum_{j=-M}^{N} x_j$. Then (4.2) yields

$$\begin{aligned}\big|\langle x, \xi \rangle\big| &\le \sum_{j=-M}^{N} \big|\langle x_j, \xi \rangle\big| \\ &\le \sum_{j=-M}^{N} 2^{\theta j} J(x_j, 2^j, X_0, X_1) 2^{-\theta j} K(\xi, 2^{-j}, X_0', X_1').\end{aligned}$$

Hence Hölder's inequality implies that

$$\begin{aligned}\big|\langle x, \xi \rangle\big| \le &\left(\sum_{j=-M}^{N} \big(2^{\theta j} J(x_j, 2^j, X_0, X_1)\big)^q \right)^{1/q} \\ &\left(\sum_{j=-\infty}^{\infty} \big(2^{-\theta j} K(\xi, 2^{-j}, X_0', X_1')\big)^{q/(q-1)} \right)^{1-1/q}.\end{aligned}$$

It follows that $\big|\langle x, \xi \rangle\big| \le \|x\|_{\theta,q}\|\xi\|_{\theta,q/(q-1)}$. Since $X_0 \cap X_1$ is dense in $(X_0, X_1)_{\theta,q}$, we obtain (4.1). In the same way we can show

$$(X_0', X_1')_{\theta,1} \subset \big((X_0, X_1)_{\theta,\infty-}\big)'.$$

Observe that X_0 and X_1 need not be Banach spaces in this part.

We next show that

$$(X_0', X_1')_{\theta,q/(q-1)} \supset \big((X_0, X_1)_{\theta,q}\big)' \tag{4.3}$$

for $q \in [1,\infty)$. To this end we show that, for every $\xi \in (X_0 \cap X_1)'$ and every $t > 0$, we have the inequality

$$\sup_{x \in X_0 \cap X_1} \frac{\bigl|\langle x, \xi\rangle\bigr|}{J(x, 1/t, X_0, X_1)} \geq K(\xi, t, X_0', X_1'). \tag{4.4}$$

Put $Y = X_0 \oplus X_1$ with the quasi-norm

$$\|(y,z)\|_Y = \max\left\{\|y\|_{X_0}, t\|z\|_{X_1}\right\}.$$

Then we have $Y' = X_0' \oplus X_1'$ with the quasi-norm

$$\|(\eta,\zeta)\|_{Y'} = \|\eta\|_{X_0'} + \frac{1}{t}\|\zeta\|_{X_1'},$$

and the space $X_0 \cap X_1$ can be identified with $Z = \{(x,x) \mid x \in X_0 \cap X_1\}$. This implies that every $\xi \in (X_0 \cap X_1)'$ can be regarded as the bounded linear functional on Z. Since Z is closed in Y, the Hahn-Banach theorem implies the existence of a functional $\rho \in Y'$ such that $\rho|_Z = \xi$ and that $\|\rho\|_{Y'} = \|\xi\|_{(X_0\cap X_1)'}$. Here we note that

$$\begin{aligned}\|\xi\|_{(X_0\cap X_1)'} &= \sup_{x \in X_0\cap X_1, x\neq 0} \frac{\bigl|\langle x, \xi\rangle\bigr|}{\max\left\{\|x\|_{X_0}, t\|x\|_{X_1}\right\}} \\ &= \sup_{x \in X_0\cap X_1, x\neq 0} \frac{\bigl|\langle x, \xi\rangle\bigr|}{J(x, 1/t, X_0, X_1)}.\end{aligned}$$

We next define $\rho_0 \in X_0'$ and $\rho_1 \in X_1'$ by $\rho_0(y) = \rho\bigl((y,0)\bigr)$ and $\rho_1(z) = \rho\bigl((0,z)\bigr)$. Then, for every $x \in X_0 \cap X_1$, we have

$$\rho_0(x) + \rho_1(x) = \rho\bigl((x,x)\bigr) = \xi(x).$$

Hence $\xi = \rho_0 + \rho_1 \in X_0' + X_1'$. Next, for every $\varepsilon > 0$, we can choose $y \in X_0$ and $z \in X_1$ such that $\|(y,0)\|_Y = \|(0,z)\|_Z = 1$ and that

$$\rho\bigl((y,0)\bigr) > \sup_{y\in X_0, y\neq 0} \frac{\rho\bigl((y,0)\bigr)}{\|(y,0)\|_Y} - \frac{\varepsilon}{2}, \quad \rho\bigl((0,z)\bigr) > \sup_{z\in X_1, z\neq 0} \frac{\rho\bigl((0,z)\bigr)}{\|(0,z)\|_Y} - \frac{\varepsilon}{2}.$$

It follows that $\|(y,z)\|_Y = 1$ and

$$\rho(y,z) > \frac{\rho\bigl((y,0)\bigr)}{\|(y,0)\|_Y} + \frac{\rho\bigl((0,z)\bigr)}{\|(0,z)\|_Y} - \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we have

$$\|\rho\|_{Y'} \geq \sup_{y\in X_0, y\neq 0} \frac{\rho\bigl((y,0)\bigr)}{\|(y,0)\|_Y} + \sup_{z\in X_1, z\neq 0} \frac{\rho\bigl((0,z)\bigr)}{\|(0,z)\|_Y}. \tag{4.5}$$

