The spin-coating process and two-phase flow for generalized Newtonian fluids

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two fluids occupying the whole space $\mathbb{R}^n = \Omega_+(t) \cup \Omega_-(t) \cup \Gamma(t)$

separated by a sharp interface $\Gamma(t) = \{(x, y) \in \mathbb{R}^n : y = h(t, x)\}$
Model equations

\[ \begin{array}{rcl}
\rho (\partial_t u + u \cdot \nabla u) - 2 \text{div} \mu(|Du|^2)Du + \nabla \pi &=& -\rho \gamma_a e_{n+1} \quad \text{in } \Omega(t) \\
\text{div} u &=& 0 \quad \text{in } \Omega(t)
\end{array} \]

- \(u\) velocity
- \(Du\) deformation tensor
- \(\pi\) pressure
- \(\rho\) density
- \(\mu\) viscosity function
- \(\gamma_a\) gravity
Model equations

\[\begin{align*}
\rho (\partial_t u + u \cdot \nabla u) - 2 \text{div} \mu (|Du|^2) Du + \nabla \pi &= -\rho \gamma_a e_{n+1} \quad \text{in } \Omega(t) \\
\text{div } u &= 0 \quad \text{in } \Omega(t) \\
- [2\mu (|Du|^2) Du - \pi] \nu &= \sigma \kappa \nu \quad \text{on } \Gamma(t) \\
[u] &= 0 \quad \text{on } \Gamma(t)
\end{align*}\]

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- \text{V normal velocity of \Gamma(t)}
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$$u(0) = u_0 \quad \text{in } \mathbb{R}^{n+1} \setminus \Gamma_0$$

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Nonlinear diffusion term

- introduce quasilinear operator $\mathcal{A}$

\[ \mathcal{A}(u)v = \sum_{j,l=1}^{n} A^{j,l}(Du) \partial_k \partial_l v \]

- $A^{i,l}_{i,k}(X) = \mu(|X|^2) \delta_{i,k} \delta_{i,j} + \mu'(|X|^2) X_{ik} X_{jl}$

- then $\mathcal{A}(u)u = -2 \text{div} \mu(|Du|^2) Du$ and $\mathcal{A}(0)u = -\mu(0) \Delta u$
Main result (two-phase flow)

Assumptions:

- $n + 2 < \rho < \infty$
- $\mu \in C^3$ with $\mu(0) > 0$
- $u_0 \in W^{2-2/\rho}_\rho, h_0 \in W^{3-2/\rho}_\rho$ satisfying suitable compatibility conditions

For all $T > 0$ there exists $\varepsilon > 0$ such that for

$$\| u_0 \|_{W^{2-2/\rho}_\rho} + \| h_0 \|_{W^{3-2/\rho}_\rho} < \varepsilon,$$
Main result (two-phase flow)

Assumptions:

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there exists a unique solution $(u, \pi, h)$ on $(0, T)$ in

$$u \in H_p^1(L_p) \cap L_p(H_p^2),$$

$$\pi \in \{q \in L_p(\hat{H}_p^1) : \|q\| \in W_p^{1/2-1/2p}(L_p) \cap L_p(W_p^{1-1/p})\},$$

$$h \in W_p^{2-1/2p}(L_p) \cap W_p^1(W_p^{2-1/p}) \cap L_p(W_p^{3-1/p}).$$
Setting for the spin-coating problem

- a fluid is applied to the center of a spinning plate and is assumed to occupy a domain $\Omega(t)$ close to a layer
- the bottom boundary is a fixed plane $\Gamma_{-}$
- the upper boundary is a free surface initially close to a plane $\Gamma_{+}(t) = \{(x, y) \in \mathbb{R}^n : y = \delta + h(t, x)\}$, $\delta > 0$
Model equations

