The Kalton-Weis Theorem in Lebesgue spaces

Lorenz v. Below

TECHNISCHE UNIVERSITÄT DARMSTADT

Functional calculus

The Kalton-Weis Theorem

Proof

Sectorial operators

Definition

An operator $A\colon D(A) \subset X \to X$ is *sectorial of type* σ if

- A is injective, densely defined, has dense range,
- $\sigma(A) \subset \overline{\Sigma}_\sigma := \operatorname{clos}\{z \in \mathbb{C} \setminus \{0\}\colon |\arg z| < \sigma\}$,
- $\|\lambda R(\lambda, A)\|_{\mathcal{L}(X)} \leq C$ for $\lambda \in \mathbb{C} \setminus \overline{\Sigma}_\sigma$.

Let $\omega(A) = \inf\{\sigma \in (0, \pi)\colon A \text{ is sectorial of type } \sigma\}$.

Spaces of holomorphic functions

Let $\mathcal{A} \subset \mathcal{L}(X)$ denote a strongly closed algebra of bounded operators.

Definition

- $H^\infty(\Sigma_\sigma)$ – the space of bounded holomorphic functions $f\colon \Sigma_\sigma \to \mathbb{C}$
- $H^\infty(\Sigma_\sigma;\mathcal{A})$ – the space of bounded holomorphic functions $F\colon \Sigma_\sigma \to \mathcal{A}$
- $H_0^\infty(\Sigma_\sigma)$ – the functions $f \in H^\infty(\Sigma_\sigma)$ with

$$|f(z)| \leq C \left(\frac{|z|}{1+|z|^2} \right)^\varepsilon$$

  for all $z \in \Sigma_\sigma$.
- $H_0^\infty(\Sigma_\sigma;\mathcal{A})$ analogously
- analogously for functions on $\Sigma_{\sigma_1} \times \cdots \times \Sigma_{\sigma_m}$

Definition

For a sectorial operator A and $F \in H_0^\infty(\Sigma_\sigma; \mathcal{A})$ define

$$F(A)u = \frac{1}{2\pi i} \int_{\Gamma_\nu} F(z)R(z, A)u \, dz$$

where $\omega(A) < \nu < \pi$, $\Gamma_\nu = \partial\Sigma_\nu$ and A commutes with the elements of $\mathcal{A}$.
Analogously for sectorial operators $A_1, \dots, A_m$ with commuting resolvents and $F \in H_0^\infty(\Sigma_{\sigma_1} \times \cdots \times \Sigma_{\sigma_m}; \mathcal{A})$ define

$$F(A_1, \dots, A_m)u = \left(\frac{1}{2\pi i}\right)^m \int_{\Gamma_{\nu_1}} \cdots \int_{\Gamma_{\nu_m}} F(z)R(z_1, A) \cdots R(z_m, A)u \, dz_m \cdots dz_1.$$

H^∞-calculus

Definition

Let A be a sectorial operator on X. We say that A has a bounded $H^\infty(\Sigma_\sigma)$-calculus if

$$H_0^\infty(\Sigma_\sigma) \to \mathcal{L}(X), \qquad f \mapsto f(A)$$

extends to a bounded operator on $H^\infty(\Sigma_\sigma)$, i. e. if

$$\|f(A)\|_{\mathcal{L}(X)} \leq C \, \|f\|_{H^\infty(\Sigma_\sigma)}$$

holds for all $f \in H_0^\infty(\Sigma_\sigma)$. Let

$$\omega_{H^\infty}(A) = \inf\{\sigma \in (0,\pi) \colon A \text{ has a bounded } H^\infty(\Sigma_\sigma)\text{-calculus}\}.$$

Joint H^∞-calculus

Definition

We say that sectorial operators $A_1, \dots, A_m$ with commuting resolvents have a joint bounded H^∞-calculus if

$$\left\| f(A_1, \dots, A_m) \right\|_{\mathcal{L}(X)} \leq C \left\| f \right\|_{H^\infty(\Sigma_{\sigma_1} \times \cdots \times \Sigma_{\sigma_m})}$$

holds for all $f \in H_0^\infty(\Sigma_{\sigma_1} \times \cdots \times \Sigma_{\sigma_m})$.

The Kalton-Weis Theorem

TECHNISCHE UNIVERSITÄT DARMSTADT

Let Ω be a σ-finite measure space and $1 < p < \infty$.

Theorem (Kalton-Weis 2001, L^p version)

Let $A_1, \dots, A_m$ be sectorial operators on $L^p(\Omega)$. Then $A_1, \dots, A_m$ have a joint bounded H^∞-calculus if and only if $A_1, \dots, A_m$ each have a bounded H^∞-calculus.

The Kalton-Weis Theorem

TECHNISCHE UNIVERSITÄT DARMSTADT

Let Ω be a σ-finite measure space and $1 < p < \infty$.