On the other hand, we have

$$\|\rho_0\|_{X_0'} = \sup_{y\in X_0, y\neq 0} \frac{\rho\bigl((y,0)\bigr)}{\|y\|_{X_0}} = \sup_{y\in X_0, y\neq 0} \frac{\rho\bigl((y,0)\bigr)}{\|(y,0)\|_Y}$$

and

$$\|\rho_1\|_{X_1'} = \sup_{z\in X_1, z\neq 0} \frac{\rho\bigl((0,z)\bigr)}{\|z\|_{X_1}} = t \sup_{z\in X_1, z\neq 0} \frac{\rho\bigl((0,z)\bigr)}{\|(0,z)\|_Y}$$

Substituting these estimates into (4.5) we obtain

$$\|\rho_0\|_{X_0'} + \frac{1}{t}\|\rho_1\|_{X_1'} \le \|\rho\|_{Y'} = \|\xi\|_{(X_0\cap X_1)'} = \sup_{x\in X_0\cap X_1, x\neq 0} \frac{\big|\langle x,\xi\rangle\big|}{J(x,1/t,X_0,X_1)}.$$

This implies (4.4).

Then we show that (4.3) for $q \in [1,\infty)$. Suppose that $\xi \in \big((X_0,X_1)_{\theta,q}\big)' \subset (X_0\cap X_1)'$. Let ε be an arbitrary small number. Then we see by (4.4) that, for every $j\in\mathbb{Z}$ there exists $x_j \in X_0\cap X_1$ such that

$$\frac{\langle x_j,\xi\rangle}{J(x_j,2^{-j},X_0,X_1)} \ge K(\xi,2^j,X_0,X_1) - \frac{\varepsilon}{2^{2+|j|}}.$$

Let $\{\alpha_j\}_{j=-\infty}^{\infty}$ be a sequence of positive numbers satisfying the condition $\{2^{-\theta j}\alpha_j\}_{j=-\infty}^{\infty} \in \ell^q$. Then we have

$$2^{-\theta j} J\left(\frac{\alpha_j x_j}{J(x_j,2^{-j},X_0,X_1)}, 2^{-j}, X_0, X_1\right) \le 2^{-\theta j}\alpha_j,$$

It follows that the sum

$$x = \sum_{j=-\infty}^{\infty} \frac{\alpha_j x_j}{J(x_j,2^{-j},X_0,X_1)}. \tag{4.6}$$

converges in $(X_0,X_1)_{\theta,q}$ and that

$$\|x\|_{\theta,q} \le C\left\{\sum_{j=-\infty}^{\infty}\big(2^{-j\theta}\alpha_j\big)^q\right\}^{1/q}.$$

We thus obtain

$$\big|\langle x,\xi\rangle\big| \le C\left\{\sum_{j=-\infty}^{\infty}\big(2^{-j\theta}\alpha_j\big)^q\right\}^{1/q}\|\xi\|_{\big((X_0,X_1)_{\theta,q}\big)'}. \tag{4.7}$$

Since the summation in (4.6) is convergent in $(X_0,X_1)_{\theta,q}$, we have

$$\begin{aligned}
\langle x,\xi\rangle &= \lim_{N\to\infty}\sum_{j=-N}^{N}\left\langle \frac{\alpha_j x_j}{J(x_j,2^{-j},X_0,X_1)},\xi\right\rangle \\
&\ge \lim_{N\to\infty}\sum_{j=-N}^{N} 2^{-j\theta}\alpha_j 2^{j\theta}\left(K(\xi,2^j,X_0',X_1') - \frac{\varepsilon}{2^{2+|j|}}\right) \\
&= \sum_{j=-\infty}^{\infty} 2^{-j\theta}\alpha_j 2^{j\theta} K(\xi,2^j,X_0',X_1') - \sum_{j=-\infty}^{\infty} 2^{-j\theta}\alpha_j\frac{\varepsilon}{2^{2+|j|}} \\
&\ge \sum_{j=-\infty}^{\infty} 2^{-j\theta}\alpha_j 2^{j\theta} K(\xi,2^j,X_0',X_1') - \varepsilon\big\|\{2^{-j\theta}\alpha_j\}_{j=-\infty}^{\infty}\big\|_{\ell^q}.
\end{aligned}$$

It follows from this inequality and (4.7) that

$$\begin{aligned}\sum_{j=-\infty}^{\infty} & 2^{-j\theta}\alpha_j 2^{j\theta} K(\xi, 2^j, X_0', X_1') \\ & \le C \left\{ \sum_{j=-\infty}^{\infty} \left(2^{-j\theta}\alpha_j\right)^q \right\}^{1/q} \|\xi\|_{\left((X_0,X_1)_{\theta,q}\right)'} + \varepsilon \left\| \{2^{-j\theta}\alpha_j\}_{j=-\infty}^{\infty} \right\|_{\ell^q}.\end{aligned}$$

Since the choice of the sequence $\{\alpha_j\}_{j=-\infty}^{\infty}$ of positive numbers such that $\{2^{-j\theta}\alpha_j\}_{j=-\infty}^{\infty} \in \ell^q$ is arbitrary, we see that the sequence $\{2^{j\theta}K(\xi, 2^j, X_0', X_1')\}_{j=-\infty}^{\infty}$ belongs to $\ell^{q/(q-1)}$, and we have the estimate

$$\left\| \{2^{j\theta}K(\xi, 2^j, X_0', X_1')\}_{j=-\infty}^{\infty} \right\|_{\ell^{q/(q-1)}} \le C \|\xi\|_{\left((X_0,X_1)_{\theta,q}\right)'} + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we conclude that $\xi \in (X_0', X_1')_{\theta,q/(q-1)}$ with the estimate $\|\xi\|_{(X_0',X_1')_{\theta,q/(q-1)}} \le C\|\xi\|_{\left((X_0,X_1)_{\theta,q}\right)'}$. This completes the proof of (4.3) for $1 \le q < \infty$. The inclusion relation $\left((X_0, X_1)_{\theta,\infty-}\right)' \subset (X_0', X_1')_{\theta,1}$ can be proved in the same way. □