- model equations for $u = (v, w)$, $n = 3$

\[
\begin{align*}
\rho(\partial_t u + u \cdot \nabla u) + A(u)u + \nabla \pi &= -\rho(2\omega \times u + \omega \times (\omega \times \chi_R(x)(x, y))) \quad \text{in } \Omega(t) \\
\text{div } u &= 0 \quad \text{in } \Omega(t)
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- (2\mu(|Du|^2)Du - \pi)\nu &= \sigma \kappa \nu \quad \text{on } \Gamma_+(t) \\
\nu &= (u|\nu) \quad \text{on } \Gamma_+(t)
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$$-(2\mu(|Du|^2)Du - \pi)\nu = \sigma \kappa \nu \quad \text{on } \Gamma_+(t)$$

$$V = (u|\nu) \quad \text{on } \Gamma_+(t)$$

$$v = c(h + \delta)^\alpha \partial_y v \quad \text{on } \Gamma_+$$

$$w = 0 \quad \text{on } \Gamma_-$$

$u(0) = u_0 \quad \text{in } \Omega(0)$

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Main result (spin-coating)

Assumptions:
- $5 < p < \infty$
- $\mu \in C^3$ with $\mu(0) > 0$
- $u_0 \in W^{2-2/p}_p$, $h_0 \in W^{3-2/p}_p$ satisfying suitable compatibility conditions

For all $T > 0$ there exists $\varepsilon > 0$ such that for

$$\|\omega \times (\omega \times \chi_R(x)(x, y))\|_{L_p(L_p)} + \|u_0\|_{W^{2-2/p}_p} + \|h_0\|_{W^{3-2/p}_p} < \varepsilon,$$
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$$h \in W^{2-1/2p}_p(L_p) \cap W^1_p(W^{2-1/p}_p) \cap L_p(W^{3-1/p}_p).$$
Known Results

Two-phase flow

- Prüß, Simonett - constant viscosity:
  - ’09 detailed investigation of the mapping properties of the boundary symbol
  - ’10 solvability of the linear and nonlinear problems (small data)
  - t.a. solvability of the linear and nonlinear problems with gravity (large data)
  - t.a. solvability of the linear and nonlinear problems with gravity (small data) and Rayleigh-Taylor instability

- Bothe, Prüß - nonlinear viscosity:
  - ’07 fixed domain, large data

- Amann - nonlinear viscosity:
  - ’94 fixed domain, small data

- Shibata, Shimizu, Solonnikov, . . .:
  - Lagrange approach to the free interface problem
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Spin-Coating

- Denk, Geissert, Hieber, Saal, Sawada:
  - '11 same setting for Newtonian fluids
Short sketch of proof I

- Hanzawa transform onto a fixed reference domain

\[
(t, x, y) \rightarrow (t, x, y + h(t, x)) \quad (t, x, y) \rightarrow \left( t, x, y \frac{h(t, x) + \delta}{\delta} \right)
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Short sketch of proof I

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- receive error terms for derivatives

\[D u \rightarrow D \tilde{u} - F_D(\partial_y \tilde{u}, \nabla' h),\]
\[\nabla \pi \rightarrow \nabla \tilde{\pi} - F_\pi(\partial_y \tilde{\pi}, \nabla h),\]
\[\text{div } u \rightarrow \text{div } \tilde{u} - F_d(\tilde{u}, h), \quad \text{etc.}\]
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- receive error terms for derivatives

\[
Du \rightarrow D\tilde{u} - F_D(\partial_y \tilde{u}, \nabla' h),
\]

\[
\nabla \pi \rightarrow \nabla \tilde{\pi} - F_\pi(\partial_y \tilde{\pi}, \nabla h),
\]

\[
\text{div } u \rightarrow \text{div } \tilde{u} - F_d(\tilde{u}, h), \quad \text{etc.}
\]