Theorem (Kalton-Weis 2001, L^p version)

Let $A_1, \ldots, A_m$ be sectorial operators on $L^p(\Omega)$. Then $A_1, \ldots, A_m$ have a joint bounded H^∞-calculus if and only if $A_1, \ldots, A_m$ each have a bounded H^∞-calculus.

Kalton-Weis:

- General Banach space with certain geometric properties, uses $\mathcal{R}$-boundedness, relies heavily on geometric assumptions

Now:

- L^p-spaces, no $\mathcal{R}$-boundedness, instead square function estimates, elementary

Square function estimates

TECHNISCHE UNIVERSITÄT DARMSTADT

Definition

We say that $\mathcal{T} \subset \mathcal{L}(L^p(\Omega))$ admits square function estimates if for every $n \in \mathbb{N}$, $T_1, \dots, T_n \in \mathcal{T}$ and $u_1, \dots, u_n \in L^p(\Omega)$ the estimate

$$\left\| \left(\sum_{j=1}^{n} |T_j u_j|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\Omega)} \leq C \left\| \left(\sum_{j=1}^{n} |u_j|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\Omega)}$$

holds. Denote the smallest such $C > 0$ with $\mathcal{R}(\mathcal{T})$.

Theorem

Assume

- *A is sectorial on $L^p(\Omega)$ with bounded $H^\infty(\Sigma_\sigma)$-calculus*
- *$\mathcal{A} \subset \mathcal{L}(L^p(\Omega))$ is a strongly closed algebra of operators commuting with A in the resolvent sense*
- *$\mathcal{F} \subset H^\infty(\Sigma_\sigma; \mathcal{A})$ such that $\{F(z) \colon z \in \Sigma_\sigma, F \in \mathcal{F}\}$ admits square function estimates.*

Then $F(A)$ is bounded for $F \in \mathcal{F}$ and $\{F(A) \colon F \in \mathcal{F}\}$ admits square function estimates with

$$\mathcal{R}(\{F(A) \colon F \in \mathcal{F}\}) \le C\mathcal{R}\left(\{F(z) \colon z \in \Sigma_\sigma, F \in \mathcal{F}\}\right).$$

The Kalton-Weis Theorem – Proof

TECHNISCHE UNIVERSITÄT DARMSTADT

For $f \in H_0^\infty(\Sigma_\sigma)$ we have

$$\begin{aligned}
\|f(A_1, \dots, A_m)\| &\le C\mathcal{R}(\{f(z_1, A_2, \dots, A_m) \colon z_1 \in \Sigma_{\sigma_1}\}) \\
&\le C\mathcal{R}(\{f(z_1, z_2, A_3, \dots, A_m) \colon z_j \in \Sigma_{\sigma_j} \text{ for } j = 1, 2\}) \\
&\vdots \\
&\le C\mathcal{R}(\{f(z_1, \dots, z_m) \colon z_j \in \Sigma_{\sigma_j} \text{ for } j = 1, \dots, m\}) \\
&\le C \|f\|_{H^\infty} .
\end{aligned}$$

An equivalent description of $F(A)$

TECHNISCHE UNIVERSITÄT DARMSTADT

Proposition

Let A have a bounded $H^\infty(\Sigma_\sigma)$-calculus and $F \in H_0^\infty(\Sigma_\sigma; \mathcal{A})$ for $\omega_{H^\infty}(A) < \sigma < \pi$. Then

$$F(A)u = \int_1^2 M(t)u \frac{dt}{t}$$

where

$$M(t)u = \frac{1}{2\pi i} \sum_\pm \sum_{k\in\mathbb{Z}} e^{\pm i(1-s)\nu} F(2^{-k}t^{-1}e^{\pm i\nu}) h_s^{\pm\nu}(2^k tA)^2 u$$

for $\omega_{H^\infty}(A) < \nu < \sigma$, $0 < s < 1$ and $u \in L^p(\Omega)$ with

$$h_s^\rho(z) = z^{s/2}(e^{i\rho} - z)^{-1/2}$$

and for all $(\alpha_k)_{k\in\mathbb{Z}} \in c_{00}(\mathbb{Z})$ and $t > 0$ we have

$$\left\| \sum_{k\in\mathbb{Z}} \alpha_k h_s^\rho(2^k tA) \right\|_{\mathcal{L}(L^p(\Omega))} \leq C \max_{k\in\mathbb{Z}} |\alpha_k| .$$

Square function estimates

Proposition

Let $(V_k)_{k\in\mathbb{N}} \subset \mathcal{L}(L^p(\Omega))$ *such that for all* $(\alpha_k)_{k\in\mathbb{N}} \in c_{00}(\mathbb{N})$ *we have*

$$\left\| \sum_{k\in\mathbb{N}} \alpha_k V_k \right\|_{\mathcal{L}(L^p(\Omega))} \leq K \max_{k\in\mathbb{N}} |\alpha_k| \,.$$

Then the following estimates hold.

$$\left\| \left(\sum_{j,k=1}^n |V_k u_j|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\Omega)} \leq K \left\| \left(\sum_{j=1}^n |u_j|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\Omega)}$$

$$\left\| \left(\sum_{j=1}^n \left| \sum_{k=1}^n V_k u_{jk} \right|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\Omega)} \leq K \left\| \left(\sum_{j,k=1}^n |u_{jk}|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\Omega)} .$$

Proof.