5. The Lorentz spaces.

We begin with the introduction of two functions defined on $(0, \infty)$ associated with a measurable function $f(x)$ on a measure space (X, μ). For $t > 0$, we put

$$\mu(|f| > s) = \mu\bigl(\{x \in X \mid |f(x)| > s\}\bigr)$$

and

$$f^*(t) = \sup\{s \in (0, \infty) \mid \mu(|f| > s) > t\}.$$

Suppose that $0 < p < \infty$. Then we have

$$\|f\|_p = \left(\int_0^\infty f^*(t)^p \, dt \right)^{1/p} = \left(\int_0^\infty \left(t^{1/p} f^*(t)\right)^p \frac{dt}{t} \right)^{1/p}.$$

On the other hand, the weak-L^p space is defined as the collection of functions $f(x)$ such that

$$\sup_{s>0} s\mu(|f| > s)^{1/p} = \sup_{t>0} t^{1/p} f^*(t) < \infty.$$

Generalizing these quantities, we define $\|f\|_{p,q}$ for $p \in (0, \infty)$ and $q \in (0, \infty]$ by

$$\|f\|_{p,q} = \begin{cases} \left(\displaystyle\int_0^\infty \left(t^{1/p} f^*(t)\right)^q \frac{dt}{t} \right)^{1/q} & \text{for } q < \infty, \\ \displaystyle\sup_{t>0} t^{1/p} f^*(t) & \text{for } q = \infty, \end{cases}$$

and let $L^{p,q}(X)$ be the collection of all measurable functions on X such that $\|f\|_{p,q} < \infty$.

The function $\|f\|_{p,q}$ does not enjoy the triangle inequality in general. For example, let $f(x) = 1 - x$ and $g(x) = x$ on $[0, 1]$. Then we have $f^*(t) = g^*(t) = f(t)$. It follows that

$$\|f\|_{1,\infty} = \|g\|_{1,\infty} = \sup_{0<t<1} t(1-t) = \frac{1}{4},$$

while $f(x) + g(x) \equiv 1$ and hence $\|f + g\|_{1,\infty} = 1$. However, suppose that

$$f^*(t) < r, \quad g^*(t) < s. \tag{5.1}$$

Then we have

$$\mu(|f| > r) < t, \quad \mu(|g| > s) < t.$$

From these inequalities and the inclusion relation

$$\begin{aligned}&\{x \in X \mid |f(x) + g(x)| > r + s\}\\ &\qquad\qquad \subset \{x \in X \mid |f(x)| > r\} \cup \{x \in X \mid |g(x)| > s\}\end{aligned}$$

it follows that $\mu(|f + g| > r + s) < 2t$, which implies

$$(f + g)^*(2t) \le r + s.$$

Since r and s satisfying (5.1) are arbitrary, we obtain

$$(f + g)^*(2t) \le f^*(t) + g^*(t). \tag{5.2}$$

This implies

$$\begin{aligned}\|f + g\|_{p,q} &\le \left(\int_0^\infty \left((2t)^{1/p}(f + g)^*(2t)\right)^q \frac{dt}{t}\right)^{1/q}\\ &\le 2^{1/p}\left(\int_0^\infty \left(t^{1/p}\{f^*(t) + g^*(t)\right)^q \frac{dt}{t}\right)^{1/q}\\ &\le 2^{1/p}C_q\left(\|f\|_{p,q} + \|g\|_{p,q}\right),\end{aligned}$$

and hence $\|\cdot\|_{p,q}$ becomes a quasi-norm. It is easy to see that $(af)^*(t) = |a|f^*(t)$ for a constant a, and hence $\|\cdot\|_{p,q}$ enjoys (1.5).

If $s \le t \le 2s$, we have

$$s^{1/p}f^*(2s) \le t^{1/p}f^*(t) \le (2s)^{1/p}f^*(s).$$

This implies that $\|\cdot\|$ is equivalent to $\left(\sum_{j=-\infty}^{\infty}\left(2^{j/p}f^*(2^j)\right)^q\right)^{1/q}$. From this fact we see that $q < r$ implies $L^{p,q} \subset L^{p,r}$. At the end of this section we give a proof of a theorem on the real interpolation between the Lorentz spaces.

Theorem 5.1. *Suppose that p_0, $p_1 \in (0, \infty)$ satisfy $p_0 \ne p_1$, and suppose that $0 < r \le \infty$ and $0 < \theta < 1$. We define $p \in (0, \infty)$ by the formula $1/p = (1 - \theta)/p_0 + \theta/p_1$. Then, for every q_0, $q_1 \in (0, \infty]$, the space $X = (L^{p_0,q_0}, L^{p_1,q_1})_{\theta,r}$ coincides with $L^{p,r}$.*

Proof. Without loss of generality we may assume $p_0 > p > p_1$.