- the kinematic condition \( V = (u|\nu) \) transforms to

\[
\partial_t h + (\nabla' h|\nu) = w
\]
nonlinear transformed system (in the case of the two-phase flow)

\[
\begin{align*}
\rho \partial_t u - \mu(0) \Delta u + \nabla \pi &= F_f(u, \pi, h) \quad \text{in } \mathbb{R}^n \\
\text{div } u &= F_d(u, h) \quad \text{in } \mathbb{R}^n \\
\partial_t h - w &= H(u, h) \quad \text{on } \mathbb{R}^{n-1} \\
-\left[\mu(0) \partial_y w\right] - \left[\mu(0) \nabla' w\right] &= G_v(u, \left[\pi\right], h) \quad \text{on } \mathbb{R}^{n-1} \\
-2\left[\mu(0) \partial_y w\right] + \left[\pi\right] - \left[\rho\right] \gamma_a h - \sigma \Delta h &= G_w(u, h) \quad \text{on } \mathbb{R}^{n-1} \\
\left[u\right] &= 0 \quad \text{on } \mathbb{R}^{n-1} \\
u(0) &= u_0 \quad \text{in } \mathbb{R}^n \\
h(0) &= h_0 \quad \text{on } \mathbb{R}^{n-1}
\end{align*}
\]
Transformed system

linear transformed system (in the case of the two-phase flow)

\[
\begin{align*}
\rho \partial_t u - \mu(0) \Delta u + \nabla \pi &= f & \text{in} & \hat{\mathbb{R}}^n \\
\text{div } u &= f_d & \text{in} & \hat{\mathbb{R}}^n \\
\partial_t h - w &= f_h & \text{on} & \mathbb{R}^{n-1} \\
-[[\mu(0) \partial_y v]] - [[\mu(0) \nabla' w]] &= g_v & \text{on} & \mathbb{R}^{n-1} \\
-2[[\mu(0) \partial_y w]] + [[\pi]] - [[\rho] \gamma_a h - \sigma \Delta h} &= g_w & \text{on} & \mathbb{R}^{n-1} \\
[u] &= 0 & \text{on} & \mathbb{R}^{n-1} \\
u(0) &= u_0 & \text{in} & \hat{\mathbb{R}}^n \\
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Transformed system

- Linear transformed system (in the case of the two-phase flow)

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\rho \partial_t u - \mu(0) \Delta u + \nabla \pi = f \quad \text{in } \mathbb{R}^n \\
\text{div } u = f_d \quad \text{in } \mathbb{R}^n \\
\partial_t h - w = f_h \quad \text{on } \mathbb{R}^{n-1} \\
- [\mu(0) \partial_y v] - [\mu(0) \nabla' w] = g_v \quad \text{on } \mathbb{R}^{n-1} \\
-2[\mu(0) \partial_y w] + [\pi] - [\rho] \gamma a h - \sigma \Delta h = g_w \quad \text{on } \mathbb{R}^{n-1} \\
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- Prüß, Simonett/Denk, Geissert, Hieber, Saal, Sawada: there exists a well-defined continuous solution operator to the linearized system \( L^{-1} : \mathcal{F} \mapsto \mathcal{E} \) under certain compatibility conditions on the initial data
the Nemytskij-operators of $\mu$ and $\mu'$ are Fréchet differentiable in suitable function spaces
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nonlinearity $N$ is essentially polynomial in the data, $\mu$ and $\mu'$ with some exceptions

Thank you for your attention.
Short sketch of proof II

- the Nemytskij-operators of $\mu$ and $\mu'$ are Fréchet differentiable in suitable function spaces

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- $N$ is continuously Fréchet differentiable with $N(0) = 0$ and $DN(0) = 0$
Short sketch of proof II

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- contraction mapping principle provides a unique solution for small data
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- Nonlinearity $N$ is essentially polynomial in the data, $\mu$ and $\mu'$ with some exceptions.
- $N$ is continuously Fréchet differentiable with $N(0) = 0$ and $DN(0) = 0$.
- Contraction mapping principle provides a unique solution for small data.

Thank you for your attention.