The spin-coating process and two-phase flow for generalized Newtonian fluids

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Setting for the two-phase flow problem

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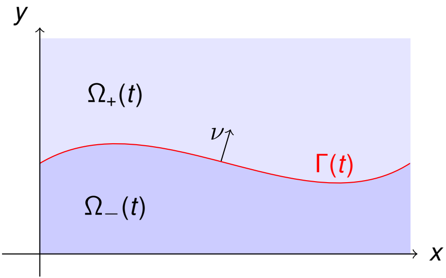
- two fluids occupying the whole space $\mathbb{R}^n = \Omega_+(t) \cup \Omega_-(t) \cup \Gamma(t)$
- separated by a sharp interface $\Gamma(t) = \{(x, y) \in \mathbb{R}^n : y = h(t, x)\}$



Model equations

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- model equations

$$\begin{aligned} \rho(\partial_t u + u.\nabla u) - 2\,\mathrm{div}\,\mu(|Du|^2)Du + \nabla\pi &= -\rho\gamma_a e_{n+1} && \text{in } \Omega(t) \\ \mathrm{div}\, u &= 0 && \text{in } \Omega(t) \end{aligned}$$

- u velocity
- Du deformation tensor
- π pressure
- ρ density
- μ viscosity function
- γ_a gravity

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Nonlinear diffusion term

- introduce quasilinear operator $\mathcal{A}$

$$\mathcal{A}(u)v = \sum_{j,l=1}^{n} A^{j,l}(Du)\partial_k \partial_l v$$

- $A^{j,l}_{i,k}(X) = \mu(|X|^2)\delta_{i,k}\delta_{i,j} + \mu'(|X|^2)X_{ik}X_{jl}$

- then $\mathcal{A}(u)u = -2\,\mathrm{div}\,\mu(|Du|^2)Du$ and $\mathcal{A}(0)u = -\mu(0)\Delta u$

Main result (two-phase flow)

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Assumptions:

- $n + 2 < p < \infty$
- $\mu \in C^3$ with $\mu(0) > 0$
- $u_0 \in W_p^{2-2/p}$, $h_0 \in W_p^{3-2/p}$ satisfying suitable compatibility conditions

For all $T > 0$ there exists $\varepsilon > 0$ such that for

$$\|u_0\|_{W_p^{2-2/p}} + \|h_0\|_{W_p^{3-2/p}} < \varepsilon,$$

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there exists a unique solution (u, π, h) on $(0, T)$ in

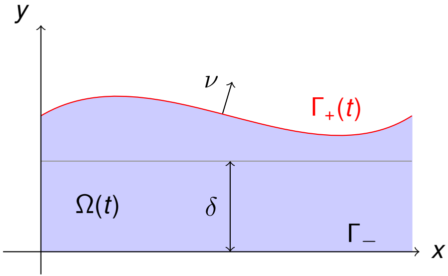
$$u \in H_p^1(L_p) \cap L_p(H_p^2),$$

$$\pi \in \{q \in L_p(\widehat{H}_p^1) : [\![q]\!] \in W_p^{1/2-1/2p}(L_p) \cap L_p(W_p^{1-1/p})\},$$

$$h \in W_p^{2-1/2p}(L_p) \cap W_p^1(W_p^{2-1/p}) \cap L_p(W_p^{3-1/p}).$$

Setting for the spin-coating problem

- a fluid is applied to the center of a spinning plate and is assumed to occupy a domain $\Omega(t)$ close to a layer
- the bottom boundary is a fixed plane Γ_-
- the upper boundary is a free surface initially close to a plane $\Gamma_+(t) = \{(x, y) \in \mathbb{R}^n : y = \delta + h(t, x)\}$, $\delta > 0$