We first show the inclusion relation $X \subset L^{p,r}$. In the proof of this relation we may assume that $q_0 = q_1 = \infty$ without loss of generality. For every $m \in \mathbb{Z}$ there exist $g_m \in L^{p_0,\infty}$ and $h_m \in L^{p_1,\infty}$ such that $f = g_m + h_m$ and that

$$\|g_m\|_{p_0,\infty} + \frac{1}{2^m}\|h_m\|_{p_1,\infty} < 2K(f, 2^m).$$

We then have

$$g_m^*(2^j) \le 2^{-j/p_0}\|g_m\|_{p_0,\infty} \le 2^{-j/p_0+1}K(f, 2^m)$$

and

$$h_m^*(2^j) \le 2^{-j/p_1}\|h_m\|_{p_1,\infty} \le 2^{-j/p_1+m+1}K(f, 2^m).$$

For every $j \in \mathbb{Z}$, let $m = m(j)$ denote the largest integer m such that

$$\left(\frac{1}{p_1} - \frac{1}{p_0}\right) j \ge m.$$

Then we have

$$j - 1 < \frac{p_0 p_1 m}{p_0 - p_1} \le j.$$

It follows that

$$-\frac{j}{p_0} + 1 \le -\frac{p_1 m}{p_0 - p_1} + 1 < -\frac{j-1}{p_0} + 1$$

and

$$-\frac{j}{p_1} + m + 1 \le -\frac{p_0 m}{p_0 - p_1} + m + 1 = -\frac{p_1 m}{p_0 - p_1} + 1 < -\frac{j-1}{p_1} + m + 1.$$

The estimate (5.2) implies the inequality $f^*(2^{j+1}) \le g_m^*(2^j) + h_m^*(2^j)$. We also have

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} = \frac{1}{p_0} + \frac{\theta(p_0 - p_1)}{p_0 p_1}.$$

It follows that

$$\begin{aligned} 2^{(j+1)\theta/p} f^*(2^{j+1}) &\le 2^{\theta/p} 2^{p_0 p_1 m/p(p_0-p_1)} \left(g_m^*(2^j) + h_m^*(2^j)\right) \\ &\le C 2^{p_1 m/(p_0-p_1)+m\theta} 2^{-p_1 m/(p_0-p_1)} K(f, 2^m) \\ &= C 2^{m\theta} K(f, 2^m). \end{aligned}$$

There exists a positive number N such that, for every $n \in \mathbb{Z}$, there exists at most N number of $j \in \mathbb{Z}$ such that $m(j) = n$. We then have

$$\sum_{j=-\infty}^{\infty} \left(2^{(j+1)\theta/p} f^*(2^{j+1})\right)^r \le CN \sum_{n=-\infty}^{\infty} \left(2^{n\theta} K(f, 2^n)\right)^r.$$

This implies $\|f\|_{p,r} \le C\|f\|_X$, and completes the proof of the inclusion relation $X \subset L^{p,r}$.

We turn to the proof of the inclusion relation $L^{p,r} \subset X$. In the proof of this relation we may assume $q < r$ without loss of generality. Choose

$s \in (0,\infty)$ so that $1/r = 1/q + 1/s$. Suppose that $f \in L^{p,r}$. For an integer ℓ, we put

$$\begin{cases} g_\ell(x) = f(x), \; h_\ell(x) = 0 & \text{if } |f(x)| \le 2^m, \\ g_\ell(x) = 0, \quad\;\; h_\ell(x) = f(x) & \text{if } |f(x)| > 2^m. \end{cases}$$

Then we have

$$\big(g_\ell^*(t), h_\ell^*(t)\big) = \begin{cases} \big(2^\ell, f^*(t)\big) & t \le \mu(|f| > 2^\ell), \\ \big(f^*(t), 0\big) & t > \mu(|f| \le 2^\ell) \end{cases}$$

It follows that

$$\begin{aligned}
&\|g_\ell\|_{p_0,q} \\
&= C\left\{\int_0^\infty \big(t^{1/p_0} g_\ell^*(t)\big)^q \,\frac{dt}{t}\right\}^{1/q} \\
&= C\left\{\int_0^{\mu(|f|>2^\ell)} \big(t^{1/p_0} 2^\ell\big)^q \,\frac{dt}{t} + \int_{\mu(|f|>2^\ell)}^\infty \big(t^{1/p_0} f^*(t)\big)^q \,\frac{dt}{t}\right\} \\
&\le C\left\{2^\ell \mu(|f|>2^\ell)^{1/p_0} + \left(\int_{\mu(|f|>2^\ell)}^\infty \big(t^{1/p_0} f^*(t)\big)^q \,\frac{dt}{t}\right)^{1/q}\right\} \\
&\le C\Bigg\{2^\ell \mu(|f|>2^\ell)^{1/p_0} + \left(\int_{\mu(|f|>2^\ell)}^\infty \big(t^{1/2(1/p+1/p_0)} f^*(t)\big)^r \,\frac{dt}{t}\right)^{1/r} \times \\
&\qquad C\left(\int_{\mu(|f|>2^\ell)}^\infty t^{(1/p_0-1/p)s/2} \,\frac{dt}{t}\right)^{1/s}\Bigg\} \\
&\le C\mu(|f|>2^\ell)^{1/p_0} f^*\big(\mu(|f|>2^\ell)\big) + \\
&\qquad C\mu(|f|>2^\ell)^{1/2(1/p_0-1/p)} \left(\int_{\mu(|f|>2^\ell)}^\infty \big(t^{1/2(1/p+1/p_0)} f^*(t)\big)^r \,\frac{dt}{t}\right)^{1/r}
\end{aligned}$$

and

$$\begin{aligned}
&\|h_\ell\|_{p_1,q} \\
&= C\left\{\int_0^{\mu(|f|>2^\ell)} \big(t^{1/p_1} f^*(t)\big)^q \,\frac{dt}{t}\right\}^{1/q} \\
&\le C\left\{\int_0^{\mu(|f|>2^\ell)} \big(t^{1/2(1/p+1/p_1)} f^*(t)\big)^r \,\frac{dt}{t}\right\}^{1/r} \times \\
&\qquad \left\{\int_0^{\mu(|f|>2^\ell)} \big(t^{1/2(1/p_1-1/p)}\big)^s \,\frac{dt}{t}\right\}^{1/s} \\
&\le C\mu(|f|>2^\ell)^{1/2(1/p_1-1/p)} \left\{\int_0^{\mu(|f|>2^\ell)} \big(t^{1/2(1/p+1/p_1)} f^*(t)\big)^r \,\frac{dt}{t}\right)^{1/r}.
\end{aligned}$$