1. $\left\|\sum_{k=1}^{n} V_k u_k\right\|_{L^p(\Omega)} \leq K \left\|\left(\sum_{k=1}^{n} |u_k|^2\right)^{\frac{1}{2}}\right\|_{L^p(\Omega)}$
2. $\left\|\left(\sum_{k=1}^{n} |V_k u|^2\right)^{\frac{1}{2}}\right\|_{L^p(\Omega)} \leq K \|u\|_{L^p(\Omega)}$
3. $\left\|\left(\sum_{j,k=1}^{n} |V_k u_j|^2\right)^{\frac{1}{2}}\right\|_{L^p(\Omega)} \leq K \left\|\left(\sum_{j=1}^{n} |u_j|^2\right)^{\frac{1}{2}}\right\|_{L^p(\Omega)}$
4. $\left\|\left(\sum_{j=1}^{n} \left|\sum_{k=1}^{n} V_k u_{jk}\right|^2\right)^{\frac{1}{2}}\right\|_{L^p(\Omega)} \leq K \left\|\left(\sum_{j,k=1}^{n} |u_{jk}|^2\right)^{\frac{1}{2}}\right\|_{L^p(\Omega)}$

□

Square function estimates

Proposition

Let $(V_k)_{k\in\mathbb{N}} \subset \mathcal{L}(L^p(\Omega))$ *such that*

$$\left\| \sum_{k\in\mathbb{N}} \alpha_k V_k \right\|_{\mathcal{L}(L^p(\Omega))} \leq K \max_{k\in\mathbb{N}} |\alpha_k|$$

holds for all $(\alpha_k)_{k\in\mathbb{N}} \in c_{00}(\mathbb{N})$. *Let* $\mathcal{T} \subset \mathcal{L}(L^p(\Omega))$ *admit square function estimates. Then*

$$\mathcal{F} = \left\{ \sum_{k=1}^{n} \alpha_k V_k T_k V_k : n \in \mathbb{N}, \alpha_k \in \mathbb{C} \text{ with } |\alpha_k| \leq 1, T_k \in \mathcal{T} \text{ for } k = 1, \ldots, n \right\}$$

admits square function estimates with $\mathcal{R}(\mathcal{F}) \leq \mathcal{R}(\mathcal{T}) K^2$.

Proof

$$\left\| \left(\sum_n \left| \sum_k \alpha_{kn} V_k T_{kn} V_k u_n \right|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\Omega)} \leq K \left\| \left(\sum_{k,n} |\alpha_{kn} T_{kn} V_k u_n|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\Omega)}$$

$$\leq \mathcal{R}(\mathcal{T}) K \left\| \left(\sum_{k,n} |V_k u_n|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\Omega)}$$

$$\leq \mathcal{R}(\mathcal{T}) K^2 \left\| \left(\sum_n |u_n|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\Omega)} .$$

□

Square functions and H^∞-calculus – Proof

TECHNISCHE UNIVERSITÄT DARMSTADT

For $\mathcal{F} \subset H_0^\infty(\Sigma_\sigma; \mathcal{A})$:

- $F(A)u = \int_1^2 M(F)(t)u \frac{\mathrm{d}t}{t}$ with $M(F)(t)u = \sum_{k\in\mathbb{Z}} F(2^{-k}t^{-1}e^{\pm i\nu})V_k(t)^2 u$.
- $\mathcal{R}(\{M(F)(t) \colon F \in \mathcal{F}\}) \leq C\mathcal{R}(\mathcal{F})$
- $\mathcal{R}(\{F(A) \colon F \in \mathcal{F}\}) \leq C\mathcal{R}(\mathcal{F})$

General case $\mathcal{F} \subset H^\infty(\Sigma_\sigma; \mathcal{A})$ via approximation. □

Applications

- Theorems of Dore-Venni type
- H^∞ implies maximal regularity if $\omega_{H^\infty} < \frac{\pi}{2}$
- $\mathcal{R}$-sectoriality implies maximal regularity if $\omega_{H^\infty} < \frac{\pi}{2}$
- Mixed Derivative Theorem
- ...

Thank you for your attention!