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- model equations for $u = (v, w)$, $n = 3$

$$\begin{aligned} \rho(\partial_t u + u.\nabla u) + \mathcal{A}(u)u + \nabla\pi &= -\rho(2\omega \times u + \omega \times (\omega \times \chi_R(x)(x, y))) && \text{in } \Omega(t) \\ \operatorname{div} u &= 0 && \text{in } \Omega(t) \end{aligned}$$

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Known Results

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Two-phase flow

- Prüß, Simonett - constant viscosity:
  - '09 detailed investigation of the mapping properties of the boundary symbol
  - '10 solvability of the linear and nonlinear problems (small data)
  - t.a. solvability of the linear and nonlinear problems with gravity (large data)
  - t.a. solvability of the linear and nonlinear problems with gravity (small data) and Rayleigh-Taylor instability

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  - '07 fixed domain, large data
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Spin-Coating

- Denk, Geissert, Hieber, Saal, Sawada:
  - '11 same setting for Newtonian fluids

Short sketch of proof I

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- Hanzawa transform onto a fixed reference domain

$$(t,x,y) \to (t,x,y+h(t,x)) \qquad (t,x,y) \to \left(t,x,y\frac{h(t,x)+\delta}{\delta}\right)$$

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- receive error terms for derivatives

$$Du \to D\bar{u} - F_D(\partial_y \bar{u}, \nabla' h),$$
$$\nabla\pi \to \nabla\bar{\pi} - F_\pi(\partial_y \bar{\pi}, \nabla h),$$
$$\operatorname{div} u \to \operatorname{div} \bar{u} - F_d(\bar{u}, h), \quad \text{etc.}$$

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$$\operatorname{div} u \to \operatorname{div} \bar{u} - F_d(\bar{u}, h), \quad \text{etc.}$$

- the kinematic condition $V = (u|\nu)$ transforms to

$$\partial_t h + (\nabla' h | v) = w$$

Transformed system

- nonlinear transformed system (in the case of the two-phase flow)

$$\begin{aligned}
\rho\partial_t u - \mu(0)\Delta u + \nabla\pi &= F_f(u,\pi,h) && \text{in } \dot{\mathbb{R}}^n \\
\operatorname{div} u &= F_d(u,h) && \text{in } \dot{\mathbb{R}}^n \\
\partial_t h - w &= H(u,h) && \text{on } \mathbb{R}^{n-1} \\
-[\![\mu(0)\partial_y v]\!] - [\![\mu(0)\nabla' w]\!] &= G_v(u,[\![\pi]\!],h) && \text{on } \mathbb{R}^{n-1} \\
-2[\![\mu(0)\partial_y w]\!] + [\![\pi]\!] - [\![\rho]\!]\gamma_a h - \sigma\Delta h &= G_w(u,h) && \text{on } \mathbb{R}^{n-1} \\
[\![u]\!] &= 0 && \text{on } \mathbb{R}^{n-1} \\
u(0) &= u_0 && \text{in } \dot{\mathbb{R}}^n \\
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- linear transformed system (in the case of the two-phase flow)

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\begin{aligned}
\rho\partial_t u - \mu(0)\Delta u + \nabla\pi &= f && \text{in } \dot{\mathbb{R}}^n \\
\operatorname{div} u &= f_d && \text{in } \dot{\mathbb{R}}^n \\
\partial_t h - w &= f_h && \text{on } \mathbb{R}^{n-1} \\
-[\![\mu(0)\partial_y v]\!] - [\![\mu(0)\nabla' w]\!] &= g_v && \text{on } \mathbb{R}^{n-1} \\
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- Prüß, Simonett/Denk, Geissert, Hieber, Saal, Sawada: there exists a well-defined continuous solution operator to the linearized system $L^{-1} : \mathbb{F} \mapsto \mathbb{E}$ under certain compatibility conditions on the initial data

Short sketch of proof II

- the Nemytskij-operators of μ and μ' are Fréchet differentiable in suitable function spaces

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- nonlinearity *N* is essentially polynomial in the data, μ and μ' with some exceptions

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- contraction mapping principle provides a unique solution for small data

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