For every $k \in \mathbb{Z}$, choose $\ell = \ell(k)$ so that $2^k \leq \mu(|f| > 2^\ell)^{1/p_1 - 1/p_0} < 2^{k+1}$. Then we have

$$\frac{1}{p_0} - \frac{1}{p} = -\theta \left(\frac{1}{p_1} - \frac{1}{p_0} \right), \quad \frac{1}{p_1} - \frac{1}{p} = (1-\theta) \left(\frac{1}{p_1} - \frac{1}{p_0} \right)$$

and

$$2^{kp_0p_1/(p_0-p_1)} \leq \mu(|f| > 2^\ell) < 2^{(k+1)p_0p_1/(p_0-p_1)}.$$

It follows that

$$\sum_{k=-\infty}^{\infty} \left(2^{\theta k} \|g_\ell\|_{p_0,q} \right)^r \leq I_1 + I_2,$$

where

$$\begin{aligned} I_2 &= C \sum_{k=-\infty}^{\infty} \left(2^{k\theta} 2^{-k\theta/2} \right)^r \int_{2^{kp_0p_1/(p_0-p_1)}}^{\infty} \left(t^{1/2(1/p+1/p_0)} f^*(t) \right)^r \frac{dt}{t} \\ &\leq C \int_{-\infty}^{\infty} \sum_{k=-\infty}^{[\log_2(1/p_1 - 1/p_0)]+1} 2^{k\theta r/2} \left(t^{1/2(1/p+1/p_0)} f^*(t) \right)^r \frac{dt}{t} \\ &\leq C \int_{-\infty}^{\infty} \left(t^{1/p} f^*(t) \right)^r \frac{dt}{t} = C \|f\|_{p,r}^r. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \theta + \frac{p_1}{p_0 - p_1} &= \frac{1/p - 1/p_0}{1/p_1 - 1/p_0} + \frac{p_1}{p_0 - p_1} = \frac{p_1(p_0 - p)}{p(p_0 - p_1)} + \frac{p_1}{p_0 - p_1} \\ &= \frac{p_1}{p_0 - p_1} \left(1 + \frac{p_0 - p}{p} \right) = \frac{p_0 p_1}{p(p_0 - p_1)}. \end{aligned}$$

It follows that

$$\begin{aligned} I_1 &= C \sum_{k=-\infty}^{\infty} \left(2^{\theta k + kp_1/(p_0-p_1)} f^*(2^{kp_0p_1/(p_0-p_1)}) \right)^r \\ &= C \sum_{k=-\infty}^{\infty} \left(2^{kp_0p_1/p(p_0-p_1)} f^*(2^{kp_0p_1/(p_0-p_1)}) \right)^r \leq C \|f\|_{p,r}^r. \end{aligned}$$

Finally, we have

$$\begin{aligned} &\sum_{k=-\infty}^{\infty} \left(2^{(\theta-1)k} \|h_k\|_{p_1,q} \right)^r \\ &= \sum_{k=-\infty}^{\infty} \left(2^{(\theta-1)k + (1-\theta)k/2} \right)^r \int_0^{2^{(k+1)p_0p_1/(p_0-p_1)}} \left(t^{1/2(1/p+1/p_1)} f^*(t) \right)^r \frac{dt}{t} \\ &= \int_0^{\infty} \sum_{k=[\log_2(1/p_1 - 1/p_0)]-1}^{\infty} 2^{(\theta-1)kr/2} \left(t^{1/2(1/p+1/p_1)} f^*(t) \right)^r \frac{dt}{t} \\ &\leq C \int_0^{\infty} \left(t^{1/p} f^*(t) \right)^r \frac{dt}{t} = C \|f\|_{p,r}^r. \end{aligned}$$

We thus conclude that

$$\begin{aligned}\|f\|_X &= \left(\sum_{k=-\infty}^{\infty}\left(2^{k\theta}K(f,2^k)\right)^r\right)^{1/r}\\ &\le \left(\sum_{k=-\infty}^{\infty}\left(2^{k\theta}\left(\|g_\ell|_{p_0,q}+2^{-k}\|h_\ell\|_{p_1,q}\right)\right)^r\right)^{1/r} \le C\|f\|_{p,r}.\end{aligned}$$

This completes the proof of the inclusion relation $L^{p,r}\subset X$. □

From Theorem 5.1 we have several important facts. First, if $1<p<\infty$ and $1\le q\le\infty$, the space $L^{p,q}$ becomes Banach spaces since we can write

$$L^{p,q}=(L^{p_0,p_0},L^{p_1,p_1})_{\theta,q}=(L^{p_0},L^{p_1})_{\theta,q}$$

with p_0, $p_1\in(1,\infty)$. Moreover, Theorem 4.2 implies that

$$\begin{aligned}(L^{p,q})' &= \left((L^{p_0},L^{p_1})_{\theta,q}\right)'\\ &= \left(L^{p_0/(p_0-1)},L^{p_1/(p_1-1)}\right)_{\theta,q/(q-1)} = L^{p/(p-1),q/(q-1)}.\end{aligned}$$

for $1<p<\infty$, $1\le q<\infty$. In the same way we can prove $(L^{p,\infty-})'=L^{p/(p-1),1}$, where $L^{p,\infty-}$ is the closure of L^p in $L^{p,\infty}$.

On the other hand, consider the Lorentz spaces on a domain of $\mathbb{R}^n$ with respect to the Lebesgue measure. Then the space L^1 is a proper subspace of $L^{1,q}$ for every $q\in(1,\infty]$. From this fact and $(L^1)'=L^\infty$, together with the property of the Lebesgue measure, we can show $(L^{1,q})'=\{0\}$. Hence the Hahn-Banach theorem implies that $L^{1,q}$ cannot be a Banach space, and this space cannot be contained in the space of distributions. It follows that these spaces cannot be applied to the study of partial differential equations directly.

We also add a remark that we can prove the interpolation relation $(L^\infty,L^{p,q})_{\theta,r}=L^{p/\theta,r}$ in the same way as in Theorem 5.1.

6. Applications the the Navier-Stokes exterior problem.

In this section we consider the following non-stationary Navier-Stokes equation on n-dimensional exterior domain Ω with smooth boundary, where $n\ge3$ and $\nabla\cdot F$ is a time-dependent external force with $F=F(t,x)$:

$$\frac{\partial u}{\partial t}=\Delta u-(u\cdot\nabla)u-\nabla\pi+\nabla\cdot F \quad \text{in } \mathbb{R}\times\Omega, \tag{6.1}$$

$$\nabla\cdot u=0 \quad \text{in } \mathbb{R}\times\Omega, \tag{6.2}$$

$$u=0 \quad \text{on } \mathbb{R}\times\partial\Omega. \tag{6.3}$$

Here u denotes a vector-valued unknown function standing for the velocity of the fluid, and π denotes another scalar-valued unknown function standing for the pressure.

The purpose of this section is to prove the following theorem.

Theorem 6.1. *Suppose that* F *is small in* $BUC\left(\mathbb{R}, \left(L^{n/2,\infty}(\Omega)\right)^{n^2}\right)$. *Then there uniquely exists a solution* u *of the system* (6.1)–(6.3) *small in* $BUC\left(\mathbb{R}, \left(L^{n,\infty}(\Omega)\right)^{n}\right)$. *Furthermore, the mapping from* F *to* u *is continuous in the function spaces above.*

This theorem implies that, if F is a time-periodic function, then u is also time-periodic with the same period. We also see that, if F is almost periodic with respect to t, then so is u in view of the continuity of the mapping.

In order to remove the function π, we introduce the Helmholtz decomposition. Suppose that Ω is one of the following:

- The whole space $\mathbb{R}^n$.
- The half space $\mathbb{R}^n_+ = \{(x_1, \dots, x_{n-1}, x_n) \mid x_n > 0\}$.
- A bounded domain with smooth boundary.
- An exterior domain with smooth boundary.

Then, for every $p \in (1, \infty)$, the space $(L^p(\Omega))^n$ admits a direct sum decomposition $L^p_\sigma(\Omega) \oplus G^p(\Omega)$, where

$$L^p_\sigma(\Omega) = \{\, u(x) \in (L^p(\Omega))^n \mid \nabla \cdot u(x) \equiv 0 \text{ in } \Omega,\ \nu \cdot u(x) \equiv 0 \text{ on } \partial\Omega\}$$

and

$$G^p(\Omega) = \{\, u(x) \in (L^p(\Omega))^n \mid u(x) = \nabla f(x) \text{ with some } f(x) \in L^p_{\mathrm{loc}}(\Omega)\}$$

Observe that $\nabla u \in L^p$ implies that u is "partly" contained in H^1_p, and hence we can define the normal trace in the Besov space $B^{-1/p}_{p,p}$. On the other hand, the function $f(x)$ in the definition of $G^p(\Omega)$ is in the homogeneous Sobolev space $\dot{H}^1_p$, and hence its trace is in $B^{1-1/p}_{p,p}$. It follows that the element of $G^p(\Omega)$ has the tangential trace in $B^{-1/p}_{p,p}$.

Let P_p denote the projection from $\left(L^p(\Omega)\right)^n$ onto $L^p_\sigma(\Omega)$ according to the direct sum decomposition above. Then P_p is a bounded linear operator, and we have the identity $P_{p_0} = P_{p_1}$ on $\left(L^{p_0}(\Omega) \cap L^{p_1}(\Omega)\right)^n$. Hence we can define the bounded projection P from the Lorentz spaces $\left(L^{p,r}(\Omega)\right)^n = \left(\left(L^{p_0}(\Omega)\right)^n, \left(L^{p_1}(\Omega)\right)^n\right)_{\theta,r}$ onto $L^{p,r}_\sigma(\Omega) = \left(L^{p_0}_\sigma(\Omega), L^{p_1}_\sigma(\Omega)\right)_{\theta,r}$, where the numbers p_0, p_1, p, r and θ are the same as in Theorem 5.1. Next, for $p \in (1, \infty)$ we see that $(u, v) = 0$ for $u \in L^p_\sigma(\Omega)$ and $v \in G^{p/(p-1)}(\Omega)$, by approximating u by test functions and integrating by parts. This equality implies that $\left(L^p_\sigma(\Omega)\right)' = L^{p/(p-1)}_\sigma(\Omega)$. From this we conclude that $\left(L^{p,q}_\sigma(\Omega)\right)' = L^{p/(p-1),q/(q-1)}_\sigma(\Omega)$ for $1 < p < \infty$ and $1 \le q < \infty$, and that $\left(L^{p,\infty-}_\sigma(\Omega)\right)' = L^{p/(p-1),1}_\sigma(\Omega)$ for $1 < p < \infty$.

The composite operator $-P\Delta$ is called the Stokes operator. It is known that the Stokes operator is a sectorial operator and generates a bounded semigroup on $L^p_\sigma(\Omega)$, and hence on $L^{p,r}_\sigma(\Omega)$.

By applying the projection P we transform the system (6.1)–(6.3) into the following evolution equation

$$\frac{du}{dt} = -Au + P\big[-(u\cdot\nabla)u + \nabla\cdot F\big]. \tag{6.4}$$

Introducing the mapping T defined by

$$T[u](t) = \int_0^\infty \exp(-sA)P\left[-\big(u(t-s)\cdot\nabla\big)u(t-s) + \nabla\cdot F(t-s)\right]\,ds,$$

we can rewrite (6.4) into $T[u] = u$. Hence, in order to prove Theorem 6.1, it suffices to find the unique fixed point $u(t)$ of T small in $\mathcal{X} = BUC\big(\mathbb{R}, L^{n,\infty}_\sigma(\Omega)\big)$ for given F small in $\mathcal{Y} = BUC\left(\mathbb{R}, \big(L^{n/2,\infty}(\Omega)\big)^{n^2}\right)$.

The main difficulty is that the improper integral in the right-hand side does not exist in the sense of the Bochner integral in general. Instead we show that the improper integral exists in the weak-$*$ topology. Namely, for $F\in\mathcal{Y}$, we define $U[F]$ by the identity

$$\langle U[F](t),\varphi\rangle = \int_0^\infty \big\langle F(t-s), \nabla\exp(-sA)\varphi\big\rangle\,ds$$

for $\varphi\in L^{n/(n-1),1}_\sigma(\Omega)$ in view of the identity $\operatorname{div} u = 0$, and put $T[u] = U[u\otimes u - F]$. Then we have the following lemma.

Lemma 6.2. *There exists a positive constant C such that the estimate*

$$\int_0^\infty \|\nabla\exp(-sA)\varphi\|_{n/(n-2),1}\,ds \le C\|\varphi\|_{n/(n-1),1},$$

holds for every $\varphi\in L^{n/(n-1),1}_\sigma(\Omega)$.

Proof. We observe the estimate

$$\|A^{1/2}\exp(-sA)\varphi\|_q \le Cs^{n/2(1/q-1/p)-1/2}\|\varphi\|_p$$

for p, q such that $1<p\le q<\infty$. Applying real interpolation

$$\|A^{1/2}\exp(-sA)\varphi\|_{q,1} \le Cs^{n/2(1/q-1/p)-1/2}\|\varphi\|_{p,1}. \tag{6.5}$$

We next observe that, it follows from the estimate

$$\|A^{1/2}u\|_{p,q} \le C\|\nabla u\|_{p,q} \text{ for } 1<p<\infty, 1\le q\le\infty$$

and the fact that the the function $u(x)$ satisfying $\nabla u(x)\in L^{p,q}$ with either $1<p<n$, $1\le q\le\infty$ or $p=n$, $q=1$ decays as $|x|\to\infty$ that the converse estimate

$$\|\nabla u\|_{p,q} \le C\|A^{1/2}u\|_{p,q} \text{ for } 1<p<n, 1\le q\le\infty \text{ or } p=n, q=1 \tag{6.6}$$

holds. Note that this estimate fails for $p=q=n$.

Applying (6.6) with $p=n/(n-2)$, $q=1$ to the left-hand side of (6.5) with $q=n/(n-2)$, we obtain

$$\|\nabla\exp(-sA)\varphi\|_{n/(n-2),1} \le Cs^{(n-3)/2-n/2p}\|\varphi\|_{p,1} \text{ for } 1<p\le n/(n-2). \tag{6.7}$$

Note that, in the case $n = 3$, the above argument holds for $p = n/(n-2) = n$ and $q = 1$.

Now we consider the mapping S from φ to the function of $s \in (0,\infty)$ defined by $\|\nabla \exp(-sA)\varphi\|_{n/(n-2),1}$. Then (6.7) implies that S is bounded from $L^{p,1}_\sigma(\Omega)$ to $L^{2p/(3p+n-np),\infty}(0,\infty)$ for $1 < p \le n/(n-2)$. Applying real interpolation once again, we see that S is bounded from

$$L^{n/(n-1),1}_\sigma(\Omega) = \left(L^{2n/(2n-1),1}_\sigma(\Omega), L^{2n/(2n-3),1}_\sigma(\Omega)\right)_{1/2,1}$$

into

$$L^1(0,\infty) = L^{1,1}(0,\infty) = \left(L^{4/5,\infty}(0,\infty), L^{4/3,\infty}(0,\infty)\right)_{1/2.1},$$

which implies the conclusion. □

This lemma immediately yields the following corollary.

Corollary 6.3. *If $F \in \mathcal{Y}$, then we have $U[F] \in \mathcal{X}$, and we have the estimate $\|U[F]\|_{\mathcal{X}} \le C\|F\|_{\mathcal{Y}}$ with a constant C.*

Proof. Lemma 6.2 yields the estimate

$$\begin{aligned}\left|\langle U[F](t),\varphi\rangle\right| &\le \int_0^\infty \left|\left\langle F(t-s), \nabla \exp(-sA)\varphi\right\rangle\right| \, ds \\ &\le C\int_0^\infty \|F(t-s)\|_{n/2,\infty}\|\nabla \exp(-sA)\varphi\rangle\|_{n/(n-2),1}\, ds \\ &\le C\|F\|_{\mathcal{Y}}\|\varphi\|_{n/(n-1),1}\end{aligned}$$

for $F \in \mathcal{Y}$. Hence the duality $\left(L^{n/(n-1),1}_\sigma\right)' = L^{n,\infty}_\sigma$ implies $\|U[F](t)\|_{n,\infty} \le C\|F\|_{\mathcal{Y}}$. Next, for every $\varepsilon > 0$, choose $\delta > 0$ such that $|t-\tau| < \delta$ implies $\|F(t) - F(\tau)\|_{n/2,\infty} < \varepsilon/C$. Next, fix a number $\theta \in (-\delta,\delta)$, and put $G(t) = F(t+\theta) - F(t)$. Then we have

$$\begin{aligned}\|U[F](t+\theta) - U[F](t)\|_{n,\infty} &= \|U[G](t)\|_{n,\infty} \\ &\le C\|G\|_{\mathcal{Y}} = C\sup_{s\in\mathbb{R}}\|F(s+\theta) - F(s)\|_{n/2,\infty} < \epsilon\end{aligned}$$

for every $t \in \mathbb{R}$. Since $\varepsilon > 0$ is arbitrary, we have $U[F](t) \in \mathcal{X}$ and $\|U[F]\|_{\mathcal{X}} \le C\|F\|_{\mathcal{Y}}$. □

For $u, v \in \mathcal{X}$ the generalized Hölder inequality implies $u\otimes u$, $v\otimes v \in \mathcal{Y}$ and the estimate

$$\begin{aligned}\|u\otimes u - v\otimes v\|_{\mathcal{Y}} &\le \|u\otimes(u-v)\|_{\mathcal{Y}} + \|(u-v)\otimes v\|_{\mathcal{Y}} \\ &\le A(\|u\|_{\mathcal{X}} + \|v\|_{\mathcal{X}})\|u-v\|_{\mathcal{X}}\end{aligned}$$

with a constant A. Hence Corollary 6.3 yields

$$\begin{aligned}&\|U[u\otimes u - F - v\otimes v + G]\|_{\mathcal{X}} \\ &\qquad \le AC(\|u\|_{\mathcal{X}} + \|v\|_{\mathcal{X}})\|u-v\|_{\mathcal{X}} + C\|F-G\|_{\mathcal{X}}.\end{aligned} \tag{6.8}$$

In particular, putting $v = G = 0$, we have

$$\|T[u]\|_{\mathcal{X}} \leq AC\|u\|_{\mathcal{X}}^2 + C\|F\|_{\mathcal{X}}, \tag{6.9}$$

and putting $F = G$, we have

$$\|T[u] - T[v]\|_{\mathcal{X}} \leq AC(\|u\|_{\mathcal{X}} + \|v\|_{\mathcal{X}})\|u - v\|_{\mathcal{X}}. \tag{6.10}$$

Suppose that $\|F\|_{\mathcal{Y}} < 1/4C^2A$. Then the equation $ACX^2 + C\|F\|_{\mathcal{Y}} = X$ has two distinct positive solutions. Let α denote the smaller one: namely,

$$\alpha = \frac{1 - \sqrt{1 - 4AC^2\|F\|_{\mathcal{Y}}}}{2AC}.$$

Then, if $\|u\|_{\mathcal{X}} \leq \alpha$, the estimate (6.9) implies

$$\|T[u]\|_{\mathcal{X}} \leq AC\alpha^2 + C\|F\|_{\mathcal{Y}} = \alpha.$$

Hence the mapping T maps $\overline{B}_{\mathcal{X}}(0, \alpha)$ into itself.

Next, for u, $v \in \overline{B}_{\mathcal{X}}(0, \alpha)$, the estimate (6.10) and the fact

$$AC(\|u\|_{\mathcal{X}} + \|v\|_{\mathcal{X}}) \leq 2AC\alpha = 1 - \sqrt{1 - 4AC^2\|F\|_{\mathcal{Y}}} < 1 \tag{6.11}$$

implies that the mapping T is a contraction mapping from $\overline{B}_{\mathcal{X}}(0, \alpha)$ into itself, which is a closed ball of a Banach space. Hence T has a unique fixed point in $\overline{B}_{\mathcal{X}}(0, \alpha)$, which is the required solution.

Finally we prove the continuity of the mapping $F \mapsto u$. Suppose that F, $G \in \mathcal{Y}$, and let u, $v \in \mathcal{X}$ denote the solutions corresponding to F and G respectively. Then we have $u = U[u \otimes u - F]$ and $v = U[v \otimes v - G]$. It follows that $u - v = U[u \otimes u - F - v \otimes v + G]$. Substituting this into (6.8) we see

$$\big(1 - AC(\|u\|_{\mathcal{X}} + \|v\|_{\mathcal{X}})\big)\|u - v\|_{\mathcal{X}} \leq \|F - G\|_{\mathcal{Y}}.$$

In view of (6.11) we obtain the required continuity. □